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WEAK DISCONTINUITY WAVES IN ISOTROPIC POROUS MEDIA FILLED BY A FLUID FLOW*

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Dedicated to Professor Constantin Fetecau on the occasion of his 80th anniversary

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Abstract

In this paper the propagation of weak discontinuity waves is investigated in a special isotropic porous medium filled by a fluid flow. To this aim we introduce a new variable related to the surface across which the first derivatives of the solutions of the governing equations undergo a jump. Following a Boillat's methodology for quasi-linear and hyperbolic systems of the first order, we have investigated, in an one-dimensional case, the propagation of one solution into an uniform unperturbed state in the first approximation, obtaining the Bernoulli's equation governing the evolution of the amplitude of the weak discontinuity wave, the wave front, the velocity along the characteristic rays, the approximated solution and the critical time. The results obtained can be applied in certain technical fields, such as biology, the medical sciences and geology.

Keywords: weak discontinuity waves, partial differential equations, approximate solutions, non-equilibrium thermodynamics, porous media filled by a fluid flow.

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1 Introduction

The study of media with thin porous channels may have relevance in the description of phenomena accompanying fluid flows in porous structures and find applications in applied sciences, like biology, medical sciences, seismic engineering. Here, we use a thermodynamic theory (see [38], [39], [41] and [42]), formulated following the methods of the extended rational thermodynamics [7], [21], [22], [23], [30], [31], [33], [45], with internal variables (see [1], [23], [28], [29], [32] and also [40]).

More precisely, in [38] and [39] introducing as internal variable the structural permeability tensor à la Kubik [24], see also [38] and [41], describing the influence of thin porous channels on the other fields, occurring inside the considered media, a model was developed for porous media filled by a fluid flow. By the analysis of the entropy inequality, using Liu's theorem [27], the constitutive theory and the rate equations for the fluxes and the porosity field were constructed as objective functions, in a special case. Smith's theorem [43] was used. In [41] and [42] constitutive relations, rate equations and other results were derived for the same media in the anisotropic case. In [10] the isotropic case was treated. In [11] a thermodynamic theory for erosion and/or deposition in elastic porous media was developed by the authors.

In [9], [12], [13], in special isotropic cases the harmonic waves propagation of coupled porosity-fluid concentration flux fields, of coupled porosity-fluid concentration flux-temperature fields and of porosity-temperature fields were investigated, respectively. In this article utilizing a model derived in [9] for perfect isotropic porous media satured by fluid flow in a particular case, the propagation of weak discontinuity waves is investigated, confining us only to the porosity, fluid concentration fields and their fluxes.

Several physical problems are described by nonlinear hyperbolic PDEs. Following A. Jeffrey in [20], the solution hypersurfaces of these are referred to as waves, because they may be interpreted as representing propagating wavefronts. The solution ahead of the wavefront may called undisturbed, whilst the solution on the other side may be regarded as a propagating disturbance wave entering in the region of the undisturbed solution. Because the closed-form solutions of nonlinear PDEs are rare, solutions are investigated, called *nonlinear waves*, that can present various types of discontinuities. As some surface is crossed, the solution and/or some of its derivatives undergo a jump: in the first case the solution presents a *shock*,

or it is a *shock wave*, in the second case we are in presence of a *discontinuity waves* (jumps of the first order derivatives) [2–4, 8, 17–20, 25, 26]. Also we can have instead of the jump smooth solutions of the non linear hyperbolic PDEs, that present a steep variation in the normal direction to the associated wavefront, called *asymptotic waves* (see [5, 6, 14–16, 36]).

The paper is organized as follows. In section 2, we present the equations of the thermodynamical model formulated in [38], used to describe the behavior of porous media filled by a fluid flow in anisotropic and isotropic cases (see [39], [41] and [42]) and we consider a particular isotropic case, worked out in [9], where the model is confined only to porosity and the fluid concentration fields and their fluxes, and some interactions among some fields are neglected. In this paper some coefficients of the governing equations are assumed to be non-constant, depending on the fields involved, obtaining a non-linear PDEs system. In sections 3, 4 and 5, the methodology established by Boillat to determine the weak discontinuity waves for quasi-linear and hyperbolic systems of the first order is illustrated, introducing a new variable related to the surface across which the first derivatives of the solutions of the governing equations undergo a jump. In sections 6 and 7, in an one-dimensional case, we study the propagation of one solution into an uniform unperturbed state in the first approximation, obtaining the associated wave front, the velocity along the characteristic rays, the Bernoulli's equation governing the evolution of the associated weak discontinuity wave amplitude and the approximated solution. Also, we calculate the critical time, that represents the instant in which the solution ceases to exist or to be unique or, from the point of view of the non-linear waves, the weak discontinuity wave amplitude becomes unbounded, and this may correspond to the formation of a shock wave.

2 The model

In the following we use the standard Cartesian tensor notation in rectangular coordinate systems and we refer to a current configuration K_t at the time t. We take into consideration the theory for a porous medium filled by a fluid flow, formulated in [38], [39], [42], [41], in the framework of rational extended thermodynamics, where we have assumed that within the medium there are present the elastic field described by the symmetric stress tensor τ_{ij} and the small-strain tensor ε_{ij}; the thermal field described by the temperature T, its gradient $T_{,i}$ and the heat flux q_i; the field of the fluid concentration c, its gradient $c_{,i}$ and its flux j_i^c; the porosity field, whose

geometric description is given by a structural permeability tensor r_{ij} à la Kubik (see [24] and for instance [41]), its gradient $r_{ij,k}$ and its flux $\mathcal{V}_{ijk}$, since the thin porous channels sometimes can self propagate because of some surrounding favorable conditions. We have chosen the following thermodynamic state vector:

$$C = \{\varepsilon_{ij}, c, T, r_{ij}, j_i^c, q_i, c_{,i}, T_{,i}, r_{ij,k}, \mathcal{V}_{ijk}\}, \tag{1}$$

where $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, with u_i the displacement field.

In the previous papers we have supposed that the elastic porous medium and the fluid, flowing inside it, constitute two-components mixture, ρ_1 being the fluid mass density and ρ_2 the density of the porous skeleton, with $\rho_1 \ll \rho_2$, so that we have $\rho = \rho_1 + \rho_2$, with ρ the density of the mixture as a whole, supposed constant.

We have considered two groups of laws. The first group is concerning the classical balance equations:

- *the continuity equation*

$$\rho\dot{c} + j^c_{i,i} = 0, \tag{2}$$

where the source term is neglected, a superimposed dot indicates the material derivative (i.e. $\frac{d}{dt} = \frac{\partial}{\partial t} + x_\gamma \frac{\partial}{\partial x_\gamma}$), a comma in lower indices indicates the spatial derivation and Einstein convention for repeated indices is used. In (2) the concentration of the fluid is defined by $c = \frac{\rho_1}{\rho}$ and its flux j_i^c is given by $j_i^c = \rho_1(v_{1i} - v_i)$, where v_{1i} is the fluid velocity and v_i is the barycentric velocity of the mixture, defined by $\rho v_i = \rho_1 v_{1i} + \rho_2 v_{2i}$, with v_{2i} the porous medium velocity.

- *the momentum balance*

$$\rho\dot{v}_i - \tau_{ji,j} - f_i = 0, \tag{3}$$

where f_i denotes a body force, disregarded in the following for the sake of simplicity;

- *the internal energy balance*

$$\rho\dot{e} - \tau_{ji}v_{i,j} + q_{i,i} - \rho h = 0, \tag{4}$$

where e is the internal energy density, h is the energy source density (neglected in the following) and $v_{i,j}$ is the gradient of the barycentric velocity.

The second group is concerning the rate equations for the porosity field r_{ij}, its flux V_{ijk}, the heat flux q_i and the fluid-concentration flux j_i^c, constructed in such a way that they obey the objectivity and frame-indifference

principles (see [34], [35], [44]). We have chosen for these evolution equations the following form

$$\overset{*}{r}_{ij} + \mathcal{V}_{ijk,k} - \mathcal{R}_{ij}(C) = 0, \tag{5}$$

$$\overset{*}{j}{}^c_i - J^c_i(C) = 0, \tag{6}$$

$$\overset{*}{q}_i - Q_i(C) = 0, \tag{7}$$

$$\overset{*}{\mathcal{V}}_{ijk} - V_{ijk}(C) = 0. \tag{8}$$

In equations (5)-(8) the quantities $\mathcal{R}_{ij}(C)$, $J^c_i(C)$, $Q_i(C)$ and $V_{ijk}(C)$ are the source-like terms which deal with the creation or annihilation of porous channels, the fluid concentration flux source, the heat source and the source term for the structural permeability field flux, respectively, and represent constitutive functions of the independent variables of the set (1), the superimposed asterisk indicates the Zaremba-Jaumann derivative (see [34], [44]), i.e.

$$\overset{*}{r}_{ij} = \dot{r}_{ij} - \Omega_{ik} r_{kj} - \Omega_{jk} r_{ik},$$

$$\overset{*}{j}{}^c_i = \dot{j}^c_i - \Omega_{ik} j^c_k, \quad \overset{*}{q}_i = \dot{q}_i - \Omega_{ik} q_k,$$

$$\overset{*}{\mathcal{V}}_{ijk} = \dot{\mathcal{V}}_{ijk} - \Omega_{il}\mathcal{V}_{ljk} - \Omega_{jl}\mathcal{V}_{ilk} - \Omega_{kl}\mathcal{V}_{ijl},$$

with $\Omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ the antisymmetric part of the velocity gradient and v_i the velocity field of the body. In [38], [39] the constitutive functions and the rate equations were obtained in special isotropic case to close the system of balance equations. The entropy inequality was analyzed by Liu's theorem (see [27]). In [42] (see also [41]) we have obtained the following rate equations in the anisotropic case for the fields r_{ij}, j^c_i, q_i and $\mathcal{V}_{ijk}$

$$\begin{aligned}\dot{r}_{ij} + \mathcal{V}_{ijk,k} = {} & \beta^1_{ijkl}\varepsilon_{kl} + \beta^2_{ijkl} r_{kl} + \beta^3_{ijk} j^c_k + \beta^4_{ijk} q_k + \beta^5_{ijklm}\mathcal{V}_{klm} + \beta^6_{ijk} c_{,k} \\ & + \beta^7_{ijk} T_{,k} + \beta^8_{ijklm} r_{kl,m},\end{aligned} \tag{9}$$

$$\tau^{j^c}_{ij}\dot{j}^c_j = -j^c_i + \xi^2_{ij} q_j + \xi^3_{ijkl}\mathcal{V}_{jkl} - \xi^4_{ij} c_{,j} + \xi^5_{ij} T_{,j} + \xi^6_{ijkl} r_{jk,l}, \tag{10}$$

$$\tau^q_{ij}\dot{q}_j = \chi^1_{ij} j^c_j - q_i + \chi^3_{ijkl}\mathcal{V}_{jkl} + \chi^4_{ij} c_{,j} - \chi^5_{ij} T_{,j} + \chi^6_{ijkl} r_{jk,l}, \tag{11}$$

$$\dot{\mathcal{V}}_{ijk} = \gamma^1_{ijkl} j^c_l + \gamma^2_{ijkl} q_l + \gamma^3_{ijklmn}\mathcal{V}_{lmn} + \gamma^4_{ijkl} c_{,l} + \gamma^5_{ijkl} T_{,l} + \gamma^6_{ijklmn} r_{lm,n}, \tag{12}$$

with $\tau^{j^c}_{ij}$ and τ^q_{ij} relaxation time tensors of the fields j^c_i and q_i, respectively. In equations (9)-(12) we have assumed the material derivative can be used instead of Zaremba-Jaumann derivative. Equations (9)-(12) describe disturbances, having finite velocity of propagation and own relaxation time to

reach the respective thermodynamic equilibrium values. Equations (9) and (12) describe the evolution of the porosity field, of its flux and in their right hand sides the source terms represent the contributions of the fields occurring inside these media. Equations (10) and (11) are the anisotropic Fick-Nonnenmacher equation for the fluid concentration flux and the anisotropic generalized Maxwell-Vernotte-Cattaneo equation for the heat flux, respectively.

2.1 Evolution equations for the concentration of the fluid and the porosity fields and their fluxes in a special case

In [9] we have treated the special case of perfect isotropic media, where their geometric, transport and thermal properties are invariant respect to all rotations and inversions of the frame axes (under orthogonal transformations). We have assumed the media into consideration are at rest and we have disregarded some cross effects among the fields confining us only to the porosity and the fluid concentration fields and their fluxes. In particular, taking into account equation (2), (10), (9) and (12), in [9] we have worked out the following system of equations

$$\rho \frac{\partial c}{\partial t} + j^c_{k,k} = 0, \tag{13}$$

$$\tau^{j^c} \frac{\partial j^c_i}{\partial t} = -j^c_i + \alpha_c r_{,i} - \rho D_c c_{,i}, \tag{14}$$

$$\frac{\partial r}{\partial t} + \mathcal{V}_{k,k} = 0, \tag{15}$$

$$\tau^{\nu} \frac{\partial \mathcal{V}_k}{\partial t} = -\mathcal{V}_k - D_\nu r_{,k} + \alpha_\nu c_{,k}, \tag{16}$$

with α_c and α_ν constant coupling coefficients, τ^{j^c} and τ^{ν} the constant relaxation times of the field j^c_i and $\mathcal{V}_{ijk}$, and D_c, D_ν the constant diffusion coefficients of the fluid concentration and porosity fields, respectively. Furthermore, in (13) the mass density was supposed constant and in equation (15) the number of the very thin porous channels was assumed locally conserved.

In this paper we suppose that in (14) and (16) the two coupling coefficients α_c and α_ν are functions of at least class C^1 of the type
$\alpha_c = \alpha_c(r)$ and $\alpha_\nu = \alpha_\nu(c)$, and we investigate the weak discontinuity waves of the PDEs system (13)-(16).

3 Weak discontinuity waves of the non-linear PDEs system into consideration

Following Boillat [2], let us observe that the system of equations (13)- (16) can be written in the following matrix form:

$$\boldsymbol{A}^{\alpha}(\boldsymbol{U})\boldsymbol{U}_{\alpha} = \boldsymbol{B}(\boldsymbol{U}) \qquad (\alpha = 0, 1, 2, 3), \tag{17}$$

where $x^0 = t$ indicates the time, x^1, x^2, x^3 are the space coordinates, $\boldsymbol{U}_{\alpha} = \dfrac{\partial \boldsymbol{U}}{\partial x^{\alpha}}$, $\boldsymbol{U}$ is the vector of the unknown functions (depending on x^{α})

$$\boldsymbol{U} = (c, j_1^c, j_2^c, j_3^c, r, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3)^T, \tag{18}$$

the vector $\boldsymbol{B}$ is defined by

$$\boldsymbol{B} = (0, -j_1^c, -j_2^c, -j_3^c, 0, -\mathcal{V}_1, -\mathcal{V}_2, -\mathcal{V}_3)^T, \tag{19}$$

and finally $\boldsymbol{A}^{\alpha}$ $(\alpha = 0, \ldots, 3)$ are the following 8×8 square matrices

$$\boldsymbol{A}^0 = \begin{pmatrix} \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau^{j^c} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau^{j^c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau^{j^c} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tau^{\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tau^{\nu} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau^{\nu} \end{pmatrix}, \tag{20}$$

$$\boldsymbol{A}^1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho D_c & 0 & 0 & 0 & -\alpha_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\alpha_{\nu} & 0 & 0 & 0 & D_{\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{21}$$

$$\boldsymbol{A}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho D_c & 0 & 0 & 0 & -\alpha_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_\nu & 0 & 0 & 0 & D_\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{22}$$

$$\boldsymbol{A}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho D_c & 0 & 0 & 0 & -\alpha_c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_\nu & 0 & 0 & 0 & D_\nu & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

In (18) and (19) the symbol $(...)^T$ means that $\boldsymbol{U}$ and $\boldsymbol{B}$ are column vectors.

Since $\boldsymbol{A}^\alpha = \boldsymbol{A}^\alpha(\boldsymbol{U})$, the PDEs system (17) is a quasi-linear system.

We suppose that the system admits a known solution in the *uniform unperturbed state* $\boldsymbol{U}^0$ that satisfy the following condition

$$\boldsymbol{A}^\alpha(\boldsymbol{U}^0)\boldsymbol{U}^0_\alpha = \boldsymbol{B}(\boldsymbol{U}^0) \qquad (\alpha = 0, 1, 2.3). \tag{24}$$

3.1 Introduction of a new variable

Let us assume (see [2], [20]) that the system (17) describes a perturbation propagating into an unperturbed state characterized by the vector $\boldsymbol{U}^0$ and the hypersurface $\mathcal{S}$

$$\varphi(t, x^1, x^2, x^3, x^4) = 0, \tag{25}$$

improperly called *wavefront* and supposed to be characterized by the fact that the first derivatives of the solution $\boldsymbol{U}$ of the system (17) undergo a jump crossing it. $\varphi(x^\alpha) = 0$ separates the region perturbed, $\varphi(x^\alpha) = 0^+$, by the unperturbed, $\varphi(x^\alpha) = 0^-$, and moves in the Euclidean space E^{3+1} (when the time flows). The solution $\boldsymbol{U}$ is supposed continuous.

We remind that the wavefront $\varphi(x^\alpha) = 0$ is still an unknown function. In order to determine it, we see that from (25) along $\mathcal{S}$ we have $\dfrac{d\varphi}{dt} = 0$, implying $\frac{\partial \varphi}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} \varphi = 0$, or, equivalently, $\frac{\partial \varphi / \partial t}{|\operatorname{grad} \varphi|} + \boldsymbol{v} \cdot \frac{\operatorname{grad} \varphi}{|\operatorname{grad} \varphi|} = 0$, with

$(\text{grad})_i = \dfrac{\partial}{\partial x^i}$ and, obviously, $\frac{\text{grad}\,\varphi}{|\text{grad}\,\varphi|} = \boldsymbol{n}$ the *normal unit vector* to the hypersurface $\mathcal{S}$, such that the previous equality reads

$$\frac{\partial\varphi/\partial t}{|\text{grad}\,\varphi|} + \boldsymbol{v}\cdot\boldsymbol{n} = 0. \tag{26}$$

Introducing the notations

$$\lambda = -\frac{\partial\varphi/\partial t}{|\text{grad}\,\varphi|}, \quad \Psi(\boldsymbol{U},\varphi_\alpha) = \varphi_t + \lambda(\boldsymbol{U},\boldsymbol{n})|\text{grad}\,\varphi|, \tag{27}$$

with

$$\varphi_\alpha = \frac{\partial\varphi}{\partial x^\alpha}, \quad \Lambda_i(\boldsymbol{U},\boldsymbol{n}) = \frac{\partial\Psi}{\partial\varphi_i},$$

we obtain the *velocity normal to the progressive wave*, λ, given by

$$\lambda(\boldsymbol{U},\boldsymbol{n}) = \boldsymbol{v}\cdot\boldsymbol{n}, \tag{28}$$

and the components of a speed, called *radial velocity*, $\boldsymbol{\Lambda}$, defined by

$$\Lambda_i(\boldsymbol{U},\boldsymbol{n}) = \frac{\partial\Psi}{\partial\varphi_i} = \lambda n_i + \frac{\partial\lambda}{\partial n_i} - \left(\boldsymbol{n}\cdot\frac{\partial\lambda}{\partial\boldsymbol{n}}\right)n_i \quad (i=1,2,3). \tag{29}$$

From equation (29) we have

$$\Lambda_i n_i = \lambda, \tag{30}$$

i.e. the velocity of propagation of the wavefront λ is the component of radial velocity $\boldsymbol{\Lambda}$ along the normal to the wavefront.

Introducing the function $\varphi = \varphi(x^\alpha)$ as new variable, continuous together with its first and second derivatives, the derivative with respect to x^α is given by

$$\boldsymbol{U}_\alpha = \boldsymbol{U}_\varphi\,\varphi_\alpha, \tag{31}$$

where $\boldsymbol{U}_\varphi = \dfrac{\partial\boldsymbol{U}}{\partial\varphi}$ and $\varphi_\alpha = \dfrac{\partial\varphi}{\partial x^\alpha}$.

Moreover, we introduce the *jump* symbol by $[\ \ ] = \lim_{\varphi\to0^+}(\ \ ) - \lim_{\varphi\to0^-}(\ \ )$, defined by the difference between the value of a quantity in the unperturbed state and the perturbed state calculated on the surface $\varphi(x^\alpha) = 0$. Thus, the jump $\boldsymbol{\Pi}$ of the normal derivative $\boldsymbol{U}_\varphi$ is given by

$$[\boldsymbol{U}] = \boldsymbol{0}, \qquad \boldsymbol{\Pi} = [\boldsymbol{U}_\varphi] = \lim_{\varphi\to0^+}(\boldsymbol{U}_\varphi) - \lim_{\varphi\to0^-}(\boldsymbol{U}_\varphi). \tag{32}$$

From equations (17), (24) and (31) we have the following relations

$$\boldsymbol{A}^\alpha(\boldsymbol{U})\boldsymbol{U}_\varphi\varphi_\alpha = \boldsymbol{B}(\boldsymbol{U}) \qquad \text{and} \qquad \boldsymbol{A}^\alpha(\boldsymbol{U}^0)\boldsymbol{U}^0_\varphi\varphi_\alpha = \boldsymbol{B}(\boldsymbol{U}^0). \tag{33}$$

Subtracting equation $(33)_2$ from equation $(33)_1$ and by computing on the surface $\varphi(x^\alpha) = 0$, where $\boldsymbol{U} = \boldsymbol{U}^0$, $\boldsymbol{A}^\alpha(\boldsymbol{U}^0) = \boldsymbol{A}^\alpha(\boldsymbol{U})$ and $\boldsymbol{B}(\boldsymbol{U}^0) = \boldsymbol{B}(\boldsymbol{U})$, we get

$$(\boldsymbol{A}^\alpha)_0\varphi_\alpha\,[\boldsymbol{U}_\varphi] = \boldsymbol{0}, \quad \text{i.e.} \quad (\boldsymbol{A}^\alpha)_0\varphi_\alpha\boldsymbol{\Pi} = \boldsymbol{0}, \tag{34}$$

where $(\boldsymbol{A}^\alpha)_0\varphi_\alpha$ represents a 8×8 matrix and equation $(34)_2$ is a homogeneous system in the 8 components of $\boldsymbol{\Pi}$.

Introducing the quantities $\boldsymbol{n}$ and λ, defined in equation $(27)_1$, the system $(34)_2$ takes the form

$$\left(\boldsymbol{A}^i n_i - \lambda \boldsymbol{A}^0\right)\boldsymbol{\Pi} = \boldsymbol{0}, \tag{35}$$

which has non-trivial solutions only if its determinant vanishes, i.e.

$$\det\left\|\boldsymbol{A}_n - \lambda \boldsymbol{A}^0\right\| = 0. \tag{36}$$

with $\boldsymbol{A}_n = \boldsymbol{A}^i n_i$.

Equation (35) shows that $\boldsymbol{\Pi}$ can be taken proportional to the right-eigenvector $\boldsymbol{d}$ of $\boldsymbol{A}_n$, corresponding to some eigenvalue λ. In the case where the eigenvalue λ has multiplicity 1, $\boldsymbol{\Pi}$ has the form [2]

$$\boldsymbol{\Pi} = \pi(x^i, t)\boldsymbol{d}. \tag{37}$$

Thus, in order to determine $\boldsymbol{\Pi}$ we have to determine the function $\pi = \pi(x^\alpha)$.

Since $\boldsymbol{A}^0$ is a non-singular matrix, the system (17) can be written in the form

$$\boldsymbol{U}_t + (\boldsymbol{A}^0)^{-1}\boldsymbol{A}^i\boldsymbol{U}_i = (\boldsymbol{A}^0)^{-1}\boldsymbol{B}(\boldsymbol{U}), \qquad (i = 1, 2, 3), \tag{38}$$

where

$$(\boldsymbol{A}^0)^{-1} = \begin{pmatrix} \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\tau^{j^c}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\tau^{j^c}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\tau^{j^c}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau^\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau^\nu} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau^\nu} \end{pmatrix} \tag{39}$$

is the inverse matrix of $\boldsymbol{A}^0$.

In the following we continue to call $\boldsymbol{A}^i$ the matrices $(\boldsymbol{A}^0)^{-1}\boldsymbol{A}^i$, and $\boldsymbol{B}$ the vector $(\boldsymbol{A}^0)^{-1}\boldsymbol{B}$, this allows the system to take the following form

$$\boldsymbol{U}_t + \boldsymbol{A}^i\boldsymbol{U}_i = \boldsymbol{B}(\boldsymbol{U}) \qquad (i = 1, 2, 3). \tag{40}$$

From equation (36) we have the following eigenvalues problem

$$\det\|\boldsymbol{A}_n - \lambda\boldsymbol{I}\| = 0. \tag{41}$$

4 Wave front and first approximation of $\boldsymbol{U}$

Let us consider the propagation in a uniform state $\boldsymbol{U}^0$. The quantity $(27)_2$ becomes zero on the wavefront $\varphi(x^\alpha) = 0$ having velocity $\lambda = \lambda^0$ (see (26)), i.e.

$$\Psi(\boldsymbol{U}^0, \varphi_\alpha) = \varphi_t + \lambda^0(\boldsymbol{U}^0, \boldsymbol{n}^0)|\mathrm{grad}\,\varphi| = \Psi^0 = 0. \tag{42}$$

To solve equation (42) the *characteristic rays*, called characteristic curves of the system (17), are introduced, defined by the following differential equations

$$\frac{dx^\alpha}{d\sigma} = \frac{\partial \Psi^0}{\partial \varphi_\alpha}, \quad \frac{d\varphi_\alpha}{d\sigma} = -\frac{\partial \Psi^0}{\partial x^\alpha}, \tag{43}$$

where σ is the time along the characteristic rays.

From equations (42) and $(43)_2$ we obtain $\dfrac{\partial \Psi^0}{\partial x^\alpha} = 0$, from which we obtain that φ_α are constant along the characteristic rays. By integration of equation $(43)_1$ one obtains

$$x^0 = t = \sigma, \tag{44}$$

$$x^i(\sigma) = x_0^i + \Lambda_i^0 \sigma, \quad (i = 1, 2, 3), \tag{45}$$

with

$$x_0^i = x^i|_{\sigma=0} \quad \text{and} \quad \Lambda_i^0 = \Lambda_i(\boldsymbol{U}^0, \boldsymbol{n}^0), \quad (i = 1, 2, 3). \tag{46}$$

If we denote by φ^0 the given initial surface, we have $\varphi|_{\sigma=0} = \varphi^0(x_0^i)$ and $\boldsymbol{n}^0$ represents the unit normal vector to the wavefront at the point x_0^i defined by

$$\boldsymbol{n}^0 = \left(\frac{\mathrm{grad}\,\varphi}{|\mathrm{grad}\,\varphi|}\right)_{\sigma=0} = \frac{\mathrm{grad}^0\,\varphi^0}{\left|\mathrm{grad}^0\,\varphi^0\right|}, \quad \text{with} \quad (\mathrm{grad}^0)_i \equiv \frac{\partial}{\partial x_0^i} \quad (i = 1, 2, 3). \tag{47}$$

Thus, $\boldsymbol{x} = \boldsymbol{x}|_{\sigma=0} + \boldsymbol{\Lambda}^0\sigma$ and since the Jacobian J of the transformation $\boldsymbol{x} \to \boldsymbol{x}|_{t=0}$ is non-vanishing, i.e.

$$J = \det\left|\frac{\partial \Lambda_k^0}{\partial x_0^i}\sigma + \delta_{ik}\right| \neq 0 \quad (i, k = 1, 2, 3), \tag{48}$$

it follows that in the first approximation we obtain

$$\varphi(\sigma, x^i) = \varphi^0(x^i - \Lambda_i^0\sigma). \tag{49}$$

Then, expanding the vector $\boldsymbol{U}$ up the first-order approximation in a neighborhood of the wavefront $\varphi(x^\alpha) = 0$ we have

$$\boldsymbol{U} = (\boldsymbol{U})_{\varphi=0^+} + \left(\frac{\partial \boldsymbol{U}}{\partial \varphi}\right)_{\varphi=0^+} \varphi + \mathcal{O}(\varphi^2), \tag{50}$$

$$\boldsymbol{U}^0 = \left(\boldsymbol{U}^0\right)_{\varphi=0^-} + \left(\frac{\partial \boldsymbol{U}^0}{\partial \varphi}\right)_{\varphi=0^-} \varphi + \mathcal{O}(\varphi^2). \tag{51}$$

Subtracting equation (51) from (50) we get

$$\boldsymbol{U} = \boldsymbol{U}^0 + \varphi \boldsymbol{\Pi} + \mathcal{O}(\varphi^2), \tag{52}$$

where $\mathcal{O}(\varphi^2)$ is the Landau's notation and represents infinitesimals of higher order respect to φ.

5 Bernoulli's equation for the amplitude of discontinuity π

In (52), following [2], the amplitude of discontinuity π satisfies *Bernoulli's equation* having the form

$$\left(\boldsymbol{l}^0 \cdot \boldsymbol{d}^0\right)\left[\frac{d\pi}{d\sigma} + \left(\nabla\Psi \cdot \boldsymbol{d}\right)^0 \pi^2 + \frac{d}{d\sigma}\ln\sqrt{J}\,\pi\right] = F^0\pi, \tag{53}$$

$$\left(\nabla\Psi \cdot \boldsymbol{d}\right)^0 = \left(|\mathrm{grad}\,\varphi|\nabla\lambda \cdot \boldsymbol{d}\right)^0, \qquad F^0 = \left(\nabla(\boldsymbol{l}\cdot\boldsymbol{B})\cdot\boldsymbol{d}\right)^0, \tag{54}$$

with $\nabla \equiv \left(\frac{\partial}{\partial c}, \frac{\partial}{\partial j_1^c}, \frac{\partial}{\partial j_2^c}, \frac{\partial}{\partial j_3^c}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \mathcal{V}_1}, \frac{\partial}{\partial \mathcal{V}_2}, \frac{\partial}{\partial \mathcal{V}_3}\right)$ and $\boldsymbol{d}^0$ the right eigenvector corresponding to the eigenvalue λ^0. We remind that the quantity $\boldsymbol{l}^0 \cdot \boldsymbol{d}^0$ is always different from zero by virtue of the hyperbolicity of the system [8].

In [2] it was seen that is possible to solve the equation (53) with the position

$$\pi = \frac{h(\sigma)}{\sqrt{J}\,\Phi(\sigma)}, \quad \text{with} \quad h(\sigma) = h^0 \exp\left[-\int_0^\sigma \frac{F^0}{(\boldsymbol{l}\cdot\boldsymbol{d})_0} d\tau\right], \tag{55}$$

$$\Phi(\sigma) = 1 + \int_0^\sigma \frac{(\nabla\Psi\cdot\boldsymbol{d})_0}{\sqrt{J}}\, h d\tau$$

and $h^0 = h(0) = \pi(0)$.

From (55) follows that if there exists a time σ_c where $\Phi(\sigma_c) = 0$, then $\pi \to \infty$, and this may correspond to the formation of a shock wave [2].

6 One-dimensional case

In this section, we treat the one-dimensional case. Confining us to the porosity and the fluid concentration fields and their fluxes, assuming that the propagation of weak discontinuity waves is along the x axis, the involved quantities depend on x_1, denoted by x, and $x_2 = x_3 = 0$, the system (13)-(16) takes the following form:

$$\frac{\partial c}{\partial t} + \frac{1}{\rho}\frac{\partial j_1^c}{\partial x} = 0, \tag{56}$$

$$\frac{\partial j_1^c}{\partial t} - \frac{\alpha_c}{\tau^{j^c}}\frac{\partial r}{\partial x} + \frac{\rho D_c}{\tau^{j^c}}\frac{\partial c}{\partial x} = -\frac{j_1^c}{\tau^{j^c}}, \tag{57}$$

$$\frac{\partial j_2^c}{\partial t} = -\frac{j_2^c}{\tau^{j^c}}, \tag{58}$$

$$\frac{\partial j_3^c}{\partial t} = -\frac{j_3^c}{\tau^{j^c}}, \tag{59}$$

$$\frac{\partial r}{\partial t} + \frac{\partial \mathcal{V}_1}{\partial x} = 0, \tag{60}$$

$$\frac{\partial \mathcal{V}_1}{\partial t} + \frac{D_\nu}{\tau^\nu}\frac{\partial r}{\partial x} - \frac{\alpha_\nu}{\tau^\nu}\frac{\partial c}{\partial x} = -\frac{\mathcal{V}_1}{\tau^\nu}, \tag{61}$$

$$\frac{\partial \mathcal{V}_2}{\partial t} = -\frac{\mathcal{V}_2}{\tau^\nu}, \tag{62}$$

$$\frac{\partial \mathcal{V}_3}{\partial t} = -\frac{\mathcal{V}_3}{\tau^\nu}. \tag{63}$$

where we recall that we have assumed $\alpha_c = \alpha_c(r)$ and $\alpha_\nu = \alpha_\nu(c)$. From the system of equations (56)-(63) we have

$$j_2^c(x,t) = (j_2^c)^0(x)e^{-\frac{t}{\tau^{j^c}}}, \qquad j_3^c(x,t) = (j_3^c)^0(x)e^{-\frac{t}{\tau^{j^c}}}, \tag{64}$$

$$\mathcal{V}_2(x,t) = (\mathcal{V}_2)^0(x)e^{-\frac{t}{\tau^\nu}}, \qquad \mathcal{V}_3(x,t) = (\mathcal{V}_3)^0(x)e^{-\frac{t}{\tau^\nu}}. \tag{65}$$

Then, we consider the following reduced system, where we have indicated j_1^c and $\mathcal{V}_1$ by the symbols j^c and $\mathcal{V}$

$$\boldsymbol{U}_t + \boldsymbol{A}(\boldsymbol{U})\boldsymbol{U}_x = \boldsymbol{B}(\boldsymbol{U}), \tag{66}$$

where

$$\boldsymbol{U} = (c, j^c, r, \mathcal{V})^T, \tag{67}$$

$$\boldsymbol{B} = \left(0, -\frac{j^c}{\tau^{j^c}}, 0, -\frac{\mathcal{V}}{\tau^\nu}, \right)^T, \tag{68}$$

and

$$\boldsymbol{A} = \begin{pmatrix} 0 & \frac{1}{\rho} & 0 & 0 \\ \frac{\rho D_c}{\tau^{j^c}} & 0 & -\frac{\alpha_c}{\tau^{j^c}} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\alpha_\nu}{\tau^\nu} & 0 & \frac{D_\nu}{\tau^\nu} & 0 \end{pmatrix}. \tag{69}$$

In the considered one dimensional case, in equation (41), being $\boldsymbol{n} = (n_1, 0, 0) = (1, 0, 0)$, we have $\boldsymbol{A}_n(\boldsymbol{U}) = \boldsymbol{A}$.

6.1 Eigenvalues and eigenvectors of the matrix $\boldsymbol{A}$

The matrix $\boldsymbol{A}$ admits the following simple eigenvalues:

$$\lambda_1^{(\pm)} = \pm\sqrt{\frac{\rho D_\nu \tau^{j^c} + \rho D_c \tau^\nu - \mathcal{M}}{2\rho\tau^{j^c}\tau^\nu}}, \quad \lambda_2^{(\pm)} = \pm\sqrt{\frac{\rho D_\nu \tau^{j^c} + \rho D_c \tau^\nu + \mathcal{M}}{2\rho\tau^{j^c}\tau^\nu}}, \tag{70}$$

where

$$\mathcal{M} = \sqrt{\left(\rho D_\nu \tau^{j^c} - \rho D_c \tau^\nu\right)^2 + 4\rho\alpha_c\alpha_\nu\tau^{j^c}\tau^\nu}. \tag{71}$$

The radicand of the quantity $\mathcal{M}$ is always positive, so $\mathcal{M}$ is real, moreover the eigenvalues $\lambda_1^{(\pm)}$ are real when the condition $\rho D_\nu \tau^{j^c} + \rho D_c \tau^\nu - \mathcal{M} \geq 0$ is valid, i.e. for

$$\alpha_c\alpha_\nu \leq \rho D_c D_\nu. \tag{72}$$

Thus, we have the propagation of discontinuity waves, with the normal velocity $\lambda_1^{(\pm)}$ when the condition (72) is valid. The eigenvalues $\lambda_2^{(\pm)}$ are always real. We observe that $\mathcal{M} \neq 0$ (so that $\lambda_1^{(\pm)} \neq \lambda_2^{(\pm)}$) because the relation $\left(\rho D_\nu \tau^{j^c} - \rho D_c \tau^\nu\right)^2 \neq -4\rho\alpha_c\alpha_\nu\tau^{j^c}\tau^\nu$ is always verified, being the left-hand member positive whereas the right-hand member is negative (the two members are equal only if they are both null but the right-hand member is different from zero by hypothesis).

The left eigenvectors $\boldsymbol{l}_1^{(\pm)}$, $\boldsymbol{l}_2^{(\pm)}$, and the right eigenvectors $\boldsymbol{d}_1^{(\pm)}$, $\boldsymbol{d}_2^{(\pm)}$

corresponding to eigenvalues $\lambda_1^{(\pm)}$, $\lambda_2^{(\pm)}$, respectively, have the form

$$\boldsymbol{l}_1^{(\pm)} = \left(\frac{\lambda_1^{(\pm)}\mathcal{N}}{2\alpha_c\tau^\nu}, \frac{\mathcal{N}}{2\rho\alpha_c\tau^\nu}, \lambda_1^{(\pm)}, 1 \right), \quad \boldsymbol{l}_2^{(\pm)} = \left(\frac{\lambda_2^{(\pm)}\mathcal{T}}{2\alpha_c\tau^\nu}, \frac{\mathcal{T}}{2\rho\alpha_c\tau^\nu}, \lambda_2^{(\pm)}, 1 \right), \tag{73}$$

$$\boldsymbol{d}_1^{(\pm)} = \left(\frac{2\alpha_c(\tau^{j^c})^2\lambda_1^{(\pm)}}{\mathcal{I}}, -\frac{\alpha_c\tau^\nu\mathcal{F}}{\tau^{j^c}\mathcal{I}}, \frac{\lambda_1^{(\pm)}\tau^\nu\mathcal{T}}{\mathcal{I}}, 1 \right)^T, \tag{74}$$

$$\boldsymbol{d}_2^{(\pm)} = \left(\frac{2\alpha_c(\tau^{j^c})^2\lambda_2^{(\pm)}}{\mathcal{Z}}, -\frac{\alpha_c\tau^\nu\mathcal{G}}{\tau^{j^c}\mathcal{Z}}, \frac{\lambda_2^{(\pm)}\tau^\nu\mathcal{N}}{\mathcal{Z}}, 1 \right)^T, \tag{75}$$

with

$$\mathcal{N} = \rho D_\nu \tau^{j^c} - \rho D_c \tau^\nu + \mathcal{M}, \quad \mathcal{T} = \rho D_\nu \tau^{j^c} - \rho D_c \tau^\nu - \mathcal{M}, \tag{76}$$

$$\mathcal{F} = \rho D_\nu \tau^{j^c} + \rho D_c \tau^\nu - \mathcal{M}, \quad \mathcal{G} = \rho D_\nu \tau^{j^c} + \rho D_c \tau^\nu + \mathcal{M}, \tag{77}$$

$$\mathcal{I} = D_\nu \left(\rho D_c \tau^\nu + \mathcal{M}\right) - \rho D_\nu^2 \tau^{j^c} - 2\alpha_c\alpha_\nu\tau^\nu, \tag{78}$$

$$\mathcal{Z} = D_\nu \left(\rho D_c \tau^\nu - \mathcal{M}\right) - \rho D_\nu^2 \tau^{j^c} - 2\alpha_c\alpha_\nu\tau^\nu.$$

The quantities $\mathcal{I}$, $\mathcal{Z}$ are assumed different than zero and this leads to the condition $\alpha_c\alpha_\nu \neq \rho D_c D_\nu$ (according to result (72)).

The discontinuity waves which are propagating with the velocity given by $\lambda_1^{(\pm)}$ and $\lambda_2^{(\pm)}$ are not exceptional waves in the sense of Lax-Boillat [2] (i.e. they can degenerate into shock waves) in the case where

$$\nabla\lambda_1^{(\pm)} \cdot \boldsymbol{d}_1^{(\pm)} = \mp\frac{\tau^\nu}{2\mathcal{M}\mathcal{I}} \left[2\rho\alpha_c^2\tau^\nu \frac{\partial\alpha_\nu}{\partial c} + \alpha_\nu\mathcal{T}\frac{\partial\alpha_c}{\partial r} \right] \neq 0, \tag{79}$$

$$\nabla\lambda_2^{(\pm)} \cdot \boldsymbol{d}_2^{(\pm)} = \pm\frac{\tau^\nu}{2\mathcal{M}\mathcal{Z}} \left[2\rho\alpha_c^2\tau^\nu \frac{\partial\alpha_\nu}{\partial c} + \alpha_\nu\mathcal{N}\frac{\partial\alpha_c}{\partial r} \right] \neq 0, \tag{80}$$

being

$$\nabla\lambda_1^{(\pm)} = \left(\mp\frac{\alpha_c}{2\mathcal{M}\lambda_1^{(\pm)}}\frac{\partial\alpha_\nu}{\partial c}, 0, \mp\frac{\alpha_\nu}{2\mathcal{M}\lambda_1^{(\pm)}}\frac{\partial\alpha_c}{\partial r}, 0 \right), \tag{81}$$

$$\nabla\lambda_2^{(\pm)} = \left(\pm\frac{\alpha_c}{2\mathcal{M}\lambda_2^{(\pm)}}\frac{\partial\alpha_\nu}{\partial c}, 0, \pm\frac{\alpha_\nu}{2\mathcal{M}\lambda_2^{(\pm)}}\frac{\partial\alpha_c}{\partial r}, 0 \right). \tag{82}$$

The scalar product between eigenvectors left and right corresponding to the same eigenvalue satisfy the following relations

$$\boldsymbol{l}_1^{(\pm)} \cdot \boldsymbol{d}_1^{(\pm)} = 1 + \frac{\mathcal{F}(\mathcal{N}+2\mathcal{M})}{2\rho\tau^{j^c}\mathcal{I}}, \quad \boldsymbol{l}_2^{(\pm)} \cdot \boldsymbol{d}_2^{(\pm)} = 1 + \frac{\mathcal{G}(3\rho D_\nu\tau^{j^c} - 3\rho D_c\tau^\nu - \mathcal{M})}{2\rho\tau^{j^c}\mathcal{Z}}, \tag{83}$$

whose values are supposed different than zero and this assumption ensures hyperbolicity of the system (66) and the propagation of the weak discontinuity waves, see (53). In the following we take into consideration $\lambda = \lambda_2^{(+)}$, corresponding to a progressive fast wave traveling to the right. Analogous results are valid for the waves propagating with the other velocities.

7 Determination of the approximated solution of the PDEs system

In this section, we consider an uniform unperturbed state in which $\boldsymbol{U}^0$, solution of the system (66), has the form

$$\boldsymbol{U}^0 = (c^0, (j^c)^0, r, (\mathcal{V})^0). \tag{84}$$

In $\boldsymbol{U}^0$ we have

$$\left(\lambda_2^{(+)}\right)^0 = \sqrt{\frac{\rho D_\nu \tau^{j^c} + \rho D_c \tau^\nu + \mathcal{M}^0}{2\rho\tau^{j^c}\tau^\nu}}, \tag{85}$$

$$\left(\boldsymbol{l}_2^{(+)}\right)^0 = \left(\frac{\left(\lambda_2^{(+)}\right)^0 \mathcal{T}^0}{2\alpha_c^0 \tau^\nu}, \frac{\mathcal{T}^0}{2\rho\alpha_c^0\tau^\nu}, \left(\lambda_2^{(+)}\right)^0, 1 \right),$$

$$\left(\boldsymbol{d}_2^{(+)}\right)^0 = \left(\frac{2\alpha_c^0 (\tau^{j^c})^2 \left(\lambda_2^{(+)}\right)^0}{\mathcal{Z}^0}, -\frac{\alpha_c^0 \tau^\nu \mathcal{G}^0}{\tau^{j^c}\mathcal{Z}^0}, \frac{\left(\lambda_2^{(+)}\right)^0 \tau^\nu \mathcal{N}^0}{\mathcal{Z}^0}, 1 \right), \tag{86}$$

$$\left(\boldsymbol{l}_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 = 1 + \frac{\mathcal{G}^0(3\rho D_\nu \tau^{j^c} - 3\rho D_c \tau^\nu - \mathcal{M}^0)}{2\rho\tau^{j^c}\mathcal{Z}^0} \tag{87}$$

and

$$\left(\nabla\lambda_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 = \frac{\tau^\nu}{2\mathcal{M}^0\mathcal{Z}^0}\left[2\rho(\alpha_c^0)^2\tau^\nu\left(\frac{\partial\alpha_\nu}{\partial c}\right)^0 + \alpha_\nu^0\mathcal{N}^0\left(\frac{\partial\alpha_c}{\partial r}\right)^0\right], \tag{88}$$

where the symbols " ^{0} " indicates that the quantities are calculated in $\boldsymbol{U}^0$.

The radial velocity along the characteristic rays is

$$\boldsymbol{\Lambda}^0(\boldsymbol{U}^0, \boldsymbol{n}^0) = \left(\lambda_2^{(+)}\right)^0 \boldsymbol{n}^0 = \left(\left(\lambda_2^{(+)}\right)^0, 0, 0\right). \tag{89}$$

and the characteristic rays are

$$\frac{dt}{d\sigma} = 1, \qquad \frac{dx_i}{d\sigma} = \frac{\partial \Psi^0}{\partial \varphi_\alpha} = \left(\lambda_2^{(+)}\right)^0, \qquad \frac{d\varphi_\alpha}{d\sigma} = 0. \tag{90}$$

By integrating $(90)_1$ we have

$$x^0 = t = \sigma, \qquad x(\sigma) = (x)_0 + \left(\lambda_2^{(+)}\right)^0 \sigma, \tag{91}$$

and the wave front takes the form

$$\varphi(x, \sigma) = \varphi^0 \left(x - \left(\lambda_2^{(+)}\right)^0 \sigma \right). \tag{92}$$

We remark that in this particular case, since $\boldsymbol{n}^0$ is constant, Λ_i^0 do not depends on x_0^i, so from (48) we deduce $J = 1$.

The amplitude π satisfies the following equation (see equation (55) with $J = 1$)

$$\left(\boldsymbol{l}_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 \left[\frac{d\pi}{d\sigma} + \left|\varphi_x^0\right| \left(\nabla \lambda_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 \pi^2 \right] = F^0 \, \pi, \tag{93}$$

where $F^0 = \left[\nabla \left(\boldsymbol{l}_2^{(+)} \cdot \boldsymbol{B}\right) \cdot \boldsymbol{d}_2^{(+)}\right]^0$.

By virtue of the hyperbolicity of the system, we have $\left(\boldsymbol{l}_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 \neq 0$.

Taking into account that

$$\boldsymbol{l}_2^{(+)} \cdot \boldsymbol{B} = -\frac{j^c \mathcal{T}}{2\rho \alpha_c \tau^{j^c} \tau^\nu} - \frac{\mathcal{V}}{\tau^\nu}, \tag{94}$$

(95)

$$\nabla \left(\boldsymbol{l}_2^{(+)} \cdot \boldsymbol{B}\right) = \Big(\frac{j^c}{\mathcal{M}} \frac{\partial \alpha_\nu}{\partial c}, \; -\frac{\mathcal{T}}{2\rho \alpha_c \tau^{j^c} \tau^\nu}, \\ \frac{j^c}{2\rho \alpha_c^2 \tau^{j^c} \tau^\nu} \left(\frac{2\rho \alpha_c \alpha_\nu \tau^{j^c} \tau^\nu}{\mathcal{M}} + \mathcal{T} \right) \frac{\partial \alpha_c}{\partial r}, \; -\frac{1}{\tau^\nu} \Big),$$

being $\nabla \equiv \left(\frac{\partial}{\partial c}, \frac{\partial}{\partial j^c}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \mathcal{V}}\right)$, in $\boldsymbol{U}^0$ we obtain

$$\left[\nabla \left(\boldsymbol{l}_2^{(+)} \cdot \boldsymbol{B}\right)\right]^0 = \Big(\frac{(j^c)^0}{\mathcal{M}^0} \left(\frac{\partial \alpha_\nu}{\partial c} \right)^0, -\frac{\mathcal{T}^0}{2\rho \alpha_c^0 \tau^{j^c} \tau^\nu}, \\ \frac{(j^c)^0}{2\rho \left(\alpha_c^0\right)^2 \tau^{j^c} \tau^\nu} \left(\frac{2\rho \alpha_c^0 \alpha_\nu^0 \tau^{j^c} \tau^\nu}{\mathcal{M}^0} + \mathcal{T}^0 \right) \left(\frac{\partial \alpha_c}{\partial r} \right)^0, -\frac{1}{\tau^\nu} \Big) \tag{96}$$

and

$$F^0 = \frac{2\alpha_c^0(\tau^{j^c})^2(j^c)^0\left(\lambda_2^{(+)}\right)^0}{\mathcal{M}^0\mathcal{Z}^0}\left(\frac{\partial\alpha_\nu}{\partial c}\right)^0 + \frac{(j^c)^0\mathcal{N}^0\left(\lambda_2^{(+)}\right)^0}{2\rho\mathcal{Z}^0\left(\alpha_c^0\right)^2\tau^{j^c}}\left(\frac{2\rho\alpha_c^0\alpha_\nu^0\tau^{j^c}\tau^\nu}{\mathcal{M}^0}+\mathcal{T}^0\right)\left(\frac{\partial\alpha_c}{\partial r}\right)^0 + \frac{\mathcal{G}^0\mathcal{T}^0}{2\rho(\tau^{j^c})^2\mathcal{Z}^0}-\frac{1}{\tau^\nu}. \tag{97}$$

Thus, we derive $\pi = \dfrac{h(\sigma)}{\Phi(\sigma)}$ (see equation (55) with $J=1$), being

$$h(\sigma) = \pi^0\exp\left[-\frac{F^0}{\left(\boldsymbol{l}_2^{(+)}\cdot\boldsymbol{d}_2^{(+)}\right)^0}\sigma\right], \tag{98}$$

with $\pi^0 = \pi(0)$,

$$\Phi(\sigma) = 1-\frac{\pi^0}{H^0}\left|\varphi_x^0\right|\left(\nabla\lambda_2^{(+)}\cdot\boldsymbol{d}_2^{(+)}\right)^0(e^{-H^0\sigma}-1), \tag{99}$$

where $H^0 = \dfrac{F^0}{\left(\boldsymbol{l}_2^{(+)}\cdot\boldsymbol{d}_2^{(+)}\right)^0}$.

From equations (98) and (99) we obtain

$$\pi(\sigma) = \frac{\pi^0H^0e^{-H^0\sigma}}{H^0-\pi^0|\varphi_x^0|\left(\nabla\lambda_2^{(+)}\cdot\boldsymbol{r}_2^{(+)}\right)^0(e^{-H^0\sigma}-1)}. \tag{100}$$

We observe that the amplitude $\pi(\sigma)$ is a limited function when

$$\lim_{\sigma\to\infty}\pi(\sigma) = \begin{cases} -\frac{H^0}{|\varphi_x^0|\left(\nabla\lambda_2^{(+)}\cdot\boldsymbol{d}_2^{(+)}\right)^0}, & \text{if } H^0<0,\\ 0, & \text{if } H^0>0. \end{cases} \tag{101}$$

Finally, by virtue of relations (84), (87), (88), (93) and (100) we derive the explicit form of the first approximation of the solution $\boldsymbol{U}$ of the system (66)

$$(c, j^c, r, \mathcal{V})^T = (c^0, (j^c)^0, r, (\mathcal{V})^0)^T$$

$$+\Gamma^0 \left(\frac{2\alpha_c^0(\tau^n)^2 \left(\lambda_2^{(+)}\right)^0}{\mathcal{Z}^0}, -\frac{\alpha_c^0 \tau^\nu \mathcal{G}^0}{\tau^{j^c}\mathcal{Z}^0}, \frac{\left(\lambda_2^{(+)}\right)^0 \tau^\nu \mathcal{N}^0}{\mathcal{Z}^0}, 1 \right)^T, \tag{102}$$

with

$$\Gamma^0 = \frac{\varphi^0\left(x - \left(\lambda_2^{(+)}\right)^0 \sigma\right) \pi^0 H^0 e^{-H^0\sigma}}{H^0 - \pi^0|\varphi_x^0| \left(\nabla\lambda_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 (e^{-H^0\sigma} - 1)}. \tag{103}$$

Thus, from (102) and (103) we have

$$c(x,\sigma) = c^0 + \frac{2\varphi^0\left(x - \left(\lambda_2^{(+)}\right)^0 \sigma\right) \pi^0 H^0 \alpha_c^0 (\tau^n)^2 \left(\lambda_2^{(+)}\right)^0 e^{-H^0\sigma}}{\mathcal{Z}^0 \left[H^0 - \pi^0|\varphi_x^0| \left(\nabla\lambda_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 (e^{-H^0\sigma} - 1)\right]}, \tag{104}$$

$$j^c(x,\sigma) = (j^c)^0 - \frac{\varphi^0\left(x - \left(\lambda_2^{(+)}\right)^0 \sigma\right) \pi^0 H^0 \alpha_c^0 \tau^\nu \mathcal{G}^0 e^{-H^0\sigma}}{\tau^{j^c}\mathcal{Z}^0 \left[H^0 - \pi^0|\varphi_x^0| \left(\nabla\lambda_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 (e^{-H^0\sigma} - 1)\right]}, \tag{105}$$

$$r(x,\sigma) = r^0 + \frac{\left(\lambda_2^{(+)}\right)^0 \varphi^0\left(x - \left(\lambda_2^{(+)}\right)^0 \sigma\right) \pi^0 H^0 \tau^\nu \mathcal{N}^0 e^{-H^0\sigma}}{\mathcal{Z}^0 \left[H^0 - \pi^0|\varphi_x^0| \left(\nabla\lambda_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 (e^{-H^0\sigma} - 1)\right]}, \tag{106}$$

$$\mathcal{V}(x,\sigma) = (\mathcal{V})^0 + \frac{\varphi^0\left(x - \left(\lambda_2^{(+)}\right)^0 \sigma\right) \pi^0 H^0 e^{-H^0\sigma}}{H^0 - \pi^0|\varphi_x^0| \left(\nabla\lambda_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0 (e^{-H^0\sigma} - 1)}. \tag{107}$$

In the case where in (100) $\Phi(\sigma_c) = 0$, with $\Phi(\sigma_c)$ given by equation (99), there exists a critical time σ_c given by

$$\sigma_c = \frac{1}{H^0} \ln V^0, \tag{108}$$

with

$$V^0 = \left(\frac{\pi^0 \left|\varphi_x^0\right| \left(\nabla\lambda_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0}{H^0 + \pi^0|\varphi_x^0| \left(\nabla\lambda_2^{(+)} \cdot \boldsymbol{d}_2^{(+)}\right)^0} \right).$$

In this case the amplitude of the weak discontinuty wave becomes unbounded and this can correspond to the formation of a shock wave [2]. In equation (108) σ_c exists in the case where

$$\frac{1}{H^0}\ln V^0 > 0 \quad \text{and} \quad V^0 > 0. \tag{109}$$

8 Conclusions

In this paper, we have presented, in the frame of extended irreversible thermodynamics with internal variables, a model for perfect isotropic porous media satured by a fluid flow in a particular case, derived in a previous paper by the authors, taking into consideration only the porosity and the fluid concentration fields and their fluxes. But, here some coefficients of the governing equations are assumed to be non-constant, depending on the fields involved, obtaining a non-linear PDEs system. Since closed-form solutions of nonlinear PDEs are rare, we have investigated the propagation of weak discontinuity waves, as approximated solutions. To this aim we have introduced a new variable related to the surface across which the first derivatives of the solutions undergo a jump. Following a Boillat's methodology for quasi-linear and hyperbolic systems of the first order, we have investigated, in an one-dimensional case, the propagation of one solution into an uniform unperturbed state in the first approximation, obtaining the Bernoulli's equation governing the evolution of the amplitude of the weak discontinuity wave, the wave front, the velocity along the characteristic rays, the approximated solution and the critical time in which the amplitude of this wave becomes unbounded, and this may correspond to the formation of a shock wave. The worked out results can be applied in several technological sectors as medical sciences, biology and geology.

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