Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 18, No. 4/2026

THE DYNAMIC STRING-AVERAGING ALGORITHM FOR INVERSE STRONGLY-MONOTONE MAPPINGS*

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DOI 10.56082/annalsarscimath.2026.4.71

Abstract

In the work by W. Takahashi and M. Toyoda (2003) it was established the weak convergence of an iteration process of solving a variational inequality problem. This variational inequality is associated with an inverse strongly-monotone mapping. In our recent research we showed that most of exact iterates of the same iterative process are approximate solutions of the variational inequality. In this paper we apply the dynamic string-averaging algorithm in order to find a common solution of a finite family of variational inequality problems, generated by inverse strongly-monotone mappings, and a finite family of fixed point problems.

Keywords: Hilbert space, iteration, nonexpansive mapping, variational inequality.

MSC: 47H05, 47H14.

1 Introduction

W. Takahashi and M. Toyoda [11] introduced and analyzed a method which finds a fixed point of a nonexpansive mapping which is also a solution of a

*Accepted for publication on July 02, 2026

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variational inequality problem generated by an inverse strongly-monotone map. The authors of [11] established the weak convergence of their algorithm. It should be mentioned that in practice only a finite number of iterates can be done. Thus it is important to know that the method generates approximate solutions and the number of iterations, which should be done needed in order to reach such approximate solutions. This goal was achieved in [12]. We showed in that work that for the method of [11] most of its exact iterates are approximate common solutions of a variational inequality associated with an inverse strongly-monotone operator. In the present paper we study, in the setting of a Hilbert space, the behavior of the sequences generated by the dynamic string-averaging algorithm for finding a common solution of a finite family of variational inequality problems and a finite family of fixed point problems. We show that most exact iterates of our algorithm are approximate common solutions of the family of variational inequalities and approximate common fixed points for the family of mappings.

Assume that $(H, \langle \cdot, \cdot \rangle)$ is a real Hilbert space equipped with an inner product which induces the complete Euclidean norm

$$\|x\| = \langle x, x \rangle^{1/2}, \ x \in H,$$

$U \subset H$ is a nonempty, convex and closed. For each $z \in H$ and each $r > 0$ set

$$B(z, r) = \{u \in H : \ \|u - z\| \leq r\}.$$

Set $I(\xi) = \xi$, $\xi \in H$. For each set $D \subset H$ and each $x \in H$ put

$$d(x, D) = \inf\{\|x - y\| : \ y \in D\}.$$

In the sequel we use the following auxiliary result [1,6–8].

Lemma 1. *Suppose that $D \subset H$ is a nonempty, convex and closed set. Then for each $\xi \in H$ there exists a unique element $P_D(\xi) \in D$ satisfying*

$$\|\xi - P_D(\xi)\| = \inf\{\|\xi - \eta\| : \ \eta \in D\}$$

and the following assertions are valid true.

1.

$$\|P_D(x_1) - P_D(x_1)\| \leq \|x_1 - x_2\| \text{ for each pair of points } x_1, x_2 \in H$$

and for any $\xi \in H$ and every $z \in D$,

$$\langle z - P_D(\xi), \xi - P_D(\xi) \rangle \leq 0,$$

$$\|z - P_D(\xi)\|^2 + \|\xi - P_D(\xi)\|^2 \le \|z - \xi\|^2.$$

2. Let $x \in H$, $\xi \in D$ and let for any element $z \in D$, we have

$$\langle z - \xi, x - \xi \rangle \le 0.$$

Then $\xi = P_D(x)$.

3. For each $\xi, \eta \in H$,

$$\|P_D(\xi) - P_D(\eta)\|^2 + \|(I - P_D)(\xi) - (I - P_D)(\eta)\|^2 \le \|\xi - \eta\|^2,$$

$$\|P_D(\xi) - P_D(\eta)\|^2 \le \langle P_D(\xi) - P_D(\eta), \xi - \eta \rangle.$$

The operator $P_D : H \to D$ is the metric projection on the convex set D. For each set $D \subset H$ and $x \in X$, set

$$d(x, D) = \inf\{\|x - y\| : \ y \in D\}.$$

Let $\mathcal{L}_1$ be a finite set of pairs (F, C) where C is a nonempty closed convex subset of U, and $F : U \to H$ and $\mathcal{L}_2$ be a finite set of mappings $T : U \to U$. We suppose that the set $\mathcal{L}_1 \cup \mathcal{L}_2$ is nonempty. (Note that one of the sets $\mathcal{L}_1$ or $\mathcal{L}_2$ may be empty.)

We suppose that for each $(F, C) \in \mathcal{L}_1$ each $u, v \in U$,

$$\langle F(u) - F(v), u - v \rangle \ge \alpha \|F(u) - F(v)\|^2 \tag{1}$$

and set for each $\epsilon \ge 0$,

$$S(F, C) = \{z \in C : \ \langle F(z), u - z \rangle \ge 0, \ u \in C\}, \tag{2}$$

$$S_\epsilon(F, C) = \{z \in U, \ B(z, \epsilon) \cap C \ne \emptyset,$$
$$\langle F(z), \xi - z \rangle \ge -\epsilon \|\xi - z\| - \epsilon \text{ for all } \xi \in C\}. \tag{3}$$

If $(F, K) \in \mathcal{L}_1$, then the operator F is called α-inverse strongly monotone [3, 5, 9, 11]. Some examples of inverse strongly-monotone mappings were considered in [11]. In particular, if K is a closed, convex subset of U and an operator $T : K \to K$ is nonexpansive, then the operator $A = I - T$ is $(1/2)$ inverse strongly monotone and $S(F, K)$ is the collection of fixed points of T.

We suppose that for every mapping $T \in \mathcal{L}_2$ the set

$$\text{Fix}(T) := \{z \in U : \ T(z) = z\} \ne \emptyset,$$

and

$$\|z - x\|^2 \ge \|z - T(x)\|^2 + \bar{c}\|x - T(x)\|^2$$

$$\text{for all } x \in U \text{ and all } z \in \text{Fix}(T). \tag{4}$$

Fix

$$0 < a \leq b < 2\alpha. \tag{5}$$

Set

$$l = \text{Card}(\mathcal{L}_1 \cup \mathcal{L}_1). \tag{6}$$

For every $T \in \mathcal{L}_2$ and every $\epsilon \geq 0$ set

$$\text{Fix}_\epsilon(T) := \{z \in U : \ \|T(z) - z\| \leq \epsilon\} \tag{7}$$

For each $\lambda \in [a, b]$ and each $(F, C) \in \mathcal{L}_1$ set

$$P_{F,C,\lambda} = P_C(I - \lambda F). \tag{8}$$

where $I : H \to H$ is the identity operator. Set

$$S := [\cap_{(F,C)\in\mathcal{L}_1} S(F, C)] \cap [\cap_{T\in\mathcal{L}_2} \text{Fix}(T)]. \tag{9}$$

We apply a dynamic string-averaging method with variable strings and weights in order to obtain a good approximative solution of the common fixed point problem. It should be mentioned that string-averaging algorithms were first introduced by Y. Censor, T. Elfving and G. T. Herman in [4] for solving a convex feasibility problem, when a given collection of sets is divided into blocks and the algorithms operate in such a manner that all the blocks are processed in parallel [2, 10].

Next we describe the dynamic string-averaging method with variable strings and weights.

By an mapping vector, we a mean a vector $T = (T_1, \ldots, T_p)$ such that for all $i = 1, \ldots, p$,

$$T_i \in \mathcal{L}_2 \cup \{P_{F,C,\lambda} : \ (F, C) \in \mathcal{L}_1, \ \lambda \in [a, b]\}. \tag{10}$$

For an mapping vector $T = (T_1, \ldots, T_q)$ set

$$p(T) = q. \tag{11}$$

Denote by $\mathcal{M}$ the collection of all pairs (Ω, w), where Ω is a finite set of mapping vectors and

$$w : \Omega \to (0, \infty) \text{ satisfies } \sum_{T\in\Omega} w(T) = 1. \tag{12}$$

The dynamic string-averaging method with variable strings and variable weights can now be described by the following algorithm.

Initialization: select an arbitrary point $x_0 \in C$.

Iterative step: given a current iteration vector x_k pick

$$(\Omega_{k+1}, w_{k+1}) \in \mathcal{M}$$

calculate the next iteration vector x_{k+1} by

$$x_{k+1} = \sum_{T=(T_1,\dots,T_{p(T)})\in\Omega_{k+1}} w_{k+1}(T)T_{p(T)}\cdots T_1(x_k).$$

Fix

$$\widehat{\Delta} \in (0, l^{-1}), \tag{13}$$

a natural number $\bar{N}$ and an integer

$$\bar{q} \geq l. \tag{14}$$

Denote by $\mathcal{M}_*$ the set of all $(\Omega, w) \in \mathcal{M}$ such that

$$p(T) \leq \bar{q} \text{ for all } T \in \Omega,$$

$$w(T) \geq \widehat{\Delta} \text{ for all } T \in \Omega. \tag{15}$$

Denote by $\mathcal{R}$ the set of all sequences $\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathcal{M}_*$ such that the following properties hold:

(Q1) for every integer $j \geq 1$ and every mapping $S \in \mathcal{L}_2$ there exist an integer $k \in \{j, \dots, j + \bar{N} - 1\}$ and $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$ such that

$$S \in \{T_1, \dots, T_{p(T)}\};$$

(Q2) for each integer $j \geq 0$ and every pair $(FC) \in \mathcal{L}_1$ there exist $k \in \{j, \dots, j + \bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_k$, $\lambda \in [a, b]$ such that $P_{F,C,\lambda} \in \{T_1, \dots, T_{p(T)}\}$.

We use the following definitions which are needed in order to study our algorithm.

Let $x \in U$ and let $T = (T_1, \dots, Tt_{p(T)})$ be an mapping vector. Define

$$A_0(x, T) = \{(y, \lambda) \in U \times R^1 :$$

$$\text{there exist a sequence } \{y_i\}_{i=0}^{p(T)} \subset U \text{ such that}$$

$$y_0 = x$$

and for all $i \in \{1, \ldots, p(T)\}$, if $T_i \in \mathcal{L}_2$, then

$$y_i = T_i(y_{i-1})$$

and if $T_i = P_{F,C,\lambda})$, where

$$(F, C) \in \mathcal{L}_1,\ \lambda \in [a, b],$$

then

$$y_i = P_C(y_{i-1}),\ y = y_{p(T)},$$
$$\lambda = \max\{\|y_i - y_{i-1}\| :\ i = 1, \ldots, p(T)\}\}. \tag{16}$$

Let $x \in U$ and let $(\Omega, w) \in \mathcal{M}$. Define

$$A(x, (\Omega, w)) = \{(y, \lambda) \in U \times R^1 :\ \text{there exist}$$

$$(y_T, \lambda_T) \in A_0(x, T),\ T \in \Omega \text{ such that}$$
$$y = \sum_{T \in \Omega} w(T) y_T,\ \lambda = \max\{\lambda_T :\ T \in \Omega\}\}. \tag{17}$$

Our goal is, for a given $\epsilon > 0$, to find a point x which is an ϵ-approximate common solution of the problems associated with the family of operators $\mathcal{L}_1 \cup \mathcal{L}_2$. We study inexact iterates of this algorithm with summable and nonsummable errors.

2 Auxiliary results

In the sequel we use the following results of [12].

Proposition 1. *Let $(A, C) \in \mathcal{L}_1$, $\tau > 0$, $u \in K$. Then $u \in S(A, C)$ if and only if*

$$P_{A,C,\tau}(u) = P_C(u - \tau A(u)) = u.$$

Proposition 2. *Let $(A, C) \in \mathcal{L}_1$, $u_1, u_2 \in U$ and $\tau > 0$. Then*

$$\|(I - \tau A)(u_1) - (I - \tau A)(u_2)\|^2 \le \|u_1 - u_2\|^2 + \tau(\tau - 2\alpha)\|A(u_1) - A(u_2)\|^2.$$

If $\tau \le 2\alpha$, then $I - \lambda A$ is a nonexpansive operator.

Lemma 2. *Assume that $(A, C) \in \mathcal{L}_1$,*

$$x \in U,\ \tau \in [a, b],\ u \in K,$$

$$u = P_C(u - \tau A(u)),\ y = P_C(x - \tau A(x)).$$

Then

$$\|y-u\|^2 \le \|x-u\|^2 + \tau(\tau-2\alpha)\|A(u)-A(x)\|^2,$$
$$\|y-u\|^2 \le \|x-u\|^2 - \|y-x+\tau(A(x)-A(u))\|^2$$

and

$$\|y-u\|^2 \le \|x-u\|^2 - 4^{-1}\|y-x\|^2 \min\{1, b^{-2}a(2\alpha-b)\}.$$

Lemma 3. *Let* $(F,K) \in \mathcal{L}_1$, $\epsilon_0 > 0$, $x \in U$, $\lambda \in [a,b]$ *and let*

$$\|x - P_K(x-\lambda F(x))\| \le \epsilon_0.$$

Then for every element $\eta \in K$, *we have*

$$\langle Fx, \eta - x\rangle \ge -\lambda^{-1}\epsilon_0\|\eta - x\| - \lambda^{-1}\epsilon_0^2 - \|F(x)\|\epsilon_0.$$

3 The main result

We prove the following result which shows that most exact iterates of our algorithm are approximate common solutions of the family of variational inequalities and approximate common fixed points for the family of mappings $\mathcal{L}_2$.

Theorem 1. *Let* $\epsilon \in (0,1]$, $M_0 \ge 1$ *be such that*

$$\{u \in S \cap B(0,M_0) :\ F(u) \in B(0,M_0),\ (F,C) \in \mathcal{L}_1\} \ne \emptyset,$$

$\epsilon_0 > 0$ *satisfy*

$$\epsilon_0 \le \min\{4^{-1}a\epsilon,\ 4^{-1}(M_0+1)^{-1}(1+4\alpha^{-1})^{-1}, \epsilon\bar{q}^{-1}(\bar{N}+1)^{-1}\}.$$

Assume that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \in \mathcal{R},$$

$\{x_i\}_{i=1}^{\infty} \subset U$, $\{\lambda_i\}_{i=1}^{\infty} \subset [0,\infty)$,

$$x_0 \in B(0,M_0)$$

and that for each integer $i \ge 1$,

$$(x_i, \lambda_i) \in A(x_{i-1}, (\Omega_i, w_i), 0).$$

Then

$$\|x_i\| \le 3M_0,\ i = 0,1,\ldots$$

and

$$Card(\{k \in \{0,1,\dots\} : \quad \text{there is } i \in \{k,\dots,\bar{N}-1\} \text{ such that } \lambda_{i+1} > \epsilon_0\})$$
$$\leq 16M_0^2\bar{N}\widehat{\Delta}^{-1}\epsilon_0^{-2}\min\{\bar{c},\ b^{-2}a(2\alpha-b)\}^{-1}.$$

Moreover, if an integer $q \geq 0$ *and*

$$\lambda_{i+1} \leq \gamma_0,\ i = q,\dots,q+\bar{N}-1,$$

then for each $k \in \{q,\dots,q+\bar{N}\}$*, for each each* $T \in \mathcal{L}_2$ *and each* $(F,C) \in \mathcal{L}_1$*,*

$$d(x_k \in Fix_\epsilon(T)) \leq \epsilon,$$

$$d(x_i, S_\epsilon(F,C)) \leq \epsilon.$$

Proof. In view of our assumptions, there exists

$$u_* \in B(0,M_0) \cap S \tag{18}$$

such that

$$\|F(u_*)\| \leq M_0,\ (F,C) \in \mathcal{L}_1. \tag{19}$$

By our assumptions and (18),

$$\|x_0 - u_*\| \leq 2M_0. \tag{20}$$

Let $k \geq 0$ be an integer. We have

$$(x_{k+1},\lambda_{k+1}) \in A(x_k,(\Omega_{k+1},w_{k+1}),0). \tag{21}$$

Equations (17) and (21) imply that there exist

$$(y_{k,T},\lambda_{k,T}) \in A_0(x_k,T),\ T \in \Omega_{k+1} \tag{22}$$

such that

$$x_{k+1} = \sum_{T\in\Omega_{k+1}} w_{k+1}(T)y_{k,T}, \tag{23}$$

$$\lambda_{k+1} = \max\{\lambda_{k,T} :\ T \in \Omega_{k+1}\}. \tag{24}$$

Let

$$T = (T_1,\dots,T_{p(T)} \in \Omega_{k+1}.$$

By (16) and (28), there exists a sequence $\{y_i^{(k,T)}\}_{i=0}^{p(T)} \subset U$ such that

$$y_0^{(k,T)} = x_k,\ y_{k,T} = y_{p(T)}^{(k,T)} \tag{25}$$

and for each $i \in \{1, \dots, p(T)\}$,

$$\text{if } T_i \in \mathcal{L}_2, \text{ then } y_i^{(k,T)} = T_i(y_{i-1}^{(k,T)}), \tag{26}$$

and if $T_i = P_{F,C,\tau}$, where

$$(F, C) \in \mathcal{L}_1, \ \tau \in [a, b],$$

then

$$y_i^{(k,T)} = P_C(y_{i-1}^{(k,T)} - \tau F(y_{i-1}^{(k,T)})), \tag{27}$$

$$\lambda_{k,T} = \max\{\|y_i^{(k,T)} - y_{i-1}^{(k,T)}\| : \ i = 1, \dots, p(T)\}. \tag{28}$$

Let $k \geq 0$ be an integer, $T \in \Omega_{k+1}$, $i \in \{1, \dots, p(T)\}$ and

$$\|y_{i-1}^{(k,T)} - u_*\| \leq 2M_0. \tag{29}$$

Assume that $T_i \in \mathcal{L}_2$. By (4), (18) and (26),

$$\|u_* - y_{i-1}^{(k,T)}\|^2 \geq \|u_* - T_i(y_{i-1}^{(k,T)})\|^2 + \bar{c}\|T_i(y_{i-1}^{(k,T)}) - y_{i-1}^{(k,T)}\|^2$$

$$= \|u_* - y_i^{(k,T)}\|^2 + \bar{c}\|y_i^{(k,T)} - y_{i-1}^{(k,T)}\|^2. \tag{30}$$

Assume that $T_i = P_{F,C,\tau}$, where

$$(F, C) \in \mathcal{L}_1, \ \tau \in [a, b].$$

In view of Proposition 2 and (18), (27),

$$\|y_i^{(k,T)} - u_*\| = \|P_{F,C,\tau}(y_i^{(k,T)}) - u_*\| \leq \|y_i^{(k,T)} - u_*\|. \tag{31}$$

It follows from (18), (27), Proposition 1 and Lemma 2 applied with

$$(A, K) = (F, C), \ x = y_{i-1}^{(k,T)}, \ y = y_i^{(k,T)}, \ u = u_*, \ \lambda = \tau$$

that

$$\|y_i^{(k,T)} - u_*\|^2 \leq \|y_{i-1}^{(k,T)} - u_*\|^2$$

$$-4^{-1}\|y_{i-1}^{(k,T)} - y_i^{(k,T)}\|^2 \min\{1, \ b^{-2}a(2\alpha - b)\}. \tag{32}$$

In both cases by (30) and (32),

$$\|y_{i-1}^{(k,T)} - u_*\|^2 \geq \|y_i^{(k,T)} - u_*\|^2$$

$$+\|y_{i-1}^{(k,T)} - y_i^{(k,T)}\|^2 \min\{\bar{c}, \ 4^{-1}, \ 4^{-1}b^{-2}a(2\alpha - b)\}. \tag{33}$$

Thus we showed that the following property holds:
(a) If $k \geq 0$ is an integer, $T \in \Omega_{k+1}$, $i \in \{1, \ldots, p(T)\}$ and

$$\|y_{i-1}^{(k,T)} - u_*\| \leq 2M_0,$$

then (33) is true.

Applying property (a) by induction we conclude that the following property holds:
(b) If $k \geq 0$ is an integer, $T \in \Omega_{k+1}$ and

$$\|x_k - u_*\| \leq 2M_0,$$

then for each $i \in \{1, \ldots, p(T)\}$,

$$\|y_i^{(k,T)} - u_*\| \leq 2M_0$$

and (33) is true.

Lemma 4. *Let $k \geq 0$ be an integer and*

$$\|x_k - u_*\| \leq 2M_0. \tag{34}$$

The for each $T \in \Omega_{k+1}$,

$$\|u_* - y_{k,T}\|^2 \leq \|u_* - x_k\|^2 - 4^{-1} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}\lambda_{k,T}^2,$$

$$\|u_* - x_{k+1}\|^2 \leq \|u_* - x_k\|^2 - 4^{-1}\widehat{\Delta} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}\lambda_{k+1}^2.$$

Proof. Let $T \in \Omega_{k+1}$. Properties (a), (b) and (34) imply that for each $i \in \{1, \ldots, p(T)\}$,

$$\|y_i^{(k,T)} - u_*\| \leq 2M_0$$

and (33) is true. It follows from (25), (28) and (33) that

$$\|u_* - x_k\|^2 - \|u_* - y_{k,T}\|^2 = \|u_* - y_0^{(k,T)}\|^2 - \|u_* - y_{p(T)}^{(k,T)}\|^2$$

$$= \sum_{i=1}^{p(T)} (\|u_* - y_{i-1}^{(k,T)}\|^2 - \|u_* - y_i^{(k,T)}\|^2)$$

$$\geq 4^{-1} \min\{\bar{c},\ b^{-2}a(2\alpha - b)\} \sum_{i=1}^{p(T)} \|y_{i-1}^{(k,T)} - v_i^{(k,T)}\|^2$$

$$\geq 4 \min\{\bar{c},\ b^{-2}a(2\alpha - b)\}\lambda_{k,T}^2. \tag{35}$$

By the convexity of the function $\|\cdot\|^2$, (12), (23), (24) and (35),

$$\|u_* - x_{k+1}\|^2 = \|u_* - \sum_{T\in\Omega_{k+1}} w_{k+1}(T)y_{k,T}\|^2$$

$$\leq \sum_{T\in\Omega_{k+1}} w_{k+1}(T)\|u_* - y_{k,T}\|^2$$

$$\leq \sum_{T\in\Omega_{k+1}} w_{k+1}(T)(\|u_* - x_k\|^2 - 4\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}\lambda_{k,T}^2)$$

$$\leq \|u_* - x_k\|^2 - 4^{-1}\widehat{\Delta}\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}\lambda_{k+1}^2.$$

Lemma 4 is proved. □

By (199), (20) and Lemma 4 applied by induction, we have

$$\|x_k - u_*\| \leq 2M_0,\ \|x_k\| \leq 3M_0,\ k = 0, 1, \dots. \tag{36}$$

Let n be a natural number, In view of (36) and Lemma 4,

$$4M_0^2 \geq \|x_0 - u_*\|^2 \geq \|x_0 - u_*\|^2 - \|x_n - u_*\|^2$$

$$= \sum_{i=0}^{n-1}(\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2)$$

$$\geq 4^{-1}\widehat{\Delta}\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}\sum_{i=0}^{n-1}\lambda_{i+1}^2$$

$$\geq 4^{-1}\widehat{\Delta}\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}\epsilon_0^2\mathrm{Card}(\{i \in \{0, \dots, n-1\} :\ \lambda_{i+1} \geq \epsilon_0\}).$$

Since the relation above holds for any natural number n we conclude that

$$\mathrm{Card}(\{i \in \{0, 1, \dots\} :\ \lambda_{i+1} \geq \epsilon_0\}) \leq 16M_0^2\widehat{\Delta}^{-1}\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1}\epsilon_0^{-2}. \tag{37}$$

Set

$$E = \{k \in \{0, 1, \dots\} :\ \text{ there is } i \in \{k, \dots, \bar{N} - 1\} \text{ such that } \lambda_{i+1} \geq \epsilon_0\}. \tag{38}$$

By (37) and (38),

$$\mathrm{Card}(E) \leq 16M_0^2\bar{N}\widehat{\Delta}^{-1}\min\{\bar{c},\ b^{-2}a(2\alpha - b)\}^{-1}\epsilon_0^{-2}.$$

Assume that $q \geq 0$ is an integer and

$$\lambda_{i+1} \leq \epsilon_0,\ i = q, \dots, q - 1 + \bar{N}. \tag{39}$$

Let $j \in \{q, \ldots, q + \bar{N} - 1\}$, $T = (T_1, \ldots, T_{p(T)}) \in \Omega_{j+1}$ and each $i \in \{1, \ldots, p(T)\}$. By (24), (28) and (29),

$$\lambda_{j,T} \le \epsilon_0, \ \|y_{i-1}^{(j,T)} - y_i^{(j,T)}\| \le \epsilon_0. \tag{40}$$

If $T_i \in \mathcal{L}_2$, then in view of (26) and (40),

$$\epsilon_0 \ge \|y_{i-1}^{(j,T)} - y_i^{(j,T)}\| = \|y_{i-1}^{(j,T)} - T_i(y_{i-1}^{(j,T)})\|,$$

$$y_{i-1}^{(j,T)} \in \mathrm{Fix}_\epsilon(T_i). \tag{41}$$

Assume that

$$\tau \in [a, b] \ (F, C) \in \mathcal{L}_1, \ T_i = P_{F,C,\tau,}. \tag{42}$$

Equations (14.120) and (14.122) imply that

$$\epsilon_0 \ge \|y_{i-1}^{(j,T)} - y_i^{(j,T)}\| = \|y_{i-1}^{(j,T)} - P_C(y_{i-1}^{(j,T)} - \tau F(y_{i-1}^{(j,T)}))\|. \tag{43}$$

It follows from (1), (18), (19) and (36) imply that

$$\|F(y_{i-1}^{(j,T)})\| \le \|F(u_*)\| + \|F(y_{i-1}^{(j,T)}) - F(u_*)\|$$

$$\le M_0 + \alpha^{-1}\|y_{i-1}^{(j,T)} - u_*\| \le M_0 + 4M_0\alpha^{-1}. \tag{44}$$

By (15), (43), (44), Proposition 1 and Lemma 3 applied with $x = y_{i-1}^{(j,T)}$, $\lambda = \tau$, $(A, K) = (F, C)$, for each $\xi \in C$,

$$\langle F(y_{i-1}^{(j,T)}), \xi - y_{i-1}^{(j,T)} \rangle \ge -\tau^{-1}\epsilon_0\|\xi - y_{i-1}^{(j,T)}\| - \tau^{-1}\epsilon_0^2 - \epsilon_0 M_0(1 + 4\alpha^{-1})$$

$$\ge -a^{-1}\epsilon_0\|\xi - y_{i-1}^{(j,T)}\| - a^{-1}\epsilon_0^2 - \epsilon_0 M_0(1 + 4\alpha^{-1})$$

$$\ge -\epsilon\|\xi - y_{i-1}^{(j,T)}\| - \epsilon,$$

$$y_{i-1}^{(j,T)} \in S_\epsilon(F, C).$$

Thus we showed that the following property holds:
(c) if $T_i \in \mathcal{L}_2$, then $y_{i-1}^{(j,T)} \in \mathrm{Fix}_\epsilon(T_i)$ and if

$$\tau \in [a, b], \ (F, C) \in \mathcal{L}_1, \ T_i = P_{F,C,\tau,},$$

then

$$y_{i-1}^{(j,T)} \in S_\epsilon(F, C).$$

By (25) and (40), for each $i, i_1.i_2 \in \{1, \dots, p(T)\}$,

$$\|y_{i_1}^{(j,T)} - y_{i_2 1}^{(j,T)}\| \le \epsilon_0 \bar{q}, \tag{45}$$

$$\|x_j - y_i^{(j,T)}\| \le \epsilon_0 \bar{q}, \ \|x_j - y_{j,T}\| \le \epsilon_0 \bar{q}. \tag{46}$$

By (12), (23), (46) and the convexity of the norm,

$$\|x_j - x_{j+1}\| = \|x_j - \sum_{T \in \Omega_{j+1}} w_{j+1}(T) y_{j,T}\|$$

$$\le \sum_{T \in \Omega_{j+1}} w_{j+1}(T) \|x_j - y_{j,T}\| \le \bar{q}\epsilon_0. \tag{47}$$

It follows from (47) that for each $j_1, j_2 \in \{q, \dots, q + \bar{N}\}$,

$$\|x_{j_1} - x_{j_2}\| \le \epsilon_0 \bar{q} \bar{N}. \tag{48}$$

Let $S \in \mathcal{L}_2$. Property (Q1) implies that there exist $j \in \{q, \dots, q + \bar{N} - 1\}$, $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ such that

$$S \in \{T_1, \dots, T_{p(T)}\}.$$

There exists $i \in \{1, \dots, p(T)\}$ such that

$$S = T_i.$$

Property (c) implies that

$$y_{i-1}^{(j,T)} \in \mathrm{Fix}_\epsilon(T_i).$$

In view (46),

$$d(x_j, \mathrm{Fix}_\epsilon(S)) \le \|x_j - y_{i-1}^{(j,T)}\| \le \bar{q}\epsilon_0. \tag{49}$$

It follows from (15), (48), (49) that for each $k \in \{q, \dots, q + \bar{N}\}$,

$$d(x_k, \mathrm{Fix}_\epsilon(S)) \le \|x_k - x_j\| + d(x_j, \mathrm{Fix}_\epsilon(S)) \le \epsilon_0(\bar{N} + 1)\bar{q} \le \epsilon.$$

Thus

$$d(x_k, \mathrm{Fix}_\epsilon(S)) \le \epsilon$$

for each $S \in \mathcal{L}_2$.

Let $(F, C) \in \mathcal{L}_1$. Property (Q2) implies that there exist $j \in \{q, \dots, q + \bar{N} - 1\}$, $\tau \in [a, b]$ and $T = (T_1, \dots, T_{p(T)}) \in \Omega_{j+1}$ such that

$$P_{F,C,\tau} \in \{T_1, \dots, T_{p(T)}\}.$$

There exists $i \in \{1, \dots, p(T)\}$ such that

$$T_i = P_{F,C,\tau}. \tag{50}$$

Property (c) and (50) imply that

$$y_{i-1}^{(j,T)} \in S_\epsilon(F, C)). \tag{51}$$

By (46) and (51),

$$d(x_j, S_\epsilon(F, C)) \le \|x_j - y_{i-1}^{(j,T)}\| \le \bar{q}\epsilon_0. \tag{52}$$

In view of (48) and (52), for each $k \in \{q, \dots, q + \bar{N}\}$,

$$d(x_k, S_\epsilon(F, C)) \le \|x_k - x_j\| + d(x_j, S_\epsilon(F, C))$$

$$\le \epsilon_0 \bar{N} \bar{q} + \bar{q}\epsilon_0 \le \epsilon.$$

Theorem 1 is proved. □

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