Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 18, No. 4/2026

THE MEAN-SQUARE STABILITY AND MEAN-SQUARE STABILIZABILITY OF THE DISCRETE-TIME MEAN-FIELD STOCHASTIC LINEAR SYSTEMS WITH PERIODIC COEFFICIENTS*

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DOI 10.56082/annalsarscimath.2026.4.339

Abstract

In this work, several aspects of the mean-square stability and of the mean-square stabilizability of a class of discrete-time mean-field stochastic linear systems (MFSLS) with periodic coefficients is addressed. To a discrete-time MFSLS one associates in a natural way a companion standard discrete-time stochastic linear system (DTSLS). Our aim is to compare the mean-field exponential stability in mean-square (MFESMS) property of the MFSLS under consideration with the exponential stability in mean-square (ESMS) property of its companion DTSLS. Even if in some already published works is asserted that the two concepts of exponential stability are equivalent, we show that this equivalence is not true. We also show that the MF mean-square stabilizability of a discrete-time controlled MFSLS is not equivalent to the mean-square stabilizability of its controlled companion DTSLS.

*Accepted for publication on October 19, 2026

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Keywords: mean-field stochastic linear systems, mean-square stability, mean-square stabilizability, periodic coefficients.
MSC: 93E15, 93B05, 93B07, 93D15, 47B65.

1 Introduction

In recent years, mean-field control theory has received increasing interest, both from a theoretical perspective as well as from engineering and finance applications. The main distinctive feature of the mean-field setting when compared with classical stochastic optimal control is that the system dynamics involves state and control, as well as their expected values.
Since the pioneering works [10,12] there have been several important contributions related to this type of stochastic systems from the control perspective [1–5, 8, 11, 14, 16].

In the present work, we consider a class of discrete-time mean-field controlled stochastic linear systems with periodic coefficients. For this kind of stochastic systems we introduce the concept of mean-field exponential stability in mean-square (MF-ESMS) (Definition 1) and the concept of mean-field stabilizability in mean-square by linear state feedback (Definition 2).
The MF-ESMS property is characterized by exponential decay of the trajectories with random initial states of the mean-field stochastic system under consideration. We show that without loss of generality, we can characterize the MF-ESMS property only by exponential decay of the trajectories starting from deterministic initial states (Proposition 1). To a mean-field stochastic linear system one may associate, in a natural way, a companion discrete-time stochastic linear system with state multiplicative and /or control multiplicative independent random disturbances. In some already published papers ([13], [15]), is asserted that the MF-ESMS property of a mean-field stochastic linear system is equivalent to the standard exponential stability in mean-square (ESMS) of its discrete-time companion stochastic linear system. Through Example 1, we show that this equivalence is not always true. However, it remains true that the ESMS property of the companion discrete-time linear stochastic system implies the MF-ESMS property of the mean-field stochastic linear system under consideration. The exponential stability in mean-square of the mean-field stochastic linear system induced by the ESMS property of its companion discrete-time linear stochastic system is named, in this work, "mean-field strong exponential stability in mean-square (MF-SESMS)". The Lyapunov type criteria for exponential stability in mean-square of the companion discrete-time linear stochastic

system provide necessary and sufficient conditions for MF-SESMS property but only sufficient conditions for the MF-ESMS property of the mean-field stochastic linear system under consideration. By the Example 2, we show that the mean-field stabilizability in mean-square of a mean-field controlled stochastic linear system is not equivalent to the standard stochastic stabilizability in mean-square by a decentralized linear state feedback.
The mean-field stabilizability of the mean-field controlled stochastic linear system induced by the stochastic stabilizability by a decentralized linear state feedback of its companion controlled discrete-time stochastic linear system is named MF strong stabilizability in mean-square of the mean-field controlled stochastic linear system. This property may be used in future research, in order to establish the existence of the maximal solution and of the stabilizing solution of a Riccati type equation involved in the solution of some optimal control problems for mean-field controlled stochastic linear systems. The developments from sections 3 and 4, of the present work, are the discrete-time counterpart of the results from the recent paper [7].

2 The problem setting

2.1 Notation

In this work $(\Omega, \mathfrak{F}, \mathcal{P})$ is a probability space where a sequence of r-dimensional independent random vectors $\{w(t)\}_{t\in\mathbb{Z}_+}$, $w(t) = (w_1(t) \;\; w_2(t) \;\; \dots \;\; w_r(t))^\top$ is defined. Here and in the sequel $\mathbb{Z}_+ = \{0, 1, 2....\}$.

We assume that the following assumption holds:

H1). $\mathbb{E}[w(t)] = 0$ and $\mathbb{E}[w(t)w^\top(t)] = I_r$ for all $t \in \mathbb{Z}_+$, where I_r is the identity matrix of size $r \times r$.

Throughout this work $\mathbb{E}[\cdot]$ denotes the mathematical expectation of a random variable, while the superscript $\top$ stands for the transpose of a matrix or a vector. For each $t \in \mathbb{Z}_+$, $\mathcal{F}_t := \{\sigma(w(s)), 0 \le s \le t\} \subset \mathfrak{F}$ denotes the smallest σ-algebra with respect to which the random vectors $(w(s)$, $0 \le s \le t$, are measurable.

If $t_0 \in \mathbb{Z}_+$, $\ell^2_{\mathcal{F}}\{t_0, \infty; \mathbb{R}^d\}$ denotes the linear space of all sequences $\{v(t)\}_{t\ge t_0}$ of $\mathbb{R}^d$ valued random vectors with the properties:

a) for each $t \ge t_0$, $v(t)$ are $\mathcal{F}_t$-measurable;

b) $\sum\limits_{t=t_0}^{\infty} \mathbb{E}[|v(t)|^2] < \infty$.

Here and in the sequel $|\cdot|$ denotes the Euclidean norm of a vector i.e. $|v| = (v^\top v)^{1/2}$ for all $v \in \mathbb{R}^d$.
As usual $\mathbb{R}^{m\times n}$ denotes the linear space of $m \times n$ real matrices. For $p \ge 2$,

$\mathcal{S}_p \subset \mathbb{R}^{p\times p}$ stands for the space of symmetric matrices of size $p\times p$. $\mathcal{S}_p^+ \subset \mathcal{S}_p$ denotes the closed, solid, normal convex cone of the positive semidefinite matrices of size $p \times p$.
If $t_0 \in \mathbb{Z}_+$, we denote $\mathcal{X}_{t_0}$ the set of all n-dimensional random vectors x_0 which are independent of all random vectors $w(t)$, $t > t_0$ and are satisfying $\mathbb{E}[|x_0|^2] < \infty$.

Remark 1. *The set $\mathcal{X}_{t_0}$ contains the n-dimensional random vectors x_0 with finite second order moments which are either $\mathcal{F}_{t_0}$-measurable, or, are independent of all random vectors $w(t)$, $t \in \mathbb{Z}_+$.*

2.2 Model description

We consider the discrete-time controlled stochastic linear system having the state space representation of the form:

$$x(t+1) = A_0(t)x(t) + \overline{A}_0(t)\mathbb{E}[x(t)] + B_0(t)u(t) + \overline{B}_0(t)\mathbb{E}[u(t)] + \quad (1)$$
$$\sum_{k=1}^{r}(A_k(t)x(t) + \overline{A}_k(t)\mathbb{E}[x(t)] + B_k(t)u(t) + \overline{B}_k(t)\mathbb{E}[u(t)])w_k(t), t \geq t_0,$$

$x(t_0) = x_0 \in \mathcal{X}_{t_0}$, where $x(t) \in \mathbb{R}^n$ are the vectors of the state parameters and $u(t) \in \mathbb{R}^m$ are the vectors of the control parameters.
In (1), $\{A_k(t)\}_{t\in\mathbb{Z}}$, $\{\overline{A}_k(t)\})_{t\in\mathbb{Z}}$ are matrix valued sequences from $\mathbb{R}^{n\times n}$ and $\{B_k(t)\}_{t\in\mathbb{Z}}$, $\{\overline{B}_k(t)\}_{t\in\mathbb{Z}}$, $0 \leq k \leq r$ are matrix valued sequences in $\mathbb{R}^{n\times m}$ which are periodic of period $\theta \geq 1$.
The set $\mathcal{U}_{ad}(t_0)$ of the admissible controls consists of all sequences $\{u(t)\}_{t\geq t_0}$ of $\mathbb{R}^m$ valued random vectors with the properties:

a) for each $t \geq t_0$, $u(t)$ is $\mathcal{F}_t$-measurable;

b) $\mathbb{E}[|u(t)|^2] < \infty$.

The controlled stochastic linear system (1) is the discrete-time controlled linear system of McKean-Vlasov type which is also known as a discrete-time mean-field controlled stochastic linear system (MFCSLS).

The study of the stochastic differential equations of McKean-Vlasov type was initiated in [10, 12]. Among the most recent papers which studied the control problems involving discrete-time stochastic linear equations of McKean-Vlasov type, are [13, 15].

For each $0 \leq t_0 < t \in \mathbb{Z}_+$, $\mathcal{G}_{t_0,t} \subset \mathfrak{F}$ stands for the smallest σ-algebra with respect to which the random vectors $x_0 \in \mathcal{X}_{t_0}$ and $w(s)$, $t_0 \leq s \leq t-1$ are measurable.

The next lemma collects several properties of the solutions of (1). Their proofs may be done recursively with respect to $t \geq t_0$.

Lemma 1. *For each $t_0 \in \mathbb{Z}_+$, $x_0 \in \mathcal{X}_{t_0}$ and $\mathbf{u} = \{u(t)\}_{t\geq t_0} \in \mathcal{U}_{ad}(t_0)$, the problem with given initial values (IVP) (1) has a unique solution $x(t; t_0, x_0, \mathbf{u})$, with the properties:*

a) $x(t; t_0, x_0, \mathbf{u})$ is $\mathcal{G}_{t_0,t}$-measurable;

b) $\mathbb{E}[|x(t; t_0, x_0, \mathbf{u})|^2] < \infty$;

c) $x(t_0; t_0, x_0, \mathbf{u}) = x_0$.

Bearing in mind the assumption **H1**) and Lemma 1 a), we obtain that the random vectors $x(t; t_0, x_0, \mathbf{u})$ and $w(t)$ are mutually independent for any $t \geq t_0$. This allows us to obtain from (1) that if $x(t) \triangleq x(t; t_0, x_0, \mathbf{u})$, $t \geq t_0$, $\{\mathbb{E}[x(t)]\}_{t\geq t_0}$ is satisfying the following IVP:

$$\mathbb{E}[x(t+1)] = (A_0(t) + \overline{A}_0(t))\mathbb{E}[x(t)] + (B_0(t) + \overline{B}_0(t))\mathbb{E}[u(t)] \tag{2a}$$

$$\mathbb{E}[x(t_0)] = \mathbb{E}[x_0]. \tag{2b}$$

Subtracting (2) from (1), we obtain:

$$\begin{aligned} &x(t+1) - \mathbb{E}[x(t+1)] = A_0(t)(x(t) - \mathbb{E}[x(t)]) + B_0(t)(u(t) - \mathbb{E}[u(t)]) \\ &+ \sum_{k=1}^{r}[A_k(t)(x(t) - \mathbb{E}[x(t)]) + (A_k(t) + \overline{A}_k(t))\mathbb{E}[x(t)] \\ &+ B_k(t)(u(t) - \mathbb{E}[u(t)]) + (B_k(t) + \overline{B}_k(t))\mathbb{E}[u(t)]]w_k(t), t \geq t_0, \end{aligned} \tag{3a}$$

$$x(t_0) - \mathbb{E}[x(t_0)] = x_0 - \mathbb{E}[x_0]. \tag{3b}$$

We set

$$x^1(t) \triangleq x(t) - \mathbb{E}[x(t)] \tag{4a}$$

$$x^2(t) \triangleq \mathbb{E}[x(t)] \tag{4b}$$

$$u^1(t) \triangleq u(t) - \mathbb{E}[u(t)] \tag{5a}$$

$$u^2(t) \triangleq \mathbb{E}[u(t)] \tag{5b}$$

$t \geq t_0$. With these notations, we obtain from (2) and (3) the following IVP:

$$\begin{aligned} &\begin{pmatrix} x^1(t+1) \\ x^2(t+1) \end{pmatrix} = \mathbb{A}_0(t)\begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} + \mathbb{B}_0(t)\begin{pmatrix} u^1(t) \\ u^2(t) \end{pmatrix} \\ &+ \sum_{k=1}^{r}(\mathbb{A}_k(t)\begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} + \mathbb{B}_k(t)\begin{pmatrix} u^1(t) \\ u^2(t) \end{pmatrix})w_k(t), t \geq t_0 \end{aligned} \tag{6a}$$

$$\begin{pmatrix} x^1(t_0) \\ x^2(t_0) \end{pmatrix} = \begin{pmatrix} x_0 - \mathbb{E}[x_0] \\ \mathbb{E}[x_0] \end{pmatrix} \tag{6b}$$

with:

$$\mathbb{A}_0(t) = \begin{pmatrix} A_0(t) & 0 \\ 0 & A_0(t) + \overline{A}_0(t) \end{pmatrix} \tag{7a}$$

$$\mathbb{A}_k(t) = \begin{pmatrix} A_k(t) & A_k(t) + \overline{A}_k(t) \\ 0 & 0 \end{pmatrix}, 1 \leq k \leq r, \tag{7b}$$

$$\mathbb{B}_0(t) = \begin{pmatrix} B_0(t) & 0 \\ 0 & B_0(t) + \overline{B}_0(t) \end{pmatrix} \tag{8a}$$

$$\mathbb{B}_k(t) = \begin{pmatrix} B_k(t) & B_k(t) + \overline{B}_k(t) \\ 0 & 0 \end{pmatrix}, 1 \leq k \leq r. \tag{8b}$$

Along with the IVP (6), we introduce the following IVP on the space $\mathbb{R}^n \times \mathbb{R}^n$:

$$\mathbf{x}(t+1) = \mathbb{A}_0(t)\mathbf{x}(t) + \mathbb{B}_0(t)\mathbf{u}(t) + \sum_{k=1}^{r} (\mathbb{A}_k(t)\mathbf{x}(t) + \mathbb{B}_k(t)\mathbf{u}(t))w_k(t), \tag{9a}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0 = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} \tag{9b}$$

where $\mathbf{x}(t) = \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^n$ and $\mathbf{u}(t) = \begin{pmatrix} u^1(t) \\ u^2(t) \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^m$, $x_0^k \in \mathcal{X}_{t_0}$, $k = 1, 2$. The IVP (6) is the particular case of the IVP (9), when $x_0^1 = x_0 - \mathbb{E}[x_0]$ and $x_0^2 = \mathbb{E}[x_0]$. In the sequel, the discrete-time controlled linear stochastic system, having the state space representation described by (6a) (or (9a), respectively) will be named the companion controlled difference stochastic linear system (CDSLS), associated to the MFCSLS (1).

When $B_k(t) = \overline{B}_k(t) = 0$, $t \in \mathbb{Z}$, $0 \leq k \leq r$, the MFCSLS (1) reduces to the uncontrolled mean-field stochastic linear system (MFSLS):

$$x(t+1) = A_0(t)x(t) + \overline{A}_0(t)\mathbb{E}[x(t)] + \sum_{k=1}^{r} (A_k(t)x(t) + \overline{A}_k(t)\mathbb{E}[x(t)])w_k(t), \tag{10a}$$

$$x(t_0) = x_0 \in \mathcal{X}_{t_0}. \tag{10b}$$

The IVPs (6) and (9) take the form:

$$\begin{pmatrix} x^1(t+1) \\ x^2(t+1) \end{pmatrix} = \mathbb{A}_0(t) \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} + \sum_{k=1}^{r} \mathbb{A}_k(t) \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} w_k(t), \tag{11a}$$

$$\begin{pmatrix} x^1(t_0) \\ x^2(t_0) \end{pmatrix} = \begin{pmatrix} x_0 - \mathbb{E}[x_0] \\ \mathbb{E}[x_0] \end{pmatrix} \tag{11b}$$

$$\mathbf{x}(t+1) = \mathbb{A}_0(t)\mathbf{x}(t) + \sum_{k=1}^{r} \mathbb{A}_k(t)\mathbf{x}(t)w_k(t), t \geq t_0 \tag{12a}$$

$$\mathbf{x_0}(t_0) = \mathbf{x}_0 = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} \tag{12b}$$

where $x_0^k \in \mathcal{X}_{t_0}$, $k = 1, 2$. The discrete-time stochastic linear system (11a) ((12a), respectively), will be named the companion discrete-time stochastic linear system (DTSLS) associated to the MFSLS (10a).

2.3 Basic definitions and the problem setting

Definition 1. *We say that the zero solution of the MFSLS (10a) or equivalently the MFSLS (10) is mean-field exponentially stable in mean square (MF-ESMS) if for all initial time $t_0 \in \mathbb{Z}_+$ and any initial state $x_0 \in \mathcal{X}_{t_0}$, the corresponding solution $x(\cdot; t_0, x_0)$ of (10a) has a decay of the form*

$$\mathbb{E}[|x(t; t_0, x_0)|^2] \leq \beta\delta^{t-t_0}\mathbb{E}[|x_0|^2] \tag{13}$$

for all $t \geq t_0$, where $\beta \geq 1$, $\delta \in (0, 1)$ are constants not depending upon t, t_0 and x_0.

Remark 2. *Unlike the nonlinear case, where the stability property refers to a given solution, in the case of a linear system, the stability property of a solution is equivalent to the stability of all solutions of that system. That is why, we may say that the MFSLS (10a) is MF-ESMS if its solutions are satisfying (13).*

Definition 2. *We say that the MFCSLS (1) is mean-field stabilizable in mean-square, or equivalently mean-field stabilizable by linear state feedback if there exist θ-periodic $\mathbb{R}^{m\times n}$ valued matrix sequences $\{F(t)\}_{t\in\mathbb{Z}}$, $\{\overline{F}(t)\}_{t\in\mathbb{Z}}$ with the property that the corresponding closed-loop MFSLS*

$$\begin{aligned} x(t+1) &= (A_0(t) + B_0(t)F(t))x(t) + [\overline{A}_0(t) + (B_0(t) + \overline{B}_0(t))\overline{F}(t) \\ &- B_0(t)F(t)]\mathbb{E}[x(t)] + \sum_{k=1}^{r}\{(A_k(t) + B_k(t)F(t))x(t) + [\overline{A}_k(t) + (B_k(t) \\ &+ \overline{B}_k(t))\overline{F}(t) - B_k(t)F(t)]\mathbb{E}[x(t)]\}w_k(t), t \geq t_0 \end{aligned} \tag{14a}$$

$$x(t_0) = x_0 \in \mathcal{X}_{t_0} \tag{14b}$$

is MF-ESMS in the sense of Definition 1.

The previous definition is a time-varying version of the Definition 3.2 from [15].

Our aim is to highlight several useful properties of the MF-ESMS of a MFSLS (10) and of the mean-field stabilizability in mean-square of a MFCSLS of the form 1). We shall emphasize the relationship between the MF-ESMS property of the MFSLS (10) and ESMS property of the corresponding companion DTSLS (12a).

We shall see that the ESMS property of the companion DTSLS (12) implies the MF-ESMS of the MFSLS (10) but the converse implication is not always true. Also, we shall show that the stochastic stabilizability in mean-square of the corresponding companion discrete-time controlled stochastic linear system implies the mean-field stabilizability in mean-square of the MFCSLS (1), but the converse implication is not always true.

Based on the particular form of the matrix coefficients of the discrete-time stochastic linear system (12a) given in (7)-(8), we obtain Lyapunov type criteria for the ESMS property of the companion DTSLS (12) of lower dimension than those obtained applying directly the already existing results from the literature.

3 The mean-field exponential stability in mean-square

3.1 An equivalent characterization of the MF-ESMS property

First we show that without loss of generality, we can characterize the MF-ESMS property of the MFSLS (10a) based only on exponential decay of its solutions starting from deterministic initial states.

Proposition 1. *Under the assumption* **H1**) *the following are equivalent:*
(i) the MFSLS (10a) is MF-ESMS in the sense of Definition 1;
(ii)

$$\mathbb{E}[|x(t;t_0,x_0)|^2] \leq \beta\delta^{t-t_0}|x_0|^2 \tag{15}$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$, *where* $\beta \geq 1$, $\delta \in (0,1)$ *are constants not depending upon* t, t_0, x_0.

Proof. The implication $(i) \Rightarrow (ii)$ is obvious because $\mathbb{R}^n \subset \mathcal{X}_{t_0}$ for any $t_0 \in \mathbb{Z}_+$. Further, we proof $(ii) \Rightarrow (i)$. Let $t_0 \in \mathbb{Z}_+$ and $\tilde{x}_0 \in \mathcal{X}_{t_0}$ be arbitrary but fixed. We have to show that the solution $x(\cdot;t_0,\tilde{x}_0)$ of the MFSLS (10a)

corresponding to this initial pair has an exponential decay as that from (13). For each $1 \leq j \leq n$, we consider $\xi_j(\cdot, t_0)$ the solution of the following IVP:

$$\xi_j(t+1) = A_0(t)\xi_j(t) + \overline{A}_0(t)\mathbb{E}[\xi_j(t)] + \sum_{k=1}^{r}(A_k(t)\xi_j(t) + \overline{A}_k(t)\mathbb{E}[\xi_j(t)])w_k(t), t \geq t_0 \tag{16a}$$

$$\xi_j(t_0) = e_j \tag{16b}$$

where $e_j \in \mathbb{R}^n$ is the j-th column of the identity matrix I_n. Comparing the IVP (16) to the IVP (10), we obtain based on the unicity of the solution of an IVP, that

$$\xi_j(t, t_0) = x(t; t_0, e_j)$$

for all $t \geq t_0$. If (ii) holds, we obtain from (15) applied in the case of the previous equality that

$$\mathbb{E}[|\xi_j(t; t_0)|^2] \leq \beta\delta^{t-t_0} \tag{17}$$

for all $t \geq t_0$.

We define the matrix $\Xi(t; t_0) = \begin{pmatrix} \xi_1(t, t_0) & \xi_2(t, t_0) & \dots & \xi_n(t, t_0) \end{pmatrix}$. Bearing in mind (16) we may infer that $\{\Xi(t; t_0)\}_{t \geq t_0}$ is the matrix valued solution of the following IVP:

$$\Xi(t+1) = A_0(t)\Xi(t) + \overline{A}_0\mathbb{E}[\Xi(t)] + \sum_{k=1}^{r}(A_k(t)\Xi(t) + \overline{A}_k(t)\mathbb{E}[\Xi(t)])w_k(t), \tag{18a}$$

$$\Xi(t_0; t_0) = I_n. \tag{18b}$$

Starting from the definition of the euclidian norm of a symmetric positive semidefinite matrix, we may write successively:

$$\|\mathbb{E}[\Xi^T(t, t_0)\Xi(t, t_0)]\|_2 = \lambda_{max}(\mathbb{E}[\Xi^T(t, t_0)\Xi(t, t_0)]) \leq Tr[\mathbb{E}[\Xi^T(t, t_0)\Xi(t, t_0)]] = \sum_{j=1}^{n}\mathbb{E}[|\xi_j(t, t_0)|^2] \tag{19}$$

where $\lambda_{max}(\cdot)$ denotes the maximal eigenvalue of a positive semidefinite symmetric matrix and $Tr[\cdot]$ stands for the trace of a matrix.

Further, (17) and (19) yield

$$\|\mathbb{E}[\Xi^T(t, t_0)\Xi(t, t_0)]\|_2 \leq n\beta\delta^{t-t_0} \tag{20}$$

for all $t \geq t_0 \geq 0$.

Let $\tilde{x}(t) := \Xi(t;t_0)\tilde{x}_0$. Employing (18), we deduce that $\tilde{x}(t)$ is the solution of the IVP (10) corresponding to the initial pair $(t_0, \tilde{x}_0)$. Hence,

$$x(t;t_0,\tilde{x}_0) = \Xi(t,t_0)\tilde{x}_0 \tag{21}$$

for all $t \geq t_0 \geq 0$, $\tilde{x}_0 \in \mathcal{X}_{t_0}$. Let $\sigma[\tilde{x}_0] \subset \mathfrak{F}$ be the smallest σ-algebra with respect to which the random vector $\tilde{x}_0$ is measurable.

Based on the properties of the conditional expectation, we obtain from (21) that

$$\begin{aligned}\mathbb{E}[|x(t;t_0,\tilde{x}_0)|^2|\sigma[\tilde{x}_0]] = \mathbb{E}[\tilde{x}_0^T\Xi^T(t,t_0)\Xi(t,t_0)\tilde{x}_0|\sigma[\tilde{x}_0]] = \\ \tilde{x}_0^T\mathbb{E}[\Xi^T(t,t_0)\Xi(t,t_0)|\sigma[\tilde{x}_0]]\tilde{x}_0.\end{aligned} \tag{22}$$

Bearing in mind the properties of the random vectors from $\mathcal{X}_{t_0}$, we may infer via (18) that the components of the matrix $\Xi^T(t,t_0)\Xi(t,t_0)$ are random vectors independent with respect to the σ-algebra $\sigma[\tilde{x}_0]$. Hence

$$\mathbb{E}[\Xi^T(t,t_0)\Xi(t,t_0)|\sigma[\tilde{x}_0]] = \mathbb{E}[\Xi^T(t,t_0)\Xi(t,t_0)] \ \ a.s. \tag{23}$$

Since $0 \leq \mathbb{E}[\Xi^T(t,t_0)\Xi(t,t_0)] \leq \|E[\Xi^T(t,t_0)\Xi(t,t_0)]\|_2 I_n$ we obtain via (20), (22), (23) that

$$\mathbb{E}[|x(t;t_0,\tilde{x}_0)|^2|\sigma[\tilde{x}_0]] \leq n\beta\delta^{t-t_0}|\tilde{x}_0|^2 \ \ a.s.$$

for all $t \geq t_0 \geq 0$, $\tilde{x}_0 \in \mathcal{X}_t$.

Taking the expectation in the previous inequality, we may conclude that $\mathbb{E}[|x(t;t_0,\tilde{x}_0)|^2] \leq n\beta\delta^{t-t_0}\mathbb{E}[|\tilde{x}_0|^2]$ for all $t \geq t_0 \geq 0$, $\tilde{x}_0 \in \mathcal{X}_{t_0}$. Thus the proof ends.

3.2 The connection between the MF-ESMS property of the MFSLS and the ESMS property of its companion DTSLS

Let $x(\cdot;t_0,x_0)$ be the solution of the MFSLS (10a) corresponding to an arbitrary initial pair (t_0,x_0), where $t_0 \in \mathbb{Z}_+$ and $x_0 \in \mathcal{X}_{t_0}$.

We set

$$x^1(t;t_0,x_0) \triangleq x(t;t_0,x_0) - \mathbb{E}[x(t;t_0,x_0)] \tag{24a}$$

$$x^2(t;t_0,x_0) \triangleq \mathbb{E}[x(t;t_0,x_0)] \tag{24b}$$

$t \geq t_0$ and obtain the solution of the IVP (11) generated by the solution $x(\cdot; t_0, x_0)$ of the IVP (10). Conversely, if $\begin{pmatrix} x^1(\cdot; t_0, x_0) \\ x^2(\cdot; t_0, x_0) \end{pmatrix}$ is a solution of the IVP (11), then $x(\cdot, t_0, x_0)$ defined by

$$x(t; t_0, x_0) \triangleq x^1(t; t_0, x_0) + x^2(t; t_0, x_0), t \geq t_0 \tag{25}$$

is the solution of the IVP (10).

Employing the equalities (24) and (25) we stabilize an equivalence between the MF-ESMS property of the MFSLS (10a) and the exponential decay of the solutions of the IVP (11) for an arbitrary $x_0 \in \mathcal{X}_{t_0}$.

Proposition 2. *Under the assumption* **H1**), *the following are equivalent:*
(i) the MFSLS (10a) is MF-ESMS in the sense of the Definition 1;
(ii) for each initial time $t_0 \in \mathbb{Z}_+$ and any initial state $x_0 \in \mathcal{X}_{t_0}$, the solution $\begin{pmatrix} x^1(\cdot; t_0, x_0) \\ x^2(\cdot; t_0, x_0) \end{pmatrix}$ of the corresponding IVP (11) has a decay of the form:

$$\mathbb{E}[|x^1(t; t_0, x_0)|^2] + \mathbb{E}[|x^2(t; t_0, x_0)|^2] \leq \tilde{\beta}\tilde{\delta}^{t-t_0}(\mathbb{E}[|x_0 - \mathbb{E}[x_0]|^2] + |\mathbb{E}[x_0]^2), \tag{26}$$

where $\tilde{\beta} \geq 1$, $\tilde{\delta} \in (0, 1)$ are constants not depending upon t, t_0, x_0.

Proof: Assuming that (i) holds, we obtain from (13), (24b), that:

$$\mathbb{E}[|x^2(t; t_0, x_0)|^2] \leq \beta\delta^{t-t_0}\mathbb{E}[|x_0|^2] \tag{27}$$

for all $t \geq t_0 \in \mathbb{Z}_+$ and $x_0 \in \mathcal{X}_{t_0}$, with $\beta \geq 1$, $\delta \in (0, 1)$ not depending upon t, t_0, x_0. Further, (25) gives

$$\mathbb{E}[|x(t; t_0, x_0)|^2] = \mathbb{E}[|x^1(t; t_0, x_0)|^2] + \mathbb{E}[|x^2(t; t_0, x_0)|^2] \tag{28}$$

for all $t \geq t_0 \in \mathbb{Z}_+$ because $\mathbb{E}[x^1(t; t_0, x_0)(x^2(t; t_0, x_0))^T] = 0$ for all $t \geq t_0$. Employing (13), (24a), (27), (28) we deduce that:

$$\mathbb{E}[|x^1(t; t_0, x_0)|^2] \leq 2\beta\delta^{t-t_0}\mathbb{E}[x_0|^2].$$

Combining this inequality to (27), we obtain:

$$\mathbb{E}[|x^1(t; t_0, x_0)|^2] + \mathbb{E}[|x^2(t; t_0, x_0)|^2] \leq 3\beta\delta^{t-t_0}\mathbb{E}[|x_0|^2] \tag{29}$$

for all $t \geq t_0 \in \mathbb{Z}_+$, $x_0 \in \mathcal{X}_{t_0}$. Bearing in mind that

$$\mathbb{E}[|x_0|^2] = \mathbb{E}[|x_0 - \mathbb{E}[x_0]|^2] + |\mathbb{E}[x_0]|^2$$

we may conclude that (29) coincides to (26), when $\tilde{\beta} = 3\beta$, $\tilde{\delta} = \delta$. Thus the validity of the implication $(i) \Rightarrow (ii)$ is confirmed. In order to check the validity of the implication $(ii) \Rightarrow (i)$ we remark that (28) allows us to deduce that (13) holds if (26) is true. Thus the proof ends.

Remark 3. *a) Even if $x^2(t;t_0,x_0)$ are deterministic vectors, we preferred to use the notation $\mathbb{E}[|x^2(t;t_0,x_0)|^2]$ in (26) in order to preserve the unit of the presentation.*

b) From (24b) we obtain, via (2) written for $B_0(t)=\overline{B}_0(t)=0$ for all $t\in\mathbb{Z}$, that $\{x^2(t;t_0,x_0)\}_{t\geq t_0}$ solves the deterministic discrete-time linear equation (DTLE) on the euclidian space $\mathbb{R}^n$:

$$x^2(t+1)=(A_0(t)+\overline{A}_0(t))x^2(t). \tag{30}$$

Invoking the implication $(i)\Rightarrow(ii)$ from Proposition 2 we may infer that the exponential stability of the zero solution of the deterministic DTLE (30) is a necessary condition for the MF-ESMS property of the MFSLS (10a).

For the reader convenience, we recall here the definition of the ESMS property of the DTSLS (12a).

Definition 3. *We say that the zero solution of the DTSLS (12a) or equivalently, the system (12a) is exponentially stable in mean-square (ESMS) if its solutions have a decay of the form*

$$\mathbb{E}[|x^1(t;t_0,\mathbf{x}_0)|^2]+\mathbb{E}[|x^2(t;t_0,\mathbf{x}_0)|^2]\leq\beta_1\delta_1^{t-t_0}(\mathbb{E}[|x_0^1|^2]+\mathbb{E}[|x_0^2|^2]) \tag{31}$$

for all $t\geq t_0\in\mathbb{Z}_+$ and any initial states $\mathbf{x}_0=\begin{pmatrix}x_0^1\\x_0^2\end{pmatrix}\in\mathcal{X}_{t_0}\times\mathcal{X}_{t_0}$. In (31), $\beta_1\geq 1$, $\delta_1\in(0,1)$ are constants not depending upon t,t_0,x_0^k, $k=1,2$.

Remark 4. *Bearing in mind that the initial states (11) are particular case of (12b) we may conclude via the implication $(ii)\Rightarrow(i)$ from Proposition 2 that the MFSLS (10a) is MF-ESMS if its companion DTSLS (12a) is ESMS. The next example shows that the converse implication is not always true.*

Example 1. *We consider the 1-dimensional discrete-time, time-invariant MFSLS:*

$$x(t+1)=a_0x(t)+\overline{a}_0\mathbb{E}[x(t)]+\sum_{k=1}^{r}(a_kx(t)+\overline{a}_k\mathbb{E}[x(t)])w_k(t),t\geq t_0 \tag{32a}$$

$$x(t_0)=x_0\in\mathbb{R} \tag{32b}$$

where $a_k,\overline{a_k}$, $0\leq k\leq r$ are real numbers subject to the following constraints:

c_1) $a_k+\overline{a}_k=0$, $1\leq k\leq r$;

c_2) $|a_0+\overline{a}_0|<1$;

$c_3)$ $\sum\limits_{k=0}^{r} a_k^2 \geq 1$,

$\{w_k(t)\}_{t\geq 0}$, $1 \leq k \leq r$, *are sequences of random variables which are satisfying the conditions from the assumption* $\mathbf{H}1$). *According to Proposition 1, for the study of the MF-ESMS property of the MFSLS (32) we may assume without loss of generality that the initial states* x_0 *are deterministic. Under the constraint* c_1), *the corresponding IVP (11) becomes:*

$$\begin{aligned} x^1(t+1) &= (a_0 + \sum_{k=1}^{r} a_k w_k)x^1(t), t \geq t_0 \\ x^1(t_0) &= 0 \end{aligned} \tag{33}$$

$$\begin{aligned} x^2(t+1) &= (a_0 + \overline{a}_0)x^2(t), t \geq t_0 \\ x^2(t_0) &= x_0. \end{aligned} \tag{34}$$

Based on the unicity of the solution of an IVP we obtain that the unique solution of the IVP (33) is $x^1(t) = 0$ *a.s.* $t \geq t_0$ *and the unique solution of (34) is* $x^2(t) = (a_0 + \overline{a}_0)^{t-t_0}x_0$, $t \geq t_0$. *Invoking* c_2) *we may infer that* $|x^2(t)|^2 \leq \delta^{t-t_0}|x_0|^2$ *for all* $t \geq t_0 \in \mathbb{Z}_+$, *where* $\delta \in (0,1)$. *In this case, the inequality (26) becomes:*

$$\mathbb{E}[|x^1(t)|^2] + \mathbb{E}[|x^2(t)|^2] \leq \delta^{t-t_0}|x_0|^2, \ \ \forall t \geq t_0.$$

The implication $(ii) \Rightarrow (i)$ *from Proposition 2 applied in this particular case, allows us to conclude that the MFSLS (32) is MF-ESMS. The companion discrete-time stochastic linear system of the form (12), associated to (32) is:*

$$\begin{aligned} x^1(t+1) &= a_0 x^1(t) + \sum_{k=1}^{r} a_k x^1(t) w_k(t), t \geq t_0 \\ x^1(t_0) &= x_0^1 \end{aligned} \tag{35}$$

$$\begin{aligned} x^2(t+1) &= (a_0 + \overline{a}_0)x^2(t), t \geq t_0 \\ x^2(t_0) &= x_0^2 \end{aligned} \tag{36}$$

where $x_0^k \in \mathcal{X}_{t_0}$, $k = 1, 2$. *Hence, the companion DTSLS associated to the MFSLS (32) is ESMS if and only if the discrete-time stochastic linear equations (35) and (36) are simultaneously ESMS.*

Employing a Lyapunov-type criterion for the characterization of the ESMS property of a discrete-time stochastic linear equation (see section 3.5.2 from

[6]) we obtain that the DTSLS (35) is ESMS if and only if its coefficients are satisfying the condition $\sum_{k=0}^{r} a_k^2 < 1$. *This is in contradiction to* c_3). *Thus, we may conclude that the MFSLS (32) is MF-ESMS but its companion discrete-time stochastic linear system is not ESMS. This contradicts Theorem 2.1 from [13] where is asserted that the MF-ESMS property of a MFSLS is equivalent to the ESMS property of its companion discrete-time standard stochastic linear system.*

Let us consider the analogous for the MFSLS (32) of the operator $\mathcal{L}$ *defined in (24) from [15]. This is* $\mathcal{L}\mathbf{X} = \begin{pmatrix} (a_0 + \overline{a}_0)^2 \overline{x} & 0 \\ 0 & \sum_{k=0}^{r} a_k^2 x \end{pmatrix}$ *for all* $\mathbf{X} = \begin{pmatrix} \overline{x} & 0 \\ 0 & x \end{pmatrix}$. *One sees that* $\lambda = \sum_{k=0}^{r} a_k^2$ *is the eigenvalue of the operator* $\mathcal{L}$ *corresponding to the eigenvector* $\mathbf{X} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Bearing in mind the constraint c_3) *we deduce that* $\lambda \geq 1$. *Hence, the spectral radius of the operator* $\mathcal{L}$ *satisfies* $\rho(\mathcal{L}) \geq 1$. *So, the MFSLS (32) is MF-ESMS, but the spectral radius of the Lyapunov-type operator defined by its companion standard discrete-time stochastic linear system is not strictly less than 1. This shows that the equivalence stated in Theorem 3.1 from [15] is not always true.*

In the next definition, we introduce a new kind of mean-field exponential stability in mean-square.

Definition 4. *We say that the MFSLS (10) is mean-field strong exponentially stable in mean-square (MF-SESMS) if its companion discrete-time SLS (12) is ESMS in the sense of Definition 3.*

Remark 5. *a) From Remark 4 and Definition 4, we may infer that the MF-SESMS property of the MFSLS (10) implies its MF-ESMS property. The Example 1 and Definition 4 shows that the converse implication is not always true.*

b) Bearing in mind that for the characterization of the ESMS property of a discrete-time stochastic linear system there exist a lot of spectral criteria and Lyapunov-type criteria, we may infer via a) that these criteria provide a set of useful sufficient conditions for characterization of the MF-ESMS property of a MFSLS.

Employing the special structure of the matrix coefficients of the DTSLS described in (7), we shall derive in the next section, a set of necessary and

sufficient conditions for the MF-SESMS property of the MFSLS (10) of lower dimension than those obtained applying directly the already existing results from the literature (see [6], [17]).

4 Criteria for MF-SESMS

The main result of this section is in the next theorem.

Theorem 1. *Under the assumption* **H1**) *the following are equivalent:*

(i) the MFSLS (10) is MF-SESMS;

(ii) the discrete-time stochastic linear equation (DTSLE):

$$x(t+1) = (A_0(t) + \sum_{k=1}^{r} A_k(t)w_k(t))x(t) \tag{37}$$

(obtained from (10a) taking $\overline{A}_k(t) = 0$, $t \in \mathbb{Z}$, $0 \leq k \leq r$*) is ESMS and the discrete-time deterministic linear equation (DTDLE):*

$$x(t+1) = (A_0(t) + \overline{A}_0(t))x(t), \quad t \in \mathbb{Z} \tag{38}$$

(obtained from (10) taking $A_k(t) = \overline{A}_k(t) = 0$, $t \in \mathbb{Z}$, $1 \leq k \leq r$*) is exponentially stable;*

(iii) there exist θ*-periodic matrix valued sequences* $\{x(t)\}_{t\in\mathbb{Z}} \subset \mathcal{S}_n^+$, $\{\overline{X}(t)\}_{t\in\mathbb{Z}} \subset \mathcal{S}_n^+$ *which are satisfying the backward time-varying linear matrix inequalities (LMIs):*

$$\begin{gathered} X(t) - \sum_{k=0}^{r} A_k^\top(t)X(t+1)A_k(t) > 0 \\ X(t) > 0, \end{gathered} \tag{39}$$

$t \in \mathbb{Z}$,

$$\begin{gathered} \overline{X}(t) - (A_0(t) + \overline{A}_0(t))^\top \overline{X}(t+1)(A_0(t) + \overline{A}_0(t)) > 0 \\ \overline{X}(t) > 0, \end{gathered} \tag{40}$$

$t \in \mathbb{Z}$;

(iv) there exist θ*-periodic matrix valued sequences* $\{Y(t)\}_{t\in\mathbb{Z}} \subset \mathcal{S}_n^+$, $\{\overline{Y}(t)\}_{t\in\mathbb{Z}} \subset \mathcal{S}_n^+$ *solving the following forward time-varying LMIs:*

$$\begin{gathered} Y(t+1) - \sum_{k=0}^{r} A_k(t)Y(t)A_k^\top(t) > 0 \\ Y(t) > 0, \end{gathered} \tag{41}$$

$t \in \mathbb{Z}$

$$\begin{gathered} \overline{Y}(t+1) - (A_0(t) + \overline{A}_0^{(}t))\overline{Y}(t)(A_0(t) + \overline{A}_0(t))^\top > 0 \\ \overline{Y}(t) > 0, \end{gathered} \tag{42}$$

$t \in \mathbb{Z}$.

Proof. Let us assume that the MFSLS (10) is MF-SESMS. This means that the solutions $\begin{pmatrix} x^1(\cdot; t_0, \mathbf{x}_0) \\ x^2(\cdot; t_o, \mathbf{x}_0) \end{pmatrix}$ of the IVP (12) have a decay rate of the form (31) for all initial time $t_0 \in \mathbb{Z}_+$ and initial state $\mathbf{x}_0 = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$ where $x_0^k \in \mathcal{X}_{t_0}$, $k = 1, 2$.

In the particular case $\mathbf{x}_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ with arbitrary $x_0 \in \mathbb{R}^n$ we have $\begin{pmatrix} x^1(\cdot; t_0, \mathbf{x}_0) \\ x^2(\cdot; t_0, \mathbf{x}_0) \end{pmatrix} = \begin{pmatrix} x^1(\cdot; t_0, x_0) \\ 0 \end{pmatrix}$ where $x^1(\cdot; t_0, x_0)$ is the solution of the DTSLE (37) corresponding to the initial pair $(t_0, x_0) \in \mathbb{Z}_+ \times \mathbb{R}^n$.
In this case, (31) gives:

$$\mathbb{E}[|x^1(t; t_0, x_0)|^2] \leq \beta_1 \delta_1^{t-t_0} |x_0|^2 \tag{43}$$

$\forall t \geq t_0 \in \mathbb{Z}_+$, $x_0 \in \mathbb{R}^n$, $\beta_1 \geq 1$, $\delta_1 \in (0, 1)$ are not depending upon t, t_0, x_0.
On the other hand, in the particular case when $\mathbf{x}_0 = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}$ with arbitrary $x_0 \in \mathbb{R}^n$, we have:
$\begin{pmatrix} x^1(\cdot; t_0, \mathbf{x}_0) \\ x^2(\cdot; t_0, \mathbf{x}_0) \end{pmatrix} = \begin{pmatrix} x^1(\cdot; t_0, \mathbf{x}_0) \\ x^2(\cdot; t_0, x_0) \end{pmatrix}$ where $x^2(\cdot; t_0, x_0)$ is the solution of the DTDLE (38) corresponding to the initial pair $(t_0, x_0) \in \mathbb{Z}_+ \times \mathbb{R}^n$.
In this case, (31) becomes:

$$\mathbb{E}[|x^1(t; t_0, \mathbf{x}_0)|^2] + \mathbb{E}[|x^2(t; t_0, x_0)|^2] \leq \beta_1 \delta_1^{t-t_0} |x_0|^2$$

for all $t \geq t_0 \in \mathbb{Z}_+$, $x_0 \in \mathbb{R}^n$. Hence,

$$\mathbb{E}[|x^2(t; t_0, x_0)|^2] \leq \beta_1 \delta_1^{t-t_0} |x_0|^2 \tag{44}$$

for all $t \geq t_0 \in \mathbb{Z}_+$, $x_0 \in \mathbb{R}^n$. Bearing in mind (43) and (44), we may infer that (ii) holds if (i) is true. Further, we show that the implication $(ii) \Rightarrow (i)$

holds. Based on (7) we rewrite (12a) as:

$$x^1(t+1) = A_0(t)x^1(t) + \sum_{k=1}^{r}(A_k(t)x^1(t) + f_k(t))w_k(t) \tag{45a}$$

$$x^2(t+1) = (A_0(t) + \overline{A}_0(t))x^2(t) \tag{45b}$$

$$x^1(t_0) = x_0^1, \quad x^2(t_0) = x_0^2, \tag{45c}$$

where $x_0^k \in \mathcal{X}_{t_0}$, $k = 1, 2$, and

$$f_k(t) \triangleq (A_k(t) + \overline{A}_k(t))x^2(t) \tag{46}$$

$t \in \mathbb{Z}_+$, $1 \leq k \leq r$. If (ii) holds, then, (43) and (44) hold too.
In this case, the solutions of (45b) satisfy

$$\mathbb{E}[|x^2(t; t_0, x_0^2)|^2] \leq \tilde{\beta}\tilde{\delta}^{t-t_0}\mathbb{E}[|x_0^2|^2] \tag{47}$$

for all $t \geq t_0 \in \mathbb{Z}_+$, $x_0^2 \in \mathcal{X}_{t_0}$.

Combining (46) and (47) we deduce that the forcing terms from (45a) are satisfying

$$\mathbb{E}[|f_k(t)|^2] \leq \gamma_k\tilde{\delta}^{t-t_0}\mathbb{E}[|x_0^2|^2] \tag{48}$$

for all $t \geq t_0 \in \mathbb{Z}_+$, where $\gamma_k = \tilde{\beta}\sup\limits_{t\in\mathbb{Z}}\|A_k(t) + \overline{A}_k(t)\|$ and $\tilde{\delta} \in (0, 1)$.
Bearing in mind (43), we may infer via Corollary 3.9 (iii) from [6] applied in the case of non-homogeneous DTSLE (45a) that its solutions are satisfying:

$$\lim_{t\to\infty}\mathbb{E}[|x^1(t; t_0, x_0^1)|^2] = 0 \tag{49}$$

for any initial time $t_0 \in \mathbb{Z}_+$ and the initial state $x_0^1 \in \mathcal{X}_{t_0}$. Employing (48) and (49) we may conclude that the solutions of the IVP (12) are satisfying

$$\lim_{t\to\infty}(\mathbb{E}[|x^1(t; t_0, \mathbf{x}_0)|^2] + \mathbb{E}[|x^2(t; t_0, \mathbf{x}_0)|^2] = 0$$

for all $t_0 \in \mathbb{Z}_+$ and any $\mathbf{x}_0 = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$ with $x_0^k \in \mathcal{X}_{t_0}$, $k = 1, 2$.

Since (12a) is a discrete-time linear equation with periodic coefficients, we may deduce via the equivalence $(i) \Leftrightarrow (v)$ from Theorem 3.10 in [6], applied in the case of the equation (12a), that its solutions are satisfying (31). According to Definition 4 we may conclude that the MFSLS (10) is MF-SESMS. Thus the validity of the implication $(ii) \Leftrightarrow (i)$ is confirmed. The equivalences $(ii) \Leftrightarrow (iii)$ and $(ii) \Leftrightarrow (iv)$ follow applying Theorem 3.11 from [6] in the case of the DTSLE (37) and DTDLE (38), respectively. Thus the proof is complete.

Remark 6. *The Lyapunov-type criteria (39)-(40) and (41)-(42), respectively, are based on the solutions of some time-varying LMIs with periodic coefficients with lower dimension than those obtained applying directly, for example, Theorem 3.11 from [6] in the case of the DTSLE (12a). Based on Remark 5 we may infer that (39)-(40) and (41)-(42) provide sets of sufficient conditions to test if a MFSLS is MF-ESMS.*

The next proposition shows that the conclusion of the Example 1 is still valid also in the time-varying context.

Proposition 3. *We assume: a) the assumption* **H1**) *holds;*

b) $A_k(t) + \overline{A}_k(t) = 0$, $t \in \mathbb{Z}$, $1 \leq k \leq r$;

c) the DTDLS (38) is exponentially stable, but the DTSLDE (37) is not ESMS.

Under these conditions, the MFSLS (10) is MF-ESMS but it is not MF-SESMS.

Proof. According to Proposition 1, without loss of generality, we may assume that $x_0 \in \mathbb{R}^n$ because $\mathbb{R}^n \subset \mathcal{X}_{t_0}$ for any $t_0 \in \mathbb{Z}_+$. Under the assumption *b*), the IVP (12) becomes:

$$\begin{gathered} x^1(t+1) = A_0(t)x^1(t) + \sum_{k=1}^{r} A_k(t)x^1(t)w_k(t), \quad t \geq t_0, \\ x^1(t_0) = 0, \end{gathered} \tag{50}$$

$$\begin{gathered} x^2(t+1) = (A_0(t) + \overline{A}_0(t))x^2(t), \quad t \geq t_0 \\ x^2(t_0) = x_0. \end{gathered} \tag{51}$$

Based on the unicity of the solution of an IVP, we may conclude that the unique solution of the IVP (50) is $x^1(t; t_0, x_0) = 0$ a.s. $t \geq t_0$ and the unique solution of the IVP (51) satisfies

$$\mathbb{E}[|x^2(t; t_0, x_0)|^2] \leq \beta\delta^{t-t_0}|x_0|^2. \tag{52}$$

In this case, (52) gives

$$\mathbb{E}[|x^1(t; t_0, x_0)|^2] + \mathbb{E}[|x^2(t; t_0, x_0)|^2] \leq \beta\delta^{t-t_0}|x_0|^2 \tag{53}$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$.

Since, $\mathbb{E}[|x(t; t_0, x_0)|^2] = \mathbb{E}[|x^1(t; t_0, x_0)|^2] + \mathbb{E}[|x^2(t; t_0, x_0)|^2]$ we obtain from (53) that

$$\mathbb{E}[|x(t; t_0, x_0)|^2] \leq \beta\delta^{t-t_0}|x_0|^2$$

for all $t \geq t_0 \in \mathbb{Z}_+$, $x_0 \in \mathbb{R}^n$, where $\beta \geq 1$, $\delta \in (0,1)$ are not depending upon t, t_0, x_0. Hence, the implication $(ii) \Rightarrow (i)$ from Proposition 2 allows us to conclude that under the considered assumptions the MFSLS (10) is MF-ESMS. It can not be MF-SESMS because the implication $(i) \Rightarrow (ii)$ from Theorem 1 would imply that the DTSLDE (37) would be ESMS, which contradicts the assumption c). Thus the proof ends.

5 The mean-fields stabilizability in mean-square

In this section we shall use the results derived in the previous sections to characterize the MF stabilizability in mean-square of the MFCSLS of the form (1).

Remark 7. *Based on Proposition 1 applied in the case of the closed-loop MFSLS (14), we may infer that, without loss of generality, in order to define the MF stabilizability in mean-square property of the MFCSLS (1) we may assume that its initial states x_0 are deterministic vectors.*

Taking the expectation in (14) we obtain:

$$\mathbb{E}[x(t+1)] = [A_0(t) + \overline{A}_0(t) + (B_0(t) + \overline{B}_0(t))\overline{F}(t)]\mathbb{E}[x(t)] \tag{54a}$$

$$\mathbb{E}[x(t_0)] = \mathbb{E}[x_0] \tag{54b}$$

for any initial time $t_0 \in \mathbb{R}^n$ and any initial state $x_0 \in \mathcal{X}_{t_0}$. To this end, we took into account that based of Lemma 1 applied in the case of the MFSLS (14), we may deduce that the random vectors $x(t)$ and $w(t)$ are mutually independent. Then, subtracting (54) from (14) we obtain:

$$\begin{aligned} &x(t+1) - \mathbb{E}[x(t+1)] = (A_0(t) + B_0(t)F(t))(x(t) - \mathbb{E}[x(t)]) + \\ &\sum_{k=1}^{r}[(A_k(t) + B_k(t)F(t))(x(t) - \mathbb{E}[x(t)]) + (A_k(t) + \overline{A}_k(t) + \\ &(B_k(t) + \overline{B}_k(t))\overline{F}(t))\mathbb{E}[x(t)]]w_k(t), \quad t \geq t_0 \end{aligned} \tag{55a}$$

$$x(t_0) - \mathbb{E}[x(t_0)] = x_0 - \mathbb{E}[x_0]. \tag{55b}$$

Employing the notation (4), we obtain from (54) and (55), the following

IVP:

$$\begin{pmatrix} x^1(t+1) \\ x^2(t+1) \end{pmatrix} = (\mathbb{A}_0(t) + \mathbb{B}_0(t)\mathbb{F}(t)) \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} + \sum_{k=1}^{r} (\mathbb{A}_k(t) + \mathbb{B}_k(t)\mathbb{F}(t)) \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} w_k(t), \quad t \geq t_0, \tag{56a}$$

$$\begin{pmatrix} x^1(t_0) \\ x^2(t_0) \end{pmatrix} = \begin{pmatrix} x_0 - \mathbb{E}[x_0] \\ \mathbb{E}[x_0] \end{pmatrix} \tag{56b}$$

for all $t_0 \in \mathbb{Z}_+$ and $x_0 \in \mathcal{X}_{t_0}$, where $\mathbb{A}_k(t), \mathbb{B}_k(t)$, $0 \leq k \leq r$ were defined in (7), (8) and

$$\mathbb{F}(t) = \begin{pmatrix} F(t) & 0 \\ 0 & \overline{\mathbb{F}}(t) \end{pmatrix} \tag{57}$$

$t \in \mathbb{Z}$. Applying Proposition 2 in the case of the pair formed by the IVP (14) and the IVP (56), we obtain via Definition 2, the following equivalence whose proof is omitted.

Corollary 1. *Under the assumption* **H1**) *the following are equivalent:*

(i) the MFCSLS (1) is MF stabilizable in mean-square;

(ii) there exist $\mathbb{R}^{m\times n}$ *valued* θ*-periodic matrix sequences* $\{F(t)\}_{t\in\mathbb{Z}}$, $\{\tilde{F}(t)\}_{t\in\mathbb{Z}}$ *with the properties that for each initial time* $t_0 \in \mathbb{Z}_+$ *and any initial state* $x_0 \in \mathcal{X}_{t_0}$, *the solutions* $\begin{pmatrix} x^1(\cdot;t_0,x_0) \\ x^2(\cdot;t_0,x_0) \end{pmatrix}$ *of the corresponding IVP (56) have a decay of the form:*

$$\mathbb{E}[|x^1(t;t_0,x_0)|^2] + \mathbb{E}[|x^2(t;t_0,x_0)|^2] \leq \beta^1 \hat{\delta}^{t-t_0} (\mathbb{E}[|x_0 - \mathbb{E}[x_0]|^2] + |\mathbb{E}[x_0]|^2) \tag{58}$$

for all $t \geq t_0$, *where* $\hat{\beta} \geq 1$, $\hat{\delta} \in (0,1)$ *are constants which are not depending upon* t, t_0, x_0.

In order to compare the mean-field stabilizability in mean-square property with a kind of stochastic stabilizability property of the DTCSLS (9) we introduce the following definition:

Definition 5. *We say that the DTCSLS (9) is* **stochastic stabilizable in mean-square by a decentralized linear state feedback** *if there exists a control of the form*

$$\mathbf{u}(t) = \mathbb{F}(t)\mathbf{x}(t) = \begin{pmatrix} F(t)x^1(t) & 0 \\ 0 & \overline{F}(t)x^2(t) \end{pmatrix} \tag{59}$$

with the properties: a) $\{F(t)\}_{t\in\mathbb{Z}}$, $\{\overline{F}(t)\}_{t\in\mathbb{Z}}$ *are* θ*-periodic* $\mathbb{R}^{m\times n}$ *matrix valued sequences; b) the closed-loop system*

$$\mathbf{x}(t+1)=(\mathbb{A}_0(t)+\mathbb{B}_0(t)\mathbb{F}(t))\mathbf{x}(t)+\sum_{k=1}^{r}(\mathbb{A}_k(t)+\mathbb{B}_k(t)\mathbb{F}(t))\mathbf{x}(t)w_k(t), \quad (60a)$$

$$\mathbf{x}(t_0)=\begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}\in\mathcal{X}_{t_0}\times\mathcal{X}_{t_0} \quad (60b)$$

(obtained plugging the control (59) to the DTCSLS (9)) is ESMS.

Remark 8. *For a given* $\mathbb{F}(t)$ *as in (57), the IVP (56) is the special case of the IVP (60) for* $x_0^1\leftarrow x_0-\mathbb{E}[x_0]$ *and* $x_0^2\leftarrow\mathbb{E}[x_0]$*. Hence, if the DTCSLS (9) is stochastic stabilizable in mean-square by a decentralized linear state feedback of the form (59), then the solutions of the IVP (56) obtained when in (6) is used the control (59), have a decay of the form (58).*

Corollary 2. *Under the assumption* **H1**) *the MFCSLS (1) is MF stabilizable in mean-square if its companion DTCSLS (9) is stochastic stabilizable in mean-square by a decentralized linear state feedback.*

Proof follows immediately combining Remark 8 with the implication $(ii)\Rightarrow(i)$ from Corollary 1.

The next example shows that the converse implication of the one from Corollary 2 is not always true.

Example 2. *We consider the 1-dimensional discrete-time time-invariant MFCSLS:*

$$\begin{aligned} x(t+1)&=a_0x(t)+\overline{a}_0\mathbb{E}[x(t)]+b_0u(t)+\overline{b}_0\mathbb{E}[u(t)]+\\ &\quad+(a_1x(t)+\overline{a}_1\mathbb{E}[x(t)]+b_1u(t)+\overline{b}_1\mathbb{E}[u(t)])w_1(t), t\geq t_0 \quad (61a)\\ x(t_0)&=x_0\in\mathbb{R} \quad (61b)\end{aligned}$$

where $a_k,\overline{a}_k,b_k,\overline{b}_k$ *are real numbers subject to the constraints:*

c_1) $a_1+\overline{a}_1=b_1+\overline{b}_1=0$, $b_0+\overline{b}_0\neq 0$;

c_2) $a_0b_0+a_1b_1=0$;

c_3) $a_0^2+a_1^2>1$.

In (61), $\{w_1(t)\}_{t\geq 0}$ *is a sequence of independent random variables with zero mean and variance 1. We show that under these conditions, the MFCSLS (61) is MF stabilizable in mean-square but its companion DTCSLS if type (9) is not stochastic stabilizable in mean-square by a decentralized linear*

state feedback. Based on Remark 7, we may infer that, without loss of generality, we can assume that the initial state x_0 is deterministic. Let $\delta \in (0,1)$ fixed and $\overline{f}$ defined from:

$$a_0 + \overline{a}_0 + (b_0 + \overline{b}_0)\overline{f} = \delta. \tag{62}$$

Let $f \in \mathbb{R}$ be arbitrary but fixed. We show that the closed-loop system obtained by coupling the control

$$u(t) = fx(t) + (\overline{f} - f)\mathbb{E}[x(t)] \tag{63}$$

$t \geq t_0$, to the MFCSLS (61) is MF-ESMS, which means that the MFCSLS (61) is MF stabilizable in mean-square.
The IVP of the form (56) obtained by coupling the control (63) to the MFCSLS (61) subject to the constraints c_1) is

$$\begin{aligned} x^1(t+1) &= (a_0 + b_0 f)x^1(t) + (a_1 + b_1 f)x^1(t)w_1(t), \quad t \geq t_0 \\ x^1(t_0) &= 0, \end{aligned} \tag{64}$$

$$\begin{aligned} x^2(t+1) &= (a_0 + \overline{a}_0 + (b_0 + \overline{b}_0)\overline{f})x^2(t), \quad t \geq t_0 \\ x^2(t_0) &= x_0. \end{aligned} \tag{65}$$

From the unicity of the solution of an IVP, we have that the unique solution of (64) is $x^1(t) = 0$ a.s. $t \geq t_0 \geq 0$. From (62) and (65) we obtain that

$$x^2(t) = \delta^{t-t_0}x_0, \quad t \geq t_0 \geq 0. \tag{66}$$

Hence, (66) gives

$$\mathbb{E}[|x^1(t;t_0,x_0)|^2] + \mathbb{E}[|x^2(t;t_0,x_0)|^2] = \delta^{t-t_0}|x_0|^2,$$

$t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}$. Bearing in mind that $\delta \in (0,1)$ we may conclude via $(ii) \Rightarrow (i)$ from Corollary 1 applied in the case of the MFCSLS (61) that this system is MF stabilizable in mean-square. Further, we show that the corresponding companion of the form (9) associated to (61) is not stochastic stabilizable in mean-square by a decentralized linear state feedback, in the sense of the Definition 5.

Let

$$\begin{pmatrix} u^1(t) \\ u^2(t) \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & \overline{f} \end{pmatrix} \begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} \tag{67}$$

be an arbitrary control of the form (59) where $f, \overline{f}$ are unknown real numbers which should be determined such that the corresponding closed-loop system to be ESMS.

The IVP of the form (60) associated to the MFCSLS (61) and to the control (67) is:

$$\begin{aligned} x^1(t+1) &= (a_0 + b_0 f)x^1(t) + (a_1 + b_1 f)x^1(t)w_1(t), \quad t \geq t_0 \\ x^1(t_0) &= x_0^1, \end{aligned} \tag{68}$$

$$\begin{aligned} x^2(t+1) &= (a_0 + \overline{a}_0 + (b_0 + \overline{b}_0)\overline{f})x^2(t), \quad t \geq t_0 \\ x^2(t_0) &= x_0^2. \end{aligned} \tag{69}$$

According to Definition 5, the DTCSLS of type (2.9) associated to the MFCSLS (61) is stochastic stabilizable in mean-square by a decentralized linear state feedback if the closed-loop DTCSLSs (68), (69) are simultaneously ESMS. Employing a Lyapunov type criteria we may infer that the DTCSLS (68) is ESMS if and only if

$$(a_0 + b_0 f)^2 + (a_1 + b_1 f)^2 - 1 < 0. \tag{70}$$

Bearing in mind the constraints c_2) we deduce that (70) is equivalent to

$$a_0^2 + a_1^2 - 1 + (b_0^2 + b_1^2)f^2 < 0. \tag{71}$$

Employing c_3) we may conclude that does not exist $f \in \mathbb{R}$ for which (71) to be satisfied. Thus, we have shown that the companion DTCSLS of the form (9) associated to the MFCSLS (61) is not stochastic stabilizable in mean-square by a decentralized linear state feedback.

Motivated by the implication stated in Corollary 2, we introduce a new kind of stabilizability in mean-square of the MFCSLS (1).

Definition 6. *We say that the MFCSLS (1) is MF strong stabilizable in mean-square if its companion DTCSLS (9) is stochastic stabilizable in mean-square by a decentralized linear state feedback.*

Remark 9. *From Corollary 2 and Definition 6, we may infer that the MF strong stabilizability in mean-square of the MFCSLS (1) implies its MF stabilizability in mean-square. From Example 2 and Definition 6 it follows that the converse implication is not always true. Hence, the necessary and sufficient conditions for MF strong stabilizability in mean-square of the MFCSLS (54) serves as sufficient conditions for the MF stabilizability in mean-square of this system.*

Theorem 2. *Under the assumption* **H1**), *the following are equivalent: (i) the MFCSLS (1) is MF strong stabilizable in mean-square;*

(ii) the discrete-time controlled stochastic linear system (DTCSLS):

$$x(t+1) = A_0(t)x(t) + B_0(t)u(t) + \sum_{k=1}^{r}(A_k(t)x(t) + B_k(t)u(t))w_k(t), \quad t \geq 0 \tag{72}$$

(obtained from (1) ignoring the mean-field part) is stochastic stabilizable in mean-square and the discrete-time controlled deterministic linear system (DTCDLS):

$$x(t+1) = (A_0(t) + \overline{A}_0(t))x(t) + (B_0(t) + \overline{B}_0(t))u(t) \tag{73}$$

(obtained from (1) taking $A_k(t) = \overline{A}_k(t) = 0$, $B_k(t) = \overline{B}_k(t) = 0$, $1 \leq k \leq r$, $t \in \mathbb{Z}$*) is stabilizable by linear state feedback.*

Proof. To prove the implication $(i) \Rightarrow (ii)$, let us remark that if (i) holds, there exist a control law of the form (59) with the property that the solutions $\begin{pmatrix} x^1(\cdot;t_0,\mathbf{x}_0) \\ x^2(\cdot;t_0,\mathbf{x}_0) \end{pmatrix}$ of the corresponding closed-loop system (60) have a decay of the form:

$$\mathbb{E}[|x^1(t;t_0,\mathbf{x}_0)|^2] + \mathbb{E}[|x^2(t;t_0,\mathbf{x}_0)|^2] \leq \beta\delta^{t-t_0}(\mathbb{E}[|x_0^1|^2] + \mathbb{E}[|x_0^2|^2]) \tag{74}$$

for all $t \geq t_0 \geq 0$, $\mathbf{x}_0 = \begin{pmatrix} (x_0^1)^\top & (x_0^2)^\top \end{pmatrix}^\top \in \mathcal{X}_{t_0} \times \mathcal{X}_{t_0}$, where $\beta \geq 1$, $\delta \in (0,1)$ are constants not depending upon t, t_0, x_0^k. Based on the unicity of the solution of an IVP we may infer that if $\mathbf{x}_0 \leftarrow \hat{\mathbf{x}}_0 = \begin{pmatrix} x_0^\top & 0^\top \end{pmatrix}^\top$ where $x_0 \in \mathbb{R}^n$, we have $\begin{pmatrix} x^1(t;t_0,\mathbf{x}_0) \\ x^2(t;t_0,\mathbf{x}_0 \end{pmatrix} = \begin{pmatrix} x^1(t;t_0,x_0) \\ 0 \end{pmatrix}$ where $t \to x^1(t;t_0,x_0)$ is the solution of the closed-loop DTSLS:

$$x^1(t+1) = (A_0(t)+B_0(t)F(t))x^1(t)+\sum_{k=1}^{r}(A_k(t)+B_k(t)F(t))x^1(t)w_k(t) \tag{75}$$

corresponding to the initial pair $(t_0, x_0) \in \mathbb{Z}_+ \times \mathbb{R}^n$. In this case (74) becomes:

$$\mathbb{E}[|x^1(t;t_0,x_0)|^2] \leq \beta\delta^{t-t_0}|x_0|^2.$$

This means that the DTSLS (75) is ESMS, or equivalently, the DTCSLS (72) is stochastic stabilizable in the sense of Definition **??** from [6]. On the other hand, if $\mathbf{x}_0 \leftarrow \check{\mathbf{x}}_0 = \begin{pmatrix} 0^\top & x_0^\top \end{pmatrix}^\top$ where $x_0 \in \mathbb{R}^n$, we have that

$\begin{pmatrix} x^1(t;t_0,\mathbf{x}_0) \\ x^2(t;t_0,\mathbf{x}_0) \end{pmatrix} = \begin{pmatrix} x^1(t;t_0,\check{\mathbf{x}}_0) \\ x^2(t;t_0,x_0) \end{pmatrix}$ where $t \to x^2(t;t_0,x_0)$ is the solution of the closed-loop deterministic linear equation:

$$x^2(t+1) = [A_0(t) + \overline{A}_0(t) + (B_0(t) + \overline{B}_0(t))\overline{F}(t)]x^2(t) \tag{76}$$

$t \geq t_0$ corresponding to the initial pair $(t_0, x_0) \in \mathbb{Z}_+ \times \mathbb{R}^n$. Employing (74) we obtain that in this case we have:

$$|x(t;t_0,x_0)|^2 \leq \beta\delta^{t-t_0}|x_0|^2$$

for all $t \geq t_0 \geq 0$, $x_0 \in \mathbb{R}^n$. This means that the discrete-time linear equation (76) is exponentially stable, or equivalent, the DTCDLE (73) is stabilizable. Thus we have shown that (ii) holds if (i) is true. The implication $(ii) \Rightarrow (i)$ may be proved in a similar way to the proof of the implication $(ii) \Rightarrow (i)$ from Theorem 1 replacing the DTSLS (37) by the DTSLS (75) and the DTDLE (38) by the closed-loop discrete-time equation (76). The details are omitted. Thus the proof ends.

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