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ZERO EPI KKM TYPE MAPS AND FIXED POINTS*

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Dedicated with admiration to Professor Constantin Fetecau

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Abstract

In this paper, we present two approaches to fixed point theory for compact KKM type maps. In particular, we consider Leray–Schauder type alternatives for these maps.

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MSC: 47H10, 54H25.

1 Introduction

This paper discusses subclasses of KKM maps (also called KKM maps for simplicity) or maps with a KKM type selection (motivated from [5, 6, 11]) and we present two approaches to fixed point theory. In the first approach using results of the author [9] (which uses the idea of null homotopy) we present fixed point results (see Theorem 8) for compact KKM type maps. This motivates our second approach which is based on the notion of a zero epi map (see [4, 7]) which enables us to present coincidence and Leray–Schauder type results for the various classes of maps mentioned above. Both approaches are based on composition and product rules so our results are new and complement previous results in the literature (see [1, 2, 8, 10, 11] and the references therein).

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Now, we describe the KKM maps which we consider in this paper. Let X be a convex subset of a Hausdorff topological vector space and Y a Hausdorff topological space. If $S, T : X \to 2^Y$ (nonempty subsets of Y) are two set valued maps such that $T(co\,(A)) \subseteq S(A)$ for each finite subset A of X then we call S a generalized KKM mapping w.r.t. T. Now the set valued map $T : X \to 2^Y$ is said to have the KKM property if for any generalized KKM map $S : X \to 2^Y$ w.r.t. T the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property (the intersection of each finite subfamily is nonempty). We let

$$KKM(X,Y) = \{T : X \to 2^Y \mid T \text{ has the } KKM \text{ property }\}.$$

First, we recall the following result [2].

Theorem 1. *Let X be a convex subset of a Hausdorff topological vector space and Y, Z be Hausdorff topological spaces.*
(i). $T \in KKM(X,Y)$ iff $T|_\triangle \in KKM(\triangle, Y)$ for each polytope $\triangle$ in X;
(ii). if $T \in KKM(X,Y)$ and $f \in C(Y,Z)$ then $f\,T \in KKM(X,Z)$;
(iii). if Y is a normal space, $\triangle$ a polytope of X and if $T : \triangle \to 2^Y$ is a set valued map such that for each $f \in C(Y, \triangle)$ we have that $f\,T$ has a fixed point in $\triangle$, then $T \in KKM(\triangle, Y)$.

Next we note a fixed point result for KKM maps in [1]. Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

Theorem 2. *Let X be an admissible convex set in a Hausdorff topological vector space E and $T \in KKM(X,X)$ be a closed compact map. Then T has a fixed point in X.*

From Theorem 1 and Theorem 2 one has the following result.

Theorem 3. *Let X be an admissible convex set in a Hausdorff topological vector space and Z a subset of a Hausdorff topological space. If $T \in KKM(X,Z)$ is a upper semicontinuous compact map with compact values and $f \in C(Z,X)$ then $T\,f$ has a fixed point in Z.*

Proof. Now $T \in KKM(X,Z)$, $f \in C(Z,X)$ and Theorem 1 (ii) implies $f\,T \in KKM(X,X)$. Also $f\,T$ is a compact upper semicontinuous map with

compact values (so $f\,F$ is a closed map). Now, Theorem 2 guarantees that $f\,T$ has a fixed point in X and consequently $T\,f$ has a fixed point in Z. □

Next, we present an analogue of Theorem 1 (ii) for $T\,f$ and this composition will be needed later.

Theorem 4. *(i). Let X be an admissible convex set in a Hausdorff topological vector space, Y a convex set in a Hausdorff topological vector space and Z a normal topological space. If $T \in KKM(X,Z)$ is an upper semicontinuous map with compact values and $f \in C(Y,X)$ then $T\,f \in KKM(Y,Z)$.*

(ii). Let X be an admissible convex set in a Hausdorff topological vector space, Y a convex set in a Hausdorff topological vector space and Z a Hausdorff topological space. Suppose $T \in KKM(X,Z)$ is an upper semicontinuous map with compact values, $f \in C(Y,X)$ and there exists a compact subset K of Z with $T(X) \subseteq K$. Then $T\,f \in KKM(Y,K)$.

Proof. (i). Note $T\,f : Y \to 2^Z$. From Theorem 1 (i), (iii) we need to show that for each polytope $\triangle$ in Y that $g(T\,f)$ has a fixed point in $\triangle$ for any $g \in C(Z,\triangle)$. Note from Theorem 1 (ii) since $T \in KKM(X,Z)$ and $g \in C(Z,\triangle)$ that $g\,T \in KKM(X,\triangle)$. Now from Theorem 3 (note there $Z = \triangle$ is compact and $g\,T : X \to 2^{\triangle}$ is a upper semicontinuous compact map with compact values) guarantees that $(g\,T)\,f$ has a fixed point in $\triangle$.

(ii). Note $T\,f : Y \to 2^K$ and from Theorem 1 (i), (iii) (note K is normal) we need to show that for each polytope $\triangle$ in Y that $g(T\,f)$ has a fixed point in $\triangle$ for any $g \in C(K,\triangle)$. The proof is exactly the same as in (i) with Z replaced by K. □

Note the following two properties (see [8]). Let C and X be convex subsets of a Hausdorff topological vector space E with $C \subseteq X$ and Y a Hausdorff topological space.

(i). If $T \in KKM(X,Y)$ then $G \equiv T|_C \in KKM(C,Y)$.

(ii). If $T \in KKM(X,Y)$, $T(X) \subseteq Z \subseteq Y$ and Z is closed in Y then $T \in KKM(X,Z)$.

A product rule for the KKM class is unknown and we note the argument given in [10] was incorrect. However a product rule is known for most subclasses of KKM maps (so the results in [10] are correct for these subclasses). As a result in this paper we consider subclasses of KKM maps (for simplicity in notation we call these KKM also) where the following Property A holds. We also assume throughout the paper that the analogue of Theorem 1 (ii), Theorem 4 (ii), properties (i) and (ii) above hold (in the relevant setting) for this subclass of KKM maps (see Remark 1).

Property A. Let I be a finite set, X be a convex set in a Hausdorff topological vector space and $\{Y_i\}_{i\in I}$ be a family of Hausdorff topological spaces. For each $i \in I$ suppose $T_i \in KKM(X, Y_i)$ is an upper semicontinuous map with compact values and let $T : X \to 2^Y$ (here $Y = \prod_{i\in I} Y_i$) be defined by $T(x) = \prod_{i\in I} T_i(x)$ for $x \in X$. Also assume for each $i \in I$ that there exists a compact set $K_i \subseteq Y_i$ with $T_i(X) \subseteq K_i$. Then $T \in KKM(X, K)$ where $K = \prod_{i\in I} K_i$.

Remark 1. *Note we also have $T \in KKM(X, Y)$; consider $i \circ T$ where $i : K \to Y$ is the inclusion map. Note Property A holds (in the relevant setting) for the usual classes of maps in the literature, for example the maps admissible with respect to Gorniewicz [3], selectionable maps [3], approximable maps etc (there are also analogues of Theorem 1 (ii), Theorem 4 (ii), properties (i) and (ii) above for these maps). Note Theorem 1 (ii), Theorem 4 (ii), and property (i) are a lot less restrictive for Gorniewicz maps, selectionable maps and approximable maps (i.e., the analogue of Theorem 4 (ii) is a lot more general for these maps). Property (ii) above is immediate for the Gorniewicz maps but the set Z is more special for the selectionable and approximable maps. As a result some of the conditions on the spaces can be removed in Section 2 for these subclasses. This will be discussed in a future paper.*

2 Fixed point theory

We begin by recalling some results of the author established in [9]. Let E be a Hausdorff topological vector space.

Definition 1. *We say $F \in KKMC(E, E)$ if $F \in KKM(E, E)$ is an upper semicontinuous compact map with compact values.*

Definition 2. *Let $F \in KKMC(E, E)$. We say $F \cong \{0\}$ in $KKMC(E, E)$ if there exists an upper semicontinuous compact map $\Omega : E \times [0, 1] \to K(E)$ with $\Omega(., \eta(.)) \in KKM(E, E)$ for any continuous map $\eta : E \to [0, 1]$, $\Omega_1 = F$ and $\Omega_0 = \{0\}$; here $K(E)$ denotes the family of nonempty, compact subsets of E and $\Omega_t(x) = \Omega(x, t)$.*

In [9], we used the idea of null homotopy to establish the following result.

Theorem 5. *Let E be a Hausdorff topological vector space and U an open subset of E with $0 \in U$. Suppose there exists an upper semicontinuous compact map $H : E \times [0, 1] \to K(E)$ with $H(., \eta(.)) \in KKM(E, E)$ for*

any continuous map $\eta : E \to [0,1]$, $H(x,0) = \{0\}$ *for each* $x \in E$ *and* $x \notin H_t(x)$ *for* $x \in \partial U$ *and* $t \in (0,1]$*; here* ∂U *denotes the boundary of* U *in* E*. In addition assume the following holds:*

$$(2.1)\quad \begin{cases} \text{for any } \Phi \in KKMC(E,E) \text{ with } \Phi \cong \{0\} \text{ in } KKMC(E,E) \\ \text{we have that } \Phi \text{ has a fixed point in } E. \end{cases}$$

Then H_1 *has a fixed point in* U*.*

A natural question to ask in Theorem 5 is if we had an upper semicontinuous compact map $\Lambda : \overline{U} \times [0,1] \to K(E)$ (here $\overline{U}$ denotes the closure of U in E) would it be possible to construct the map H in the statement of Theorem 5. In [9], we obtained the following result.

Theorem 6. *Let* E *be a Hausdorff topological vector space,* U *an open subset of* E *with* $0 \in U$*, and suppose there exists a retraction (continuous)* $r : E \to \overline{U}$*. Also assume there exists an upper semicontinuous compact map* $\Lambda : \overline{U} \times [0,1] \to K(E)$ *with* $\Lambda(r(.)\,, \eta(.)) \in KKM(E,E)$ *for any continuous map* $\eta : E \to [0,1]$, $\Lambda(x,0) = \{0\}$ *for each* $x \in \overline{U}$ *and* $x \notin \Lambda_t(x)$ *for* $x \in \partial U$ *and* $t \in (0,1]$*. In addition assume* (2.1) *holds. Then* Λ_1 *has a fixed point in* U*.*

Remark 2. *In the proof of Theorem* 6 *we used Theorem* 5 *with* $H(x,t) = \Lambda(r(x),t)$ *for* $(x,t) \in E \times [0,1]$*.*

We consider the subclass of KKM maps (again for simplicity we call these KKM) for the remainder of Section 2 and we assume Theorem 1 (ii), Theorem 4 (ii), properties (i) and (ii) in Section 1 and Property A hold for the subclass. The new idea here is to consider the natural homotopy between H_1 and the zero map in Definition 2 and Theorem 5 (and the corresponding one in Theorem 6) to generate applicable fixed point theory for the map H_1 in Theorem 5 (and also Λ_1 in Theorem 6). This is possible because of Section 1 where the idea here is to use appropriately Theorem 1 (ii), Theorem 4 (ii) and Property A (for the subclasses of KKM mentioned above and we use the appropriate $\cong$ in that subclass; note the proof of the analogue of Theorem 5 and Theorem 6 for this subclass is exactly the same as in [9]). We first consider Theorem 5 and then Theorem 6.

Theorem 7. *Let* E *be a Hausdorff topological vector space and* U *an open subset of* E *with* $0 \in U$*. Let* $F \in KKMC(E,E)$ *and assume the following conditions hold:*

$$(2.2)\qquad x \notin t\,F(x) \text{ for } x \in \partial U \text{ and } t \in (0,1]$$

and

(2.3) *any* $\Phi \in KKMC(E,E)$ *has a fixed point in* E.

Then F *has a fixed point in* U.

Proof. Let $H(x,t) = t\,F(x)$ for $(x,t) \in E \times [0,1]$. Note $H : E \times [0,1] \to K(E)$ is an upper semicontinuous compact map with compact values. Let $\eta : E \to [0,1]$ be a continuous map (i.e. $\eta \in C(E,[0,1])$). We claim $H(\,.\,,\eta(\,.\,)) = \eta(\,.\,)\,F(\,.\,) \in KKM(E,E)$. To see this first note that there exists a compact subset K of E with $F(E) \subseteq K$. Let $\Psi : E \to 2^{[0,1]\times E}$ be given by $\Psi(x) = J(x) \times F(x)$, $x \in E$ where $J(x) = \eta(x)$. From Property A (with $T_1 = J$, $T_2 = F$, $X = E$, $K_1 = [0,1]$ and $K_2 = K$) we have that $\Psi \in KKM(E,[0,1] \times K)$ is an upper semicontinuous compact (note $\Psi(E) \subseteq [0,1]\times K$) map with compact values. Let $p : [0,1]\times K \to E$ be given by $p(t,x) = t\,x$ for $(t,x) \in [0,1] \times K$ and note $p \in C([0,1] \times K, E)$. From Theorem 1 (ii) we see that $H(\,.\,,\eta(\,.\,)) = \eta(\,.\,)\,F(\,.\,) = p\,\Psi(\,.\,) \in KKM(E,E)$, so our claim is true. Also note $H_0 = \{0\}$, $H_1 = F$ and $x \notin H_t(x)$ for $x \in \partial U$ and $t \in [0,1]$ from (2.2). Thus we have constructed a H as in Theorem 5.

We next note if $\Phi \in KKMC(E,E)$ then $\Phi \cong \{0\}$ in $KKMC(E,E)$ (i.e. we need to show that there exists an upper semicontinuous compact map $\Omega : E \times [0,1] \to K(E)$ with $\Omega(\,.\,,\eta(\,.\,)) \in KKM(E,E)$ for any continuous $\eta : E \to [0,1]$, $\Omega_1 = \Phi$ and $\Omega_0 = \{0\}$). To see this let $\Omega(t,x) = t\,\Phi(x)$ for $(x,t) \in E \times [0,1]$ and exactly the same argument as above with H replaced by Ω and F replaced by Φ guarantees this. Thus (2.3) guarantees (2.1). Thus, our result now follows from Theorem 5. □

Remark 3. *Note if* E *was admissible then Theorem* 2 *guarantees that* (2.3) *holds.*

Let E be a Hausdorff topological vector space and U an open convex subset of E.

Definition 3. *We say* $F \in KKMC(\overline{U},E)$ *if* $F \in KKM(\overline{U},E)$ *is an upper semicontinuous compact map with compact values.*

Theorem 8. *Let* E *be a Hausdorff topological vector space,* U *an open convex subset of* E, $0 \in U$, $\overline{U}$ *an admissible subset of* E *and assume there exists a retraction (continuous)* $r : E \to \overline{U}$. *Let* $F \in KKMC(\overline{U},E)$ *and suppose* (2.2) *and* (2.3) *hold. Then* F *has a fixed point in* U.

Proof. Let $\Lambda(x,t) = t\,F(x)$ for $(x,t) \in \overline{U}\times[0,1]$. Note $\Lambda : \overline{U}\times[0,1] \to K(E)$ is an upper semicontinuous compact map with compact values. Let $\eta : E \to$

$[0,1]$ be a continuous map and we claim $\Lambda(r(\,.\,),\eta(\,.\,)) = \eta(\,.\,)\,F\,r(\,.\,) \in KKM(E,E)$. First note that there exists a compact subset K of E with $F(\overline{U}) \subseteq K$ and note $F \in KKM(\overline{U},E)$ (is an upper semicontinuous compact map with compact values), $r \in C(E,\overline{U})$ so Theorem 4 (ii) (note $\overline{U}$ is admissible) guarantees that $F\,r \in KKM(E,K)$ is an upper semicontinuous compact map with compact values. Let $\Psi : E \to 2^{[0,1]\times E}$ be given by $\Psi(x) = J(x) \times F\,r(x)$, $x \in E$ where $J(x) = \eta(x)$. From Property A (with $T_1 = J$, $T_2 = F\,r$, $X = E$, $K_1 = [0,1]$ and $K_2 = K$) we have that $\Psi \in KKM(E,[0,1] \times K)$ is an upper semicontinuous compact with compact values. Let $p : [0,1] \times K \to E$ be given by $p(t,x) = t\,x$ for $(t,x) \in [0,1] \times K$ and note $p \in C([0,1] \times K, E)$. From Theorem 1 (ii) we see that $\Lambda(r(\,.\,),\eta(\,.\,)) = \eta(\,.\,)\,F\,r(\,.\,) = p\,\Psi(\,.\,) \in KKM(E,E)$, so our claim is true. Also note $\Lambda_0 = \{0\}$, $\Lambda_1 = F$ and $x \notin \Lambda_t(x)$ for $x \in \partial U$ and $t \in [0,1]$ from (2.2). Thus we have constructed a Λ as in Theorem 6. Also (2.1) holds as in Theorem 7. Thus, our result follows from Theorem 5. □

Remark 4. *If E is a locally convex topological vector space and U is an open convex subset of E then $\overline{U}$ is a retract of E and also note $\overline{U}$ is an admissible subset of E . We either note Dugundji's extension theorem [3] or alternatively let $r : E \to \overline{U}$ be given by*

$$r(x) = \frac{x}{\max\{1,\eta(x)\}} \quad \text{for} \quad x \in E$$

where η is the Minkowski functional on $\overline{U}$ i.e. $\eta(x) = \inf\{\alpha :\ x \in \alpha\,\overline{U}\}$.

The theory above now motivates a theory of zero epi compact KKM maps which we now present. The classes of maps considered was motivated from the KLU [5], $HLPY$ [6] and Wu [11] maps. We will consider three classes (the arguments are very similar in each class).

Let E be a Hausdorff topological vector space (so completely regular) and U an open convex subset of E.

Definition 4. *We say $F \in A(\overline{U},E)$ if $F : \overline{U} \to C(E)$ is upper semicontinuous; here $C(E)$ denotes the family of nonempty closed subsets of E.*

Definition 5. *We say $F \in A_{\partial U}(\overline{U},E)$ if $F \in A(\overline{U},E)$ and $0 \notin F(x)$ for $x \in \partial U$.*

Definition 6. *We say $G \in csKKMC(\overline{U},E)$ if $G : \overline{U} \to 2^E$ and there exists a $\Phi \in KKMC(\overline{U},E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$.*

Definition 7. *We say* $G \in csKKMC_0(\overline{U}, E)$ *if* $G : \overline{U} \to 2^E$ *and there exists a* $\Phi \in KKMC(\overline{U}, E)$ *with* $\Phi(x) \subseteq co\,(G(x))$ *for* $x \in \overline{U}$ *and* $\Phi(x) = \{0\}$ *for* $x \in \partial U$.

Now we present the notion of 0–epi maps [4, 7].

Definition 8. *Let* $F \in A_{\partial U}(\overline{U}, E)$. *We say* F *is* 0*–epi if for any map* $G \in csKKMC_0(\overline{U}, E)$ *and any* $\Phi \in KKMC(\overline{U}, E)$ *with* $\Phi(x) \subseteq co\,(G(x))$ *for* $x \in \overline{U}$ *and* $\Phi(x) = \{0\}$ *for* $x \in \partial U$ *there exists a* $x \in U$ *with* $F(x) \cap \Phi(x) \neq \emptyset$.

Remark 5. *(i). In Definition 8 note* $\emptyset \neq F(x) \cap \Phi(x) \subseteq F(x) \cap co\,(G(x))$ *so there exists a* $x \in U$ *with* $F(x) \cap co\,(G(x)) \neq \emptyset$.

(ii). In Definition 8 (take $G = 0$*) note there exists a* $x \in U$ *with* $0 \in F(x)$.

We begin with a coincidence result.

Theorem 9. *Let* E *be a Hausdorff topological vector space,* U *an open convex subset of* E, $G \in csKKMC(\overline{U}, E)$ *and* $F \in A_{\partial U}(\overline{U}, E)$ *be* 0*–epi. In addition for any* $\Phi \in KKMC(\overline{U}, E)$ *with* $\Phi(x) \subseteq co\,(G(x))$ *for* $x \in \overline{U}$ *suppose*

$$\left\{x \in \overline{U} : \ F(x) \cap t\,\Phi(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

does not intersect ∂U. *There there exists an* $x \in U$ *with* $F(x) \cap co\,(G(x)) \neq \emptyset$.

Proof. Let $\Phi \in KKMC(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and let

$$K = \left\{x \in \overline{U} : \ F(x) \cap t\,\Phi(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

Note $K \neq \emptyset$ (take $t = 0$ and note Remark 5 (ii)), K is closed (recall F and Φ are upper semicontinuous) and compact (note Φ is compact). By assumption $K \cap \partial U = \emptyset$. Now since E is completely regular then there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $J(x) = \mu(x)\,G(x)$ and $R(x) = \mu(x)\,\Phi(x)$ for $x \in \overline{U}$. Note $R(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ (since for a fixed $x \in \overline{U}$ note $\mu(x)\,\Phi(x) \subseteq \mu(x)\,co\,(G(x)) = co\,(\mu(x)\,G(x))$ because $co(t\,A) = t\,co(A)$). Also note $R : \overline{U} \to 2^E$ is an upper semicontinuous compact map with compact values and $R \in KKM(\overline{U}, E)$. To see this note that there exists a compact subset K_0 of E with $\Phi(\overline{U}) \subseteq K_0$. Let $\Psi : \overline{U} \to 2^{[0,1]\times E}$ be given by $\Psi(x) = J_0(x) \times \Phi(x)$, $x \in \overline{U}$ where $J_0(x) = \mu(x)$. From Property A (with $T_1 = J_0$, $T_2 = \Phi$, $X = \overline{U}$, $K_1 = [0,1]$ and $K_2 = K_0$) we have that $\Psi \in KKM(\overline{U}, [0,1] \times K_0)$ is an upper semicontinuous compact (note $\Psi(\overline{U}) \subseteq [0,1] \times K_0$) map with compact values. Let $p : [0,1] \times K_0 \to E$ be given by $p(t,x) = t\,x$ for $(t,x) \in [0,1] \times K_0$

and note $p \in C([0,1] \times K_0, E)$. From Theorem 1 (ii) we see that $R(\,.\,) = \mu(\,.\,)\,\Phi(\,.\,) = p\,\Psi(\,.\,) \in KKM(\overline{U}, E)$ is an upper semicontinuous compact map with compact values. Next notice $J \in csKKMC_0(\overline{U}, E)$ since if $x \in \partial U$ then $R(x) = \mu(x)\,\Phi(x) = \{0\}$ (note $\mu(\partial U) = 0$). Now since F is 0–epi then there exists a $y \in U$ with $F(y) \cap R(y) \neq \emptyset$. Note $y \in K$ so $\mu(y) = 1$ i.e. $F(y) \cap \Phi(y) \neq \emptyset$ so $F(y) \cap co\,(G(y)) \neq \emptyset$. □

Next, we present a normalization result.

Theorem 10. *Let E be an admissible Hausdorff topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E and assume there exists a retraction $r : E \to \overline{U}$. Then the identity mapping i is 0–epi.*

Proof. Note $i(x) = x$ for $x \in \overline{U}$ and $i \in A_{\partial U}(\overline{U}, E)$ (note $0 \notin i(x) = x$ for $x \in \partial U$ since $0 \in U$). Consider any $G \in csKKMC_0(\overline{U}, E)$ and any $\Phi \in KKMC(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$. Let

$$\Omega = \left\{x \in \overline{U} : \;\; x \in \lambda\,\Phi(x) \;\; \text{ for some } \;\; \lambda \in [0,1]\right\}.$$

Now $\Omega \neq \emptyset$ (note $0 \in U$) and Ω is closed and compact. Note $\Omega \cap \partial U = \emptyset$ since if there exists a $x \in \partial U$ and $x \in \Omega$ then for some $\lambda \in [0,1]$ we have $x \in \lambda\,\Phi(x) = \{0\}$, so $x = 0$ which is a contradiction since $0 \in U$. Now there exists a continuous map $\mu : E \to [0,1]$ with $\mu(E \backslash U) = 0$ and $\mu(\Omega) = 1$. Let $J(x) = \mu(x)\,\Phi(r(x))$. Note $J : E \to 2^E$ is an upper semicontinuous compact map with compact values and $J \in KKM(E, E)$. To see this first note that there exists a compact subset K of E with $\Phi(\overline{U}) \subseteq K$ so $\Phi\, r(E) \subseteq \Phi(\overline{U}) \subseteq K$. Note $\Phi \in KKM(\overline{U}, E)$, $r \in C(E, \overline{U})$ so Theorem 4 (ii) (note $\overline{U}$ is admissible) guarantees that $\Phi\, r \in KKM(E, K)$ is an upper semicontinuous compact map with compact values. Let $\Psi : E \to 2^{[0,1] \times E}$ be given by $\Psi(x) = J_0(x) \times \Phi\, r(x)$, $x \in E$ where $J_0(x) = \mu(x)$. From Property A (with $T_1 = J_0$, $T_2 = \Phi\, r$, $X = E$, $K_1 = [0,1]$ and $K_2 = K$) we have that $\Psi \in KKM(E, [0,1] \times K)$ is an upper semicontinuous compact (note $\Psi(E) \subseteq [0,1] \times K$) with compact values. Let $p : [0,1] \times K \to E$ be given by $p(t,x) = t\,x$ for $(t,x) \in [0,1] \times K$ and note $p \in C([0,1] \times K, E)$. From Theorem 1 (ii) we see that $J(\,.\,) = \mu(\,.\,)\,\Phi\, r(\,.\,) = p\,\Psi(\,.\,) \in KKM(E, E)$ is an upper semicontinuous compact with compact values. Now, Theorem 2 guarantees that there exists a $x \in E$ with $x \in J(x)$. If $x \in E \backslash U$ then $\mu(x) = 0$, so $x \in \{0\}$ i.e. $x = 0$, a contradiction since $0 \in U$. Thus $x \in U$ so $x \in \mu(x)\,\Phi(r(x)) = \mu(x)\,\Phi(x)$, so $x \in \Omega$ and so $\mu(x) = 1$. Thus $x \in \Phi(x)$ i.e. $i(x) \cap \Phi(x) \neq \emptyset$. Consequently i is 0–epi. □

Now, Theorem 9 and Theorem 10 immediately gives us a Leray–Schauder type alternative.

Theorem 11. *Let E be an admissible Hausdorff topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E and assume there exists a retraction $r : E \to \overline{U}$. Suppose $G \in csKKMC(\overline{U}, E)$ and $x \notin t\, co\, G(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then there exists a $x \in \overline{U}$ with $x \in co\, G(x)$.*

Proof. Suppose $x \notin co\, G(x)$ for $x \in \partial U$ (otherwise we are finished). Then $x \notin t\, co\, G(x)$ for $x \in \partial U$ and $t \in [0,1]$. Consider any $\Phi \in KKMC(\overline{U}, E)$ with $\Phi(x) \subseteq co\,(G(x))$ for $x \in \overline{U}$. Then

$$\left\{x \in \overline{U} : \ i(x) \cap t\,\Phi(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

does not intersect ∂U since if there exists a $x \in \partial U$ with $i(x)\,(= x) \in t\Phi(x)$ for some $t \in [0,1]$ then $x \in t\, co\, G(x)$, which is a contradiction. Now, i is 0–epi (Theorem 10), so Theorem 9 (with $F = i$) guarantees a $x \in U$ with $i(x) \cap co\, G(x) \neq \emptyset$ i.e. $x \in co\, G(x)$. □

From the ideas above we could also consider the following class of maps.

Definition 9. *We say $G \in sKKMC(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists a $\Phi \in KKMC(\overline{U}, E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$.*

Definition 10. *We say $G \in sKKMC_0(\overline{U}, E)$ if $G : \overline{U} \to 2^E$ and there exists a $\Phi \in KKMC(\overline{U}, E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$.*

Definition 11. *Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is 0–epi if for any map $G \in sKKMC_0(\overline{U}, E)$ and any $\Phi \in KKMC(\overline{U}, E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ and $\Phi(x) = \{0\}$ for $x \in \partial U$ there exists a $x \in U$ with $F(x) \cap \Phi(x) \neq \emptyset$.*

The same proofs with minor adjustments (i.e. removing "*co*") in Theorem's 9, 10 and 11 establish the following results.

Theorem 12. *Let E be a Hausdorff topological vector space, U an open convex subset of E, $G \in sKKMC(\overline{U}, E)$ and $F \in A_{\partial U}(\overline{U}, E)$ be 0–epi (in the sense of Definition 11). In addition for any $\Phi \in KKMC(\overline{U}, E)$ with $\Phi(x) \subseteq G(x)$ for $x \in \overline{U}$ suppose*

$$\left\{x \in \overline{U} : \ F(x) \cap t\,\Phi(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

does not intersect ∂U. There there exists an $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.

Theorem 13. *Let E be an admissible Hausdorff topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E and assume there exists a retraction $r : E \to \overline{U}$. Then the identity mapping i is 0–epi (in the sense of Definition 11).*

Theorem 14. *Let E be an admissible Hausdorff topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E and assume there exists a retraction $r : E \to \overline{U}$. Suppose $G \in sKKMC(\overline{U}, E)$ and $x \notin t\,G(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then there exists a $x \in \overline{U}$ with $x \in G(x)$.*

One could also consider the following class of maps.

Definition 12. *We say $G \in KKMC_0(\overline{U}, E)$ if $G \in KKMC(\overline{U}, E)$ with $G(x) = \{0\}$ for $x \in \partial U$.*

Definition 13. *Let $F \in A_{\partial U}(\overline{U}, E)$. We say F is 0–epi if for any map $G \in KKMC_0(\overline{U}, E)$ there exists a $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.*

Then we have the following results.

Theorem 15. *Let E be a Hausdorff topological vector space, U an open convex subset of E, $G \in KKMC(\overline{U}, E)$ and $F \in A_{\partial U}(\overline{U}, E)$ be 0–epi (in the sense of Definition 13). In addition suppose*

$$\left\{x \in \overline{U} : \ F(x) \cap t\,G(x) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

does not intersect ∂U. There there exists an $x \in U$ with $F(x) \cap G(x) \neq \emptyset$.

Theorem 16. *Let E be an admissible Hausdorff topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E and assume there exists a retraction $r : E \to \overline{U}$. Then the identity mapping i is 0–epi (in the sense of Definition 13).*

Theorem 17. *Let E be an admissible Hausdorff topological vector space, U an open convex subset of E, $0 \in U$, $\overline{U}$ an admissible subset of E and assume there exists a retraction $r : E \to \overline{U}$. Suppose $G \in KKMC(\overline{U}, E)$ and $x \notin t\,G(x)$ for $x \in \partial U$ and $t \in (0,1)$. Then there exists a $x \in \overline{U}$ with $x \in G(x)$.*

Remark 6. *It is also easy to extend Theorem 5, Theorem 6, Theorem 7 and Theorem 8 to $csKKMC$ and $sKKMC$ maps.*

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