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# STRUCTURAL STABILITY OF MICROPOLAR FLUID FLOW IN A CURVED PIPE WITH DARCY–LAPWOOD–FORCHHEIMER–BRINKMAN MODEL*

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**Abstract**

In this article, we address the structural stability of the Darcy–Lapwood–Forchheimer–Brinkman model for a micropolar fluid flowing through a curved pipe. A rigorous analysis is carried out to prove the continuous dependence of solutions of the model on several physical and geometric parameters such as the Reynolds number, curvature ratio, Forchheimer coefficient, and micropolarity parameter.

**Keywords:** Darcy–Lapwood–Forchheimer–Brinkman model, micropolar fluid, curved pipe, structural stability, continuous dependence.
**MSC:** 35Q35, 76A05, 76S05, 35B30.

## 1 Introduction and mathematical model

A central question in mathematical modelling is whether solutions depend continuously on variations in coefficients, boundary conditions, or the gov-

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erning partial differential equations themselves. This continuous dependence of the solutions with respect to the model's structural components is known as structural stability. For real life problems, variations in coefficients of the governing partial differential equations correspond to changes in the constitutive parameters, therefore, establishing the continuous dependence is crucial for validating the mathematical model's real-world applicability. A comprehensive treatment of structural stability and related continuous dependence results can be found in the monographs by Ames and Straughan [1] and Straughan [19]. For Newtonian fluids, structural stability of the flows through porous media with the Darcy and Brinkman model has been extensively investigated by Morro and Straughan [13], Ames and Payne [2], and Payne and Straughan [14–16]. For the Brinkman–Forchheimer model, related stability results of Newtonian fluid flows were established by Payne and Straughan [17], Çelebi et al. [6,7], and Liu [11]. Łukaszewicz [12] established continuous dependence of two-dimensional micropolar fluid flows on the microrotation viscosity coefficient. To the best knowledge of the authors, structural stability of the flows in curved pipe is not yet explored. Therefore, in this paper, we investigate the structural stability of micropolar fluid flow in a curved pipe with Darcy-Lapwood-Forchheimer-Brinkman model on the pertinent physical and geometric parameters. The mathematical model of the flow problem under consideration is presented in Subsection 1.1. *A priori* estimates that are required for establishing structural stability are presented in Section 2. Main theorems establishing the structural stability of the flow model are presented, along with their proofs, in Section 3.

## 1.1 Mathematical model

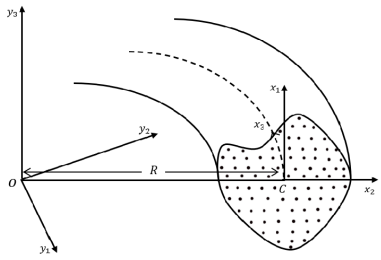



Figure 1: Flow Geometry.

We consider the flow of a micropolar fluid within a stationary curved pipe of arbitrary cross-section of diameter $2a$, which is characterized by a

constant centerline radius $R > a$. A typical section of the curved pipe is shown in Figure 1. We assume that the pipe's cross-section, $\Omega \subset \mathbb{R}^2$, is a bounded domain with Lipschitz boundary $\partial\Omega$. The flow is driven by a constant pressure gradient along the axis of the pipe and it is assumed to be axisymmetric. Under these assumptions, using toroidal coordinate system [3], the partial differential equations governing the flow of micropolar fluid through porous medium with Darcy-Lapwood-Forchheimer-Brinkman model are given by [8–10, 18]

$$\nabla \cdot (H\mathbf{u}) = 0, \tag{1}$$

$$\begin{aligned} &ReH(\mathbf{u}\cdot\nabla)\mathbf{u} - Re\delta u_3^2\hat{a}_2 + Re\delta u_2u_3\hat{a}_3 \\ &= \mathbf{f} - ReH\nabla p + ReG\hat{a}_3 + mH\nabla\times\boldsymbol{\nu} + m\delta\nu_3\hat{a}_1 + (1+m)\nabla\cdot(H\nabla\mathbf{u}) \\ &\quad -(1+m)\frac{\delta^2}{H}u_2\hat{a}_2 - (1+m)\frac{\delta^2}{H}u_3\hat{a}_3 - \sigma^2H\mathbf{u} - Re\sigma C_FH|\mathbf{u}|\mathbf{u}, \end{aligned} \tag{2}$$

$$\begin{aligned} &ReHJ(\mathbf{u}\cdot\nabla)\boldsymbol{\nu} - ReJ\delta u_3\nu_3\hat{a}_2 + ReJ\delta u_3\nu_2\hat{a}_3 \\ &= \mathbf{l} - 2mH\boldsymbol{\nu} + mH\nabla\times\mathbf{u} + m\delta u_3\hat{a}_1 + n_1\nabla\cdot(H\nabla\boldsymbol{\nu}) - n_1\frac{\delta^2}{H}(\nu_2\hat{a}_2+\nu_3\hat{a}_3) \\ &\quad +(n_2+n_3)H\nabla(\nabla\cdot\boldsymbol{\nu}) + (n_2+n_3)\delta\nabla\nu_2 - (n_2+n_3)\frac{\delta^2}{H}\nu_2\hat{a}_2 \end{aligned} \tag{3}$$

in $\Omega$,

where, the velocity vector $\mathbf{u} = (u_1, u_2, u_3)$ and the microrotation vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ are independent of $x_3$, $Re = \rho Ua/\mu$ is the Reynolds number, $\delta = a/R \in [0, 1)$ is the curvature ratio, $\sigma = a/\sqrt{K}$ is the porosity parameter, $H = 1 + \delta x_2$ is the scale factor, $G$ is the constant axial pressure gradient so that the total pressure is $p(x_1, x_2) - G\,x_3$, $m = \frac{k}{\mu}$ is the micropolarity parameter, $J$ is the nondimensional gyration parameter, and $n_1, n_2$ and $n_3$ are nondimensional gyro-viscosity parameters. We assume that $\min\{n_1, n_2+n_3\} > 0$. The constants $\mu$ and $k$ represents the usual Newtonian viscosity and the microrotation viscosity. $\rho$ is the density of the fluid, $K$ is the permeability of the porous medium and $C_F$ is Forchheimer coefficient. The functions $\mathbf{f}$ and $\mathbf{l}$ are the prescribed dimensionless body force and dimensionless bode couple, respectively, $U$ is the characteristic velocity and $\{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$ is the orthonormal basis of the toroidal coordinate system [3,18]. The differential operators $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, 0\right)$ and $\nabla\cdot = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$.

Since the curved pipe is stationary, the no-slip and the hyperstick boundary conditions to be satisfied are, respectively,

$$u_i = 0 \quad \text{and} \quad \nu_i = 0 \quad \text{on } \partial\Omega \quad \text{for } i = 1, 2, 3. \tag{4}$$

## 2 *A priori* estimates

The authors have established the existence and uniqueness of a solutions to the boundary value problem (BVP) (1)-(4) in [18]. More precisely, it was proved that if $\mathbf{f}, \mathbf{l} \in (L^2(\Omega))^3$ then for sufficiently small values of $\delta$, there exists $(\mathbf{u}, p, \boldsymbol{\nu}) \in \mathcal{X} \times L_0^2(\Omega) \times \left(H_0^1(\Omega)\right)^3$ such that the triplet $(\mathbf{u}, p, \boldsymbol{\nu})$ satisfy (1)-(4) in the sense of distributions, where

$$\mathcal{X} = \left\{ \mathbf{u} = (u_1, u_2, u_3) \in \left(H_0^1(\Omega)\right)^3 : \frac{\partial (H u_1)}{\partial x_1} + \frac{\partial (H u_2)}{\partial x_2} = 0 \right\}.$$

It was also proved that the solution $(\mathbf{u}, p, \boldsymbol{\nu})$ is unique for sufficiently small $Re$ and $m$. *A priori* estimates for $\mathbf{u}$ and $\boldsymbol{\nu}$, which were presented in [18], can be simplified as

$$\|\mathbf{u}\|_{1,2,\Omega} = \|\nabla \mathbf{u}\|_2 \leq \frac{\mathfrak{C}_1}{(1+m)H_0}\Big(m(1+\delta)\|\boldsymbol{\nu}\|_{1,2,\Omega} + \|\mathbf{F}\|_2 + \|\mathbf{f}\|_2\Big) \equiv E_{\mathbf{u}} \quad (5)$$

and

$$\|\boldsymbol{\nu}\|_{1,2,\Omega} = \|\nabla \boldsymbol{\nu}\|_2 \leq \frac{\mathfrak{C}_2 \left(\frac{m(1+\delta)}{(1+m)H_0}\Big(\|\mathbf{F}\|_2 + \|\mathbf{f}\|_2\Big) + \|\mathbf{l}\|_2\right)}{\Big(n_1 H_0 - \mathfrak{C}_3 \delta (n_2 + n_3)\Big)} \equiv E_{\boldsymbol{\nu}}, \quad (6)$$

where $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ are positive constants depending only on $\Omega$, $\mathbf{F} = (0, 0, Re\, G)$ and $H_0 = \inf_{\Omega} H$.

To derive an estimate for $p$, we express equation (2) as

$$Re \nabla p = (1+m)\nabla^2 \mathbf{u} + \mathfrak{F}, \quad (7)$$

in which

$$\mathfrak{F} = \frac{\mathbf{f}}{H} + \frac{\mathbf{F}}{H} - Re(\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{Re\delta}{H}(u_3^2 \hat{a}_2 - u_2 u_3 \hat{a}_3) + \frac{\delta}{H}\frac{\partial \mathbf{u}}{\partial x_2}$$
$$-(1+m)\frac{\delta^2}{H^2}(u_2 \hat{a}_2 + u_3 \hat{a}_3) - \sigma^2 \mathbf{u} - Re\sigma C_F |\mathbf{u}|\mathbf{u} + m\nabla \times \boldsymbol{\nu} + \frac{m\delta}{H}\nu_3 \hat{a}_1. \quad (8)$$

From the expression of $\mathfrak{F}$, it is clear that $\mathfrak{F} \in \left(H^{-1}(\Omega)\right)^3$; therefore, using Lemma IV.1.9 of [4], we get

$$Re\|p\|_2 \leq \mathfrak{C}_4\Big((1+m)\|\mathbf{u}\|_{1,2,\Omega} + \|\mathfrak{F}\|^*\Big), \quad (9)$$

where $\mathfrak{C}_4$ is a positive constant depending only on $\Omega$ and $\|\cdot\|^*$ denotes the operator norm in $\left(H^{-1}(\Omega)\right)^3$.
Now, we find $\|\mathfrak{F}\|^*$ to get appropriate estimate for $p$. For $\boldsymbol{v} \in \left(H_0^1(\Omega)\right)^3$,

$$\langle \mathfrak{F}, \boldsymbol{v} \rangle_{H^{-1},H_0^1} = \int_\Omega \left( \frac{\mathbf{f}}{H} + \frac{\mathbf{F}}{H} - Re(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{Re\delta}{H}(u_3^2\hat{a}_2 - u_2u_3\hat{a}_3) + \frac{\delta}{H}\frac{\partial \mathbf{u}}{\partial x_2} \right.$$
$$\left. -(1+m)\frac{\delta^2}{H^2}(u_2\hat{a}_2 + u_3\hat{a}_3) - \sigma^2\mathbf{u} - Re\sigma C_F|\mathbf{u}|\mathbf{u} + m\nabla\times\boldsymbol{\nu} + \frac{m\delta}{H}\nu_3\hat{a}_1 \right) \cdot \boldsymbol{v}\, dx \tag{10}$$

Using Sobolev embedding [5] and Hölder's inequality, we obtain

$$\left|\langle \mathfrak{F}, \boldsymbol{v} \rangle_{H^{-1},H_0^1}\right| \le \mathfrak{C}_5 \left[ \frac{1}{H_0}\left(\|\mathbf{f}\|_2 + \|\mathbf{F}\|_2\right) + Re\left(1 + \sigma C_F + \frac{\delta}{H_0}\right)\|\nabla\mathbf{u}\|_2^2 \right.$$
$$\left. + \left(\sigma^2 + \frac{\delta}{H_0} + (1+m)\frac{\delta^2}{H_0^2}\right)\|\nabla\mathbf{u}\|_2 + m\left(1 + \frac{\delta}{H_0}\right)\|\nabla\boldsymbol{\nu}\|_2 \right] \|\nabla\boldsymbol{v}\|_2,$$

and hence

$$\|\mathfrak{F}\|^* \le \mathfrak{C}_5 \left[ \frac{1}{H_0}\left(\|\mathbf{f}\|_2 + \|\mathbf{F}\|_2\right) + Re\left(1 + \sigma C_F + \frac{\delta}{H_0}\right) E_{\mathbf{u}}^2 \right.$$
$$\left. + \left(\sigma^2 + \frac{\delta}{H_0} + (1+m)\frac{\delta^2}{H_0^2}\right) E_{\mathbf{u}} + m\left(1 + \frac{\delta}{H_0}\right) E_{\boldsymbol{\nu}} \right], \tag{11}$$

where $\mathfrak{C}_5$ is a positive constant depending only on $\Omega$.

Using (11) in (9), we get an estimate for $p$ as

$$\|p\|_2 \le E_p, \tag{12}$$

$$E_p = \mathfrak{C}_6 \left[ \frac{1}{H_0 Re}\left(\|\mathbf{f}\|_2 + \|\mathbf{F}\|_2\right) + \left(1 + \sigma C_F + \frac{\delta}{H_0}\right) E_{\mathbf{u}}^2 \right.$$
$$\left. + \frac{1}{Re}\left(\sigma^2 + \frac{\delta}{H_0} + (1+m)\left(1 + \frac{\delta^2}{H_0^2}\right)\right) E_{\mathbf{u}} + \frac{m}{Re}\left(1 + \frac{\delta}{H_0}\right) E_{\boldsymbol{\nu}} \right],$$

where $\mathfrak{C}_6$ is a positive constant depending only on $\Omega$.

Using *a priori* estimates (5), (6) and (12), in the next section, we investigate the structural stability of the flow problem (1)-(4); particularly, we establish the continuous dependence of the solution on the key physical and geometric parameters, namely, the Reynolds number ($Re$), the curvature ratio ($\delta$), the Forchheimer coefficient ($C_F$), and the micropolarity parameter ($m$).

## 3 Structural stability

We assume that $\mathbf{f}, \mathbf{l} \in \left(L^2(\Omega)\right)^3$ and the values of $\delta$, $Re$ and $m$ are sufficiently small to ensure the existence and uniqueness of the weak solutions of the problem (1)-(4). Throughout this section, we denote $\mathfrak{C}$ as a generic positive constant depending only on $\Omega$.

The main results of the paper are presented in the following four subsections; indeed, Theorems 1, 2, 3 and 4 present the continuous dependence of the solution on parameters $Re$, $\delta$, $C_F$ and $m$, respectively.

### 3.1 Continuous dependence on the Reynolds number $(Re)$

Let $(\mathbf{u}, p, \boldsymbol{\nu})$ and $(\mathbf{v}, q, \boldsymbol{\eta})$ be the solutions of the BVP (1)-(4) corresponding to the Reynolds numbers $Re_1$ and $Re_2$, respectively. Then the difference $(\mathbf{w}, \pi, \boldsymbol{\phi}) := (\mathbf{u} - \mathbf{v}, p - q, \boldsymbol{\nu} - \boldsymbol{\eta})$ satisfies the BVP

$$\nabla \cdot (H\mathbf{w}) = 0, \tag{13}$$

$$\begin{aligned}
&Re_1 H\Big((\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v}\Big) + \widetilde{Re}H(\mathbf{v}\cdot\nabla)\mathbf{v} - Re_1\delta(u_3+v_3)w_3\hat{a}_2 \\
&-\widetilde{Re}\delta v_3^2\hat{a}_2 + Re_1\delta(w_2u_3 + v_2w_3)\hat{a}_3 + \widetilde{Re}\delta v_2v_3\hat{a}_3 \\
&= \widetilde{Re}\, G\hat{a}_3 - Re_1 H\nabla\pi - \widetilde{Re}H\nabla q + mH\nabla\times\boldsymbol{\phi} + m\delta\phi_3\hat{a}_1 \\
&+(1+m)\nabla\cdot(H\nabla\mathbf{w}) - (1+m)\frac{\delta^2}{H}(w_2\hat{a}_2 + w_3\hat{a}_3) - \sigma^2 H\mathbf{w} \\
&-Re_1\sigma C_F H(|\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v}) - \widetilde{Re}\sigma C_F H|\mathbf{v}|\mathbf{v},
\end{aligned} \tag{14}$$

$$\begin{aligned}
&Re_1 HJ\Big((\mathbf{u}\cdot\nabla)\boldsymbol{\phi} + (\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\Big) + \widetilde{Re}HJ(\mathbf{v}\cdot\nabla)\boldsymbol{\eta} - Re_1 J\delta u_3\phi_3\hat{a}_2 \\
&-Re_1 J\delta w_3\eta_3\hat{a}_2 - \widetilde{Re}J\delta v_3\eta_3\hat{a}_2 + Re_1 J\delta u_3\phi_2\hat{a}_3 + Re_1 J\delta w_3\eta_2\hat{a}_3 \\
&+\widetilde{Re}J\delta v_3\eta_2\hat{a}_3 \\
&= -2mH\boldsymbol{\phi} + mH\nabla\times\mathbf{w} + m\delta w_3\hat{a}_1 + n_1\nabla\cdot(H\nabla\boldsymbol{\phi}) - \frac{n_1\delta^2}{H}(\phi_2\hat{a}_2 + \phi_3\hat{a}_3) \\
&+(n_2+n_3)H\nabla(\nabla\cdot\boldsymbol{\phi}) + (n_2+n_3)\delta\nabla\boldsymbol{\phi} - (n_2+n_3)\frac{\delta^2}{H}\phi_2\hat{a}_2
\end{aligned} \tag{15}$$

in $\Omega$,

and

$$\mathbf{w} = \mathbf{0} = \boldsymbol{\phi} \text{ on } \partial\Omega, \tag{16}$$

where $\mathbf{w} = (w_1, w_2, w_3)$, $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, $w_i = u_i - v_i$, $\phi_i = \nu_i - \eta_i$ $(i = 1, 2, 3)$ and $\widetilde{Re} = Re_1 - Re_2$.

The main result regarding the continuous dependence on the Reynolds number $(Re)$ is stated in the following theorem.

**Theorem 1.** *If $(\mathbf{w}, \pi, \boldsymbol{\phi})$ is the solution of (13)-(16), then, for sufficiently small values of $Re_1$, $m$ and $\delta$, $\mathbf{w} \to \mathbf{0}$ in $\mathcal{X}$ and $\boldsymbol{\phi} \to \mathbf{0}$ in $(H_0^1(\Omega))^3$ as $Re_2 \to Re_1$.*

*Proof.* Taking the $L^2(\Omega)$-inner product of (14) with $\mathbf{w}$, we get

$$\begin{aligned}
&Re_1 \int_\Omega H\big((\mathbf{u}\cdot\nabla)\mathbf{w}\big)\cdot\mathbf{w}\,dx + Re_1 \int_\Omega H\big((\mathbf{w}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx \\
&+\widetilde{Re}\int_\Omega H\big((\mathbf{v}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx - Re_1\delta\int_\Omega v_3 w_3 w_2\,dx - \widetilde{Re}\delta\int_\Omega v_3^2 w_2\,dx \\
&+Re_1\delta\int_\Omega v_2 w_3^2\,dx + \widetilde{Re}\delta\int_\Omega v_2 v_3 w_3\,dx \\
&= \widetilde{Re}\,G\int_\Omega w_3\,dx - Re_1\int_\Omega H\nabla\pi\cdot\mathbf{w}\,dx - \widetilde{Re}\int_\Omega H\nabla q\cdot\mathbf{w}\,dx \\
&+m\int_\Omega H(\nabla\times\boldsymbol{\phi})\cdot\mathbf{w}\,dx + m\delta\int_\Omega \phi_3 w_1\,dx \\
&+(1+m)\int_\Omega \nabla\cdot(H\nabla\mathbf{w})\cdot\mathbf{w}\,dx - (1+m)\delta^2\int_\Omega \frac{1}{H}(w_2^2+w_3^2)\,dx \\
&-\sigma^2\int_\Omega H\mathbf{w}\cdot\mathbf{w}\,dx - Re_1\sigma C_F\int_\Omega H(|\mathbf{u}|\mathbf{u}-|\mathbf{v}|\mathbf{v})\cdot\mathbf{w}\,dx \\
&-\widetilde{Re}\sigma C_F\int_\Omega H|\mathbf{v}|\mathbf{v}\cdot\mathbf{w}\,dx. \qquad (17)
\end{aligned}$$

Since $(|\mathbf{u}|u_i - |\mathbf{v}|v_i)w_i = \frac{1}{2}(|\mathbf{u}|+|\mathbf{v}|)w_i^2 + \frac{1}{2}(|\mathbf{u}|-|\mathbf{v}|)^2(|\mathbf{u}|+|\mathbf{v}|)$ and $H$ is a positive function, we have $\int_\Omega H(|\mathbf{u}|\mathbf{u}-|\mathbf{v}|\mathbf{v})\cdot\mathbf{w}\,dx \geq 0$. Also, considering the nonnegativity of $\int_\Omega \frac{1}{H}(w_2^2+w_3^2)\,dx$ together with (13) and the fact that $\int_\Omega H\big((\mathbf{u}\cdot\nabla)\mathbf{w}\big)\cdot\mathbf{w}\,dx = 0$, the equality (17) becomes

$$\begin{aligned}
&(1+m)H_0\|\nabla\mathbf{w}\|_2^2 + \sigma^2 H_0\|\mathbf{w}\|_2^2 \leq \left|Re_1\int_\Omega H\big((\mathbf{w}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx\right| \\
&+\left|Re_1\delta\int_\Omega v_3 w_3 w_2\,dx\right| + \left|Re_1\delta\int_\Omega v_2 w_3^2\,dx\right| + \left|\widetilde{Re}\int_\Omega H\big((\mathbf{v}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx\right| \\
&+\left|\widetilde{Re}\delta\int_\Omega v_3^2 w_2\,dx\right| + \left|\widetilde{Re}\delta\int_\Omega v_2 v_3 w_3\,dx\right| + \left|\widetilde{Re}\,G\int_\Omega w_3\,dx\right|
\end{aligned}$$

$$+\left|\widetilde{Re}\sigma C_F\int_\Omega H|\mathbf{v}|\mathbf{v}\cdot\mathbf{w}\,dx\right|+\left|m\int_\Omega H(\nabla\times\boldsymbol{\phi})\cdot\mathbf{w}\,dx\right|+\left|m\delta\int_\Omega \phi_3 w_1\,dx\right|. \tag{18}$$

Using Hölder's inequality, Sobolev inequality and the bound (5), the terms of the right hand side of (18) can be estimated as follows:

$$\begin{aligned}\left|Re_1\int_\Omega H\big((\mathbf{w}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx\right| &\le Re_1H_1\|\mathbf{w}\|_4\|\nabla\mathbf{v}\|_2\|\mathbf{w}\|_4\\ &\le \mathfrak{C}Re_1E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2^2,\end{aligned}$$

$$\begin{aligned}\left|Re_1\delta\int_\Omega v_3w_3w_2\,dx\right| &\le Re_1\delta\|\mathbf{v}\|_2\|\mathbf{w}\|_4^2\\ &\le \mathfrak{C}Re_1\delta E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2^2,\end{aligned}$$

$$\begin{aligned}\left|\widetilde{Re}\int_\Omega H\big((\mathbf{v}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx\right| &\le |\widetilde{Re}|H_1\|\mathbf{v}\|_4\|\nabla\mathbf{v}\|_2\|\mathbf{w}\|_4\\ &\le \mathfrak{C}|\widetilde{Re}|E_{\mathbf{v}}^2\|\nabla\mathbf{w}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\widetilde{Re}\delta\int_\Omega v_3^2w_2\,dx\right| &\le |\widetilde{Re}|\delta\|\mathbf{v}\|_4^2\|\mathbf{w}\|_2\\ &\le \mathfrak{C}|\widetilde{Re}|\delta E_{\mathbf{v}}^2\|\nabla\mathbf{w}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\widetilde{Re}\,G\int_\Omega w_3\,dx\right| &\le |\widetilde{Re}||G||\Omega|\|\mathbf{w}\|_2\\ &\le \mathfrak{C}|\widetilde{Re}|\|\nabla\mathbf{w}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\widetilde{Re}\sigma C_F\int_\Omega H|\mathbf{v}|\mathbf{v}\cdot\mathbf{w}\,dx\right| &\le |\widetilde{Re}|\sigma C_FH_1\||\mathbf{v}|\mathbf{v}\|_2\|\mathbf{w}\|_2\\ &\le \mathfrak{C}|\widetilde{Re}|\sigma C_F\|\nabla\mathbf{v}\|_2^2\|\nabla\mathbf{w}\|_2 \text{ (see [18])}\\ &\le \mathfrak{C}|\widetilde{Re}|\sigma C_FE_{\mathbf{v}}^2\|\nabla\mathbf{w}\|_2,\end{aligned}$$

$$\begin{aligned}\left|m\int_\Omega H(\nabla\times\boldsymbol{\phi})\cdot\mathbf{w}\,dx\right| &\le mH_1\|\nabla\times\boldsymbol{\phi}\|_2\|\mathbf{w}\|_2\\ &\le \mathfrak{C}m\|\nabla\boldsymbol{\phi}\|_2\|\nabla\mathbf{w}\|_2,\end{aligned}$$

and

$$\left|m\delta\int_\Omega \phi_3 w_1\,dx\right| \le m\delta\|\boldsymbol{\phi}\|_2\|\mathbf{w}\|_2 \le \mathfrak{C}m\delta\|\nabla\boldsymbol{\phi}\|_2\|\nabla\mathbf{w}\|_2.$$

The preceding estimates transform the inequality (18) into

$$(1+m)H_0\|\nabla\mathbf{w}\|_2^2 \le \mathfrak{C}\Big\{Re_1(1+\delta)E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2^2 + |\widetilde{Re}|\Big((1+\delta+\sigma C_F)E_{\mathbf{v}}^2+1\Big)\|\nabla\mathbf{w}\|_2 + m(1+\delta)\|\nabla\boldsymbol{\phi}\|_2\|\nabla\mathbf{w}\|_2\Big\}. \tag{19}$$

Taking the $L^2(\Omega)$-inner product of (15) with $\boldsymbol{\phi}$, we get

$$\begin{aligned}
&Re_1J\int_\Omega H\big((\mathbf{u}\cdot\nabla)\boldsymbol{\phi}\big)\cdot\boldsymbol{\phi}\,dx + Re_1J\int_\Omega H\big((\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\big)\cdot\boldsymbol{\phi}\,dx\\
&+\widetilde{Re}J\int_\Omega H\big((\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\big)\cdot\boldsymbol{\phi}\,dx - Re_1J\delta\int_\Omega w_3\eta_3\phi_2\,dx - \widetilde{Re}J\delta\int_\Omega v_3\eta_3\phi_2\,dx\\
&+Re_1J\delta\int_\Omega w_3\eta_2\phi_3\,dx + \widetilde{Re}J\delta\int_\Omega v_3\eta_2\phi_3\,dx\\
&\le -2m\int_\Omega H\boldsymbol{\phi}\cdot\boldsymbol{\phi}\,dx + m\int_\Omega H(\nabla\times\mathbf{w})\cdot\boldsymbol{\phi}\,dx + m\delta\int_\Omega w_3\phi_1\,dx\\
&+n_1\int_\Omega \nabla\cdot(H\nabla\boldsymbol{\phi})\cdot\boldsymbol{\phi}\,dx - n_1\delta^2\int_\Omega\frac{1}{H}(\phi_2^2+\phi_3^2)\,dx\\
&+(n_2+n_3)\int_\Omega H\nabla(\nabla\cdot\boldsymbol{\phi})\cdot\boldsymbol{\phi}\,dx + (n_2+n_3)\delta\int_\Omega(\nabla\boldsymbol{\phi})\cdot\boldsymbol{\phi}\,dx\\
&-(n_2+n_3)\delta^2\int_\Omega\frac{1}{H}\phi_2^2\,dx.
\end{aligned}\tag{20}$$

Integration by parts allows us to replace the integral $(n_2+n_3)\int_\Omega H\nabla(\nabla\cdot\boldsymbol{\phi})\cdot\boldsymbol{\phi}\,dx$ by $-(n_2+n_3)\int_\Omega H(\nabla\cdot\boldsymbol{\phi})^2\,dx - (n_2+n_3)\delta\int_\Omega\phi_2(\nabla\cdot\boldsymbol{\phi})\,dx$ in the inequality (20); subsequently, using the nonnegativity of $\int_\Omega H(\nabla\cdot\boldsymbol{\phi})^2\,dx$, $\int_\Omega\frac{1}{H}(\phi_2^2+\phi_3^2)\,dx$ and $\int_\Omega\frac{1}{H}\phi_2^2\,dx$ and the fact that $\int_\Omega H\big((\mathbf{u}\cdot\nabla)\boldsymbol{\phi}\big)\cdot\boldsymbol{\phi}\,dx = 0$, we obtain

$$\begin{aligned}
n_1H_0\|\nabla\boldsymbol{\phi}\|_2^2 + 2mH_0\|\boldsymbol{\phi}\|_2^2 &\le \left|Re_1J\int_\Omega H\big((\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\big)\cdot\boldsymbol{\phi}\,dx\right|\\
&+\left|Re_1J\delta\int_\Omega w_3\eta_3\phi_2\,dx\right| + \left|Re_1J\delta\int_\Omega w_3\eta_2\phi_3\,dx\right|\\
&+\left|(n_2+n_3)\delta\int_\Omega\phi_2(\nabla\cdot\boldsymbol{\phi})\,dx\right| + \left|(n_2+n_3)\delta\int_\Omega(\nabla\boldsymbol{\phi})\cdot\boldsymbol{\phi}\,dx\right|\\
&+\left|\widetilde{Re}J\int_\Omega H\big((\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\big)\cdot\boldsymbol{\phi}\,dx\right| + \left|\widetilde{Re}J\delta\int_\Omega v_3\eta_3\phi_2\,dx\right| + \left|\widetilde{Re}J\delta\int_\Omega v_3\eta_2\phi_3\,dx\right|
\end{aligned}$$

$$+\left|m\int_\Omega H(\nabla\times\mathbf{w})\cdot\boldsymbol{\phi}\,dx\right|+\left|m\delta\int_\Omega w_3\phi_1\,dx\right|. \tag{21}$$

The terms present in right hand side of (21) can be estimated as

$$\begin{aligned}\left|Re_1J\int_\Omega H\big((\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\big)\cdot\boldsymbol{\phi}\,dx\right|&\le Re_1JH_1\|\mathbf{w}\|_4\|\nabla\boldsymbol{\eta}\|_2\|\boldsymbol{\phi}\|_4\\&\le \mathfrak{C}Re_1JE_{\boldsymbol{\eta}}\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2,\end{aligned}$$

$$\begin{aligned}\left|Re_1J\delta\int_\Omega w_3\eta_3\phi_2\,dx\right|&\le Re_1J\delta\|\mathbf{w}\|_4\|\boldsymbol{\eta}\|_2\|\boldsymbol{\phi}\|_4\\&\le \mathfrak{C}Re_1J\delta E_{\boldsymbol{\eta}}\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2,\end{aligned}$$

$$\begin{aligned}\left|(n_2+n_3)\delta\int_\Omega \phi_2(\nabla\cdot\boldsymbol{\phi})\,dx\right|&\le (n_2+n_3)\delta\|\boldsymbol{\phi}\|_2\|\nabla\cdot\boldsymbol{\phi}\|_2\\&\le \mathfrak{C}(n_2+n_3)\delta\|\nabla\boldsymbol{\phi}\|_2^2,\end{aligned}$$

$$\begin{aligned}\left|(n_2+n_3)\delta\int_\Omega (\nabla\boldsymbol{\phi})\cdot\boldsymbol{\phi}\,dx\right|&\le (n_2+n_3)\delta\|\nabla\boldsymbol{\phi}\|_2\|\boldsymbol{\phi}\|_2\\&\le \mathfrak{C}(n_2+n_3)\delta\|\nabla\boldsymbol{\phi}\|_2^2,\end{aligned}$$

$$\begin{aligned}\left|\widetilde{Re}J\int_\Omega H\big((\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\big)\cdot\boldsymbol{\phi}\,dx\right|&\le |\widetilde{Re}|JH_1\|\mathbf{v}\|_4\|\nabla\boldsymbol{\eta}\|_2\|\boldsymbol{\phi}\|_4\\&\le \mathfrak{C}|\widetilde{Re}|JE_{\mathbf{v}}E_{\boldsymbol{\eta}}\|\nabla\boldsymbol{\phi}\|_2,\end{aligned}$$

$$\begin{aligned}\left|\widetilde{Re}J\delta\int_\Omega v_3\eta_3\phi_2\,dx\right|&\le |\widetilde{Re}|J\delta\|\mathbf{v}\|_4\|\boldsymbol{\eta}\|_2\|\boldsymbol{\phi}\|_4\\&\le \mathfrak{C}|\widetilde{Re}|J\delta E_{\mathbf{v}}E_{\boldsymbol{\eta}}\|\nabla\boldsymbol{\phi}\|_2,\end{aligned}$$

$$\begin{aligned}\left|m\int_\Omega H(\nabla\times\mathbf{w})\cdot\boldsymbol{\phi}\,dx\right|&\le mH_1\|\nabla\times\mathbf{w}\|_2\|\boldsymbol{\phi}\|_2\\&\le \mathfrak{C}m\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2,\end{aligned}$$

and

$$\begin{aligned}\left|m\delta\int_\Omega w_3\phi_1\,dx\right|&\le m\delta\|\mathbf{w}\|_2\|\boldsymbol{\phi}\|_2\\&\le \mathfrak{C}m\delta\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2.\end{aligned}$$

Combining all these bounds with (21), we obtain

$$n_1 H_0 \|\nabla \boldsymbol{\phi}\|_2^2 \leq \mathfrak{C}\Big\{(1+\delta)(m+Re_1 JE_{\boldsymbol{\eta}})\|\nabla \mathbf{w}\|_2 \|\nabla \boldsymbol{\phi}\|_2 + (n_2+n_3)\delta \|\nabla \boldsymbol{\phi}\|_2^2 + |\widetilde{Re}| J(1+\delta)E_{\mathbf{v}} E_{\boldsymbol{\eta}} \|\nabla \boldsymbol{\phi}\|_2 \Big\}. \tag{22}$$

Adding (19) and (22), we get

$$\begin{aligned}
&(1+m)H_0\|\nabla \mathbf{w}\|_2^2 + n_1 H_0 \|\nabla \boldsymbol{\phi}\|_2^2 \\
&\leq \mathfrak{C}\Big\{Re_1(1+\delta)E_{\mathbf{v}}\|\nabla \mathbf{w}\|_2^2 + |\widetilde{Re}|\Big((1+\delta+\sigma C_F)E_{\mathbf{v}}^2+1\Big)\|\nabla \mathbf{w}\|_2 \\
&\quad +(1+\delta)(m+Re_1 JE_{\boldsymbol{\eta}})\|\nabla \mathbf{w}\|_2\|\nabla \boldsymbol{\phi}\|_2 + (n_2+n_3)\delta\|\nabla \boldsymbol{\phi}\|_2^2 \\
&\quad +|\widetilde{Re}| J(1+\delta)E_{\mathbf{v}}E_{\boldsymbol{\eta}}\|\nabla \boldsymbol{\phi}\|_2\Big\}.
\end{aligned} \tag{23}$$

By Young's inequality, we obtain

$$\begin{aligned}
&(1+m)H_0\|\nabla \mathbf{w}\|_2^2 + n_1 H_0 \|\nabla \boldsymbol{\phi}\|_2^2 \\
&\leq \frac{(1+m)H_0}{2}\|\nabla \mathbf{w}\|_2^2 + \frac{n_1 H_0}{2}\|\nabla \boldsymbol{\phi}\|_2^2 + \mathfrak{C}(n_2+n_3)\delta\|\nabla \boldsymbol{\phi}\|_2^2 \\
&\quad +\mathfrak{C}\Big\{Re_1(1+\delta)E_{\mathbf{v}} + \frac{(1+\delta)^2(m+Re_1 JE_{\boldsymbol{\eta}})^2}{n_1 H_0}\Big\}\|\nabla \mathbf{w}\|_2^2 \\
&\quad +\mathfrak{C}\left\{\frac{\Big((1+\delta+\sigma C_F)E_{\mathbf{v}}^2+1\Big)^2}{(1+m)H_0} + \frac{J^2(1+\delta)^2 E_{\mathbf{v}}^2 E_{\boldsymbol{\eta}}^2}{n_1 H_0}\right\}\widetilde{Re}^2.
\end{aligned} \tag{24}$$

After simplification, (24) yields

$$\begin{aligned}
&\left\{\frac{(1+m)H_0}{2} - \mathfrak{C}\left(Re_1(1+\delta)E_{\mathbf{v}} + \frac{(1+\delta)^2(m+Re_1 JE_{\boldsymbol{\eta}})^2}{n_1 H_0}\right)\right\}\|\nabla \mathbf{w}\|_2^2 \\
&\quad +\left\{\frac{n_1 H_0}{2} - \mathfrak{C}(n_2+n_3)\delta\right\}\|\nabla \boldsymbol{\phi}\|_2^2 \\
&\leq \mathfrak{C}\left\{\frac{\Big((1+\delta+\sigma C_F)E_{\mathbf{v}}^2+1\Big)^2}{(1+m)H_0} + \frac{J^2(1+\delta)^2 E_{\mathbf{v}}^2 E_{\boldsymbol{\eta}}^2}{n_1 H_0}\right\}\widetilde{Re}^2.
\end{aligned} \tag{25}$$

It follows from inequality (25) that the required result holds whenever $Re_1 \leq \epsilon_1$, $m \leq \epsilon_2$ and $\delta \leq \epsilon_3$ for some positive constants $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$. □

**Remark 1.** *The convergence of* $\mathbf{v}$ *to* $\mathbf{u}$ *in* $\mathcal{X}$ *and* $\boldsymbol{\eta}$ *to* $\boldsymbol{\nu}$ *in* $\left(H_0^1(\Omega)\right)^3$ *as* $Re_2 \to 0$ *follows from the inequality* (25) *when* $Re_1 = 0$. *Therefore, it can be concluded that the BVP* (1)-(4) *admits a regular perturbation with respect to the Reynolds number* $Re$.

### 3.2 Continuous dependence on the curvature ratio ($\delta$)

Suppose that $(\mathbf{u}, p, \boldsymbol{\nu})$ and $(\mathbf{v}, q, \boldsymbol{\eta})$ are the solutions of the BVP (1)-(4) corresponding to the curvature ratios $\delta_1$ and $\delta_2$, respectively. Let $(\mathbf{w}, \pi, \boldsymbol{\phi}) := (\mathbf{u} - \mathbf{v}, p - q, \boldsymbol{\nu} - \boldsymbol{\eta})$. Then $(\mathbf{w}, \pi, \boldsymbol{\phi})$ satisfies the BVP

$$\nabla \cdot (H_{\delta_1}\mathbf{w}) + \tilde{\delta}\nabla \cdot (x_2\mathbf{v}) = 0, \tag{26}$$

$$\begin{aligned}
&ReH_{\delta_1}\Big((\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v}\Big) + \tilde{\delta}Rex_2(\mathbf{v}\cdot\nabla)\mathbf{v} - Re\delta_1 w_3(u_3+v_3)\hat{a}_2 \\
&\quad -\tilde{\delta}Rev_3^2\hat{a}_2 + Re\delta_1(w_2u_3 + v_2w_3)\hat{a}_3 + \tilde{\delta}Rev_2v_3\hat{a}_3 \\
&= -ReH_{\delta_1}\nabla\pi - \tilde{\delta}Rex_2\nabla q + mH_{\delta_1}\nabla\times\boldsymbol{\phi} + \tilde{\delta}mx_2\nabla\times\boldsymbol{\eta} + m\delta_1\phi_3\hat{a}_1 \\
&\quad +\tilde{\delta}m\eta_3\hat{a}_1 + (1+m)\nabla\cdot(H_{\delta_1}\nabla\mathbf{w}) + \tilde{\delta}(1+m)\nabla\cdot(x_2\nabla\mathbf{v}) \\
&-(1+m)\frac{\delta_1^2}{H_{\delta_1}}(w_2\hat{a}_2 + w_3\hat{a}_3) - \tilde{\delta}(1+m)\left(\frac{\delta_1+\delta_2}{H_{\delta_1}} - \frac{\delta_2^2x_2}{H_{\delta_1}H_{\delta_2}}\right)(v_2\hat{a}_2 + v_3\hat{a}_3) \\
&\quad -\sigma^2H_{\delta_1}\mathbf{w} - \tilde{\delta}\sigma^2x_2\mathbf{v} - Re\sigma C_FH_{\delta_1}(|\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v}) - \tilde{\delta}Re\sigma C_Fx_2|\mathbf{v}|\mathbf{v},
\end{aligned} \tag{27}$$

$$\begin{aligned}
&ReH_{\delta_1}J\Big((\mathbf{u}\cdot\nabla)\boldsymbol{\phi} + (\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\Big) + \tilde{\delta}ReJx_2(\mathbf{v}\cdot\nabla)\boldsymbol{\eta} - ReJ\delta_1(u_3\phi_3 + w_3\eta_3)\hat{a}_2 \\
&\quad -\tilde{\delta}ReJv_3\eta_3\hat{a}_2 + ReJ\delta_1(u_3\phi_2 + w_3\eta_2)\hat{a}_3 + \tilde{\delta}ReJv_3\eta_2\hat{a}_3 \\
&= -2mH_{\delta_1}\boldsymbol{\phi} - 2\tilde{\delta}mx_2\boldsymbol{\eta} + mH_{\delta_1}\nabla\times\mathbf{w} + \tilde{\delta}mx_2\nabla\times\mathbf{v} + m\delta_1w_3\hat{a}_1 + \tilde{\delta}mv_3\hat{a}_1 \\
&\quad +n_1\nabla\cdot(H_{\delta_1}\nabla\boldsymbol{\phi}) + \tilde{\delta}n_1\nabla\cdot(x_2\nabla\boldsymbol{\eta}) - n_1\frac{\delta_1^2}{H_{\delta_1}}(\phi_2\hat{a}_2 + \phi_3\hat{a}_3) \\
&-\tilde{\delta}n_1\left(\frac{\delta_1+\delta_2}{H_{\delta_1}} - \frac{\delta_2^2x_2}{H_{\delta_1}H_{\delta_2}}\right)(\eta_2\hat{a}_2 + \eta_3\hat{a}_3) + (n_2+n_3)H_{\delta_1}\nabla(\nabla\cdot\boldsymbol{\phi}) \\
&\quad +\tilde{\delta}(n_2+n_3)x_2\nabla(\nabla\cdot\boldsymbol{\eta}) + (n_2+n_3)\delta_1\nabla\phi_2 + \tilde{\delta}(n_2+n_3)\nabla\eta_2 \\
&-(n_2+n_3)\frac{\delta_1^2}{H_{\delta_1}}\phi_2\hat{a}_2 - \tilde{\delta}(n_2+n_3)\left(\frac{\delta_1+\delta_2}{H_{\delta_1}} - \frac{\delta_2^2x_2}{H_{\delta_1}H_{\delta_2}}\right)\eta_2\hat{a}_2
\end{aligned} \tag{28}$$

in $\Omega$,

and

$$\mathbf{w} = \mathbf{0} = \boldsymbol{\phi} \text{ on } \partial\Omega, \tag{29}$$

where $\mathbf{w} = (w_1, w_2, w_3)$, $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, $w_i = u_i - v_i$, $\phi_i = \nu_i - \eta_i$ $(i = 1, 2, 3)$, $\tilde{\delta} = \delta_1 - \delta_2$ and $H_{\delta_j} = 1 + \delta_j x_2$ $(j = 1, 2)$.

The principal result concerning the continuous dependence on the curvature ratio ($\delta$) is presented below:

**Theorem 2.** *Let* $(\mathbf{w}, \pi, \boldsymbol{\phi})$ *be the solution of the boundary value problem* (26)-(29). *Then* $\mathbf{w} \to \mathbf{0}$ *in* $\mathcal{X}$ *and* $\boldsymbol{\phi} \to \mathbf{0}$ *in* $(H_0^1(\Omega))^3$ *as* $\delta_2 \to \delta_1$ *for sufficiently small values of* $Re$, $m$ *and* $\delta_1$.

*Proof.* Taking the $L^2(\Omega)$-inner product on both sides of (27) with $\mathbf{w}$, we get

$$
\begin{aligned}
&Re \int_\Omega H_{\delta_1}\big((\mathbf{u}\cdot\nabla)\mathbf{w}\big)\cdot\mathbf{w}\,dx + Re\int_\Omega H_{\delta_1}\big((\mathbf{w}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx\\
&+\tilde{\delta}Re\int_\Omega x_2\big(\mathbf{v}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx - Re\delta_1\int_\Omega (u_3+v_3)w_3w_2\,dx\\
&-\tilde{\delta}Re\int_\Omega v_3^2w_2\,dx + Re\delta_1\int_\Omega (w_2u_3+v_2w_3)w_3\,dx + \tilde{\delta}Re\int_\Omega v_2v_3w_3\,dx\\
&= -Re\int_\Omega H_{\delta_1}\nabla\pi\cdot\mathbf{w}\,dx - \tilde{\delta}Re\int_\Omega x_2\nabla q\cdot\mathbf{w}\,dx + m\int_\Omega H_{\delta_1}(\nabla\times\boldsymbol{\phi})\cdot\mathbf{w}\,dx\\
&+\tilde{\delta}m\int_\Omega x_2(\nabla\times\boldsymbol{\eta})\cdot\mathbf{w}\,dx + m\delta_1\int_\Omega \phi_3w_1\,dx + \tilde{\delta}m\int_\Omega \eta_3w_1\,dx\\
&+(1+m)\int_\Omega \nabla\cdot(H_{\delta_1}\nabla\mathbf{w})\cdot\mathbf{w}\,dx + \tilde{\delta}(1+m)\int_\Omega \nabla\cdot(x_2\nabla\mathbf{v})\cdot\mathbf{w}\,dx\\
&-(1+m)\delta_1^2\int_\Omega \frac{1}{H_{\delta_1}}(w_2^2+w_3^2)\,dx - \tilde{\delta}(1+m)(\delta_1+\delta_2)\int_\Omega \frac{1}{H_{\delta_1}}(v_2w_2+v_3w_3)\,dx\\
&+\tilde{\delta}\delta_2^2(1+m)\int_\Omega \frac{x_2}{H_{\delta_1}H_{\delta_2}}(v_2w_2+v_3w_3)\,dx - \sigma^2\int_\Omega H_{\delta_1}\mathbf{w}\cdot\mathbf{w}\,dx\\
&-\tilde{\delta}\sigma^2\int_\Omega x_2\mathbf{v}\cdot\mathbf{w}\,dx - Re\sigma C_F H_{\delta_1}\int_\Omega (|\mathbf{u}|\mathbf{u}-|\mathbf{v}|\mathbf{v})\cdot\mathbf{w}\,dx\\
&-\tilde{\delta}Re\sigma C_F\int_\Omega x_2|\mathbf{v}|\mathbf{v}\cdot\mathbf{w}\,dx. \qquad (30)
\end{aligned}
$$

Using the identity $\int_\Omega H_{\delta_1}\big((\mathbf{u}\cdot\nabla)\mathbf{w}\big)\cdot\mathbf{w}\,dx = 0$, and the nonnegativity of the integrals $\int_\Omega \frac{1}{H_{\delta_1}}(w_2^2+w_3^2)\,dx$ and $\int_\Omega(|\mathbf{u}|\mathbf{u}-|\mathbf{v}|\mathbf{v})\cdot\mathbf{w}\,dx$, the equality (30) gives

$$
\begin{aligned}
&(1+m)\left\|\sqrt{H_{\delta_1}}\nabla\mathbf{w}\right\|_2^2 + \sigma^2\left\|\sqrt{H_{\delta_1}}\mathbf{w}\right\|_2^2 \le \left|Re\int_\Omega H_{\delta_1}\big((\mathbf{w}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx\right|\\
&+\left|\tilde{\delta}Re\int_\Omega x_2\big(\mathbf{v}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx\right| + \left|Re\delta_1\int_\Omega v_3w_3w_2\,dx\right| + \left|\tilde{\delta}Re\int_\Omega v_3^2w_2\,dx\right|\\
&+\left|Re\delta_1\int_\Omega v_2w_3^2\,dx\right| + \left|\tilde{\delta}Re\int_\Omega v_2v_3w_3\,dx\right| + \left|Re\int_\Omega H_{\delta_1}\nabla\pi\cdot\mathbf{w}\,dx\right|\\
&+\left|\tilde{\delta}Re\int_\Omega x_2\nabla q\cdot\mathbf{w}\,dx\right| + \left|m\int_\Omega H_{\delta_1}(\nabla\times\boldsymbol{\phi})\cdot\mathbf{w}\,dx\right| + \left|\tilde{\delta}m\int_\Omega x_2(\nabla\times\boldsymbol{\eta})\cdot\mathbf{w}\,dx\right|
\end{aligned}
$$

$$+\left|m\delta_1\int_\Omega \phi_3 w_1\,dx\right|+\left|\tilde{\delta}m\int_\Omega \eta_3 w_1\,dx\right|+\left|\tilde{\delta}(1+m)\int_\Omega \nabla\cdot(x_2\nabla\mathbf{v})\cdot\mathbf{w}\,dx\right|$$
$$+\left|\tilde{\delta}(1+m)(\delta_1+\delta_2)\int_\Omega \frac{1}{H_{\delta_1}}(v_2w_2+v_3w_3)\,dx\right|$$
$$+\left|\tilde{\delta}\delta_2^2(1+m)\int_\Omega \frac{x_2}{H_{\delta_1}H_{\delta_2}}(v_2w_2+v_3w_3)\,dx\right|+\left|\tilde{\delta}\sigma^2\int_\Omega x_2\mathbf{v}\cdot\mathbf{w}\,dx\right|$$
$$+\left|\tilde{\delta}Re\sigma C_F\int_\Omega x_2|\mathbf{v}|\mathbf{v}\cdot\mathbf{w}\,dx\right|. \tag{31}$$

By Hölder's and Sobolev inequalities together with (26), the terms of the right-hand side of (31) can be estimated as follows:

$$\left|Re\int_\Omega H_{\delta_1}\big((\mathbf{w}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{w}\,dx\right|\le \mathfrak{C}ReE_{\mathbf{v}}\|\nabla\mathbf{w}\|_2^2,$$

$$\left|\tilde{\delta}Re\int_\Omega x_2\big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\big)\cdot\mathbf{w}\,dx\right|\le \mathfrak{C}|\tilde{\delta}|ReE_{\mathbf{v}}^2\|\nabla\mathbf{w}\|_2,$$

$$\left|Re\delta_1\int_\Omega v_3w_3w_2\,dx\right|\le \mathfrak{C}Re\delta_1E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2^2,$$

$$\left|\tilde{\delta}Re\int_\Omega v_3^2w_2\,dx\right|\le \mathfrak{C}|\tilde{\delta}|ReE_{\mathbf{v}}^2\|\nabla\mathbf{w}\|_2,$$

$$\begin{aligned}\left|Re\int_\Omega H_{\delta_1}\nabla\pi\cdot\mathbf{w}\,dx\right|&=\left|Re\int_\Omega \pi\nabla\cdot(H_{\delta_1}\mathbf{w})\,dx\right|\\&=\left|Re\int_\Omega \pi\big(-\tilde{\delta}\nabla\cdot(x_2\mathbf{v})\big)\,dx\right|\ \text{(by (26))}\\&\le|\tilde{\delta}|Re\|\pi\|_2\|x_2\nabla\cdot\mathbf{v}+v_2\|_2\\&\le\mathfrak{C}|\tilde{\delta}|Re(\|p\|_2+\|q\|_2)\|\nabla\mathbf{v}\|_2\\&\le\mathfrak{C}|\tilde{\delta}|Re(E_p+E_q)E_{\mathbf{v}},\end{aligned}$$

$$\begin{aligned}\left|\tilde{\delta}Re\int_\Omega x_2\nabla q\cdot\mathbf{w}\,dx\right|&=\left|\tilde{\delta}Re\int_\Omega q\nabla\cdot(x_2\mathbf{w})\,dx\right|\\&=\left|\tilde{\delta}Re\int_\Omega q(x_2\nabla\cdot\mathbf{w}+w_2)\,dx\right|\\&\le|\tilde{\delta}|Re\|q\|_2\big(\|x_2\|_\infty\|\nabla\cdot\mathbf{w}\|_2+\|\mathbf{w}\|_2\big)\\&\le\mathfrak{C}|\tilde{\delta}|ReE_q\|\nabla\mathbf{w}\|_2,\end{aligned}$$

$$\left|m\int_\Omega H_{\delta_1}(\nabla\times\boldsymbol{\phi})\cdot\mathbf{w}\,dx\right|\le\mathfrak{C}m\|\nabla\boldsymbol{\phi}\|_2\|\nabla\mathbf{w}\|_2,$$

$$\left|\tilde{\delta}m\int_\Omega x_2(\nabla\times\boldsymbol{\eta})\cdot\mathbf{w}\,dx\right|\le\mathfrak{C}|\tilde{\delta}|mE_{\boldsymbol{\eta}}\|\nabla\mathbf{w}\|_2,$$

$$\left|m\delta_1\int_\Omega\phi_3 w_1\,dx\right|\le\mathfrak{C}m\delta_1\|\nabla\boldsymbol{\phi}\|_2\|\nabla\mathbf{w}\|_2,$$

$$\left|\tilde{\delta}m\int_\Omega\eta_3 w_1\,dx\right|\le\mathfrak{C}|\tilde{\delta}|mE_{\boldsymbol{\eta}}\|\nabla\mathbf{w}\|_2,$$

$$\begin{aligned}\left|\tilde{\delta}(1+m)\int_\Omega\nabla\cdot(x_2\nabla\mathbf{v})\cdot\mathbf{w}\,dx\right|&=\left|\tilde{\delta}(1+m)\int_\Omega x_2\nabla\mathbf{v}:\nabla\mathbf{w}\,dx\right|\\&\le\mathfrak{C}|\tilde{\delta}|(1+m)E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2,\end{aligned}$$

$$\left|\tilde{\delta}(1+m)(\delta_1+\delta_2)\int_\Omega\frac{1}{H_{\delta_1}}(v_2w_2+v_3w_3)\,dx\right|\le\mathfrak{C}|\tilde{\delta}|(1+m)\frac{(\delta_1+\delta_2)}{H^0_{\delta_1}}E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2,$$

$$\left|\tilde{\delta}\delta_2^2(1+m)\int_\Omega\frac{x_2}{H_{\delta_1}H_{\delta_2}}(v_2w_2+v_3w_3)\,dx\right|\le\mathfrak{C}|\tilde{\delta}|(1+m)\frac{\delta_2^2}{H^0_{\delta_1}H^0_{\delta_2}}E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2,$$

$$\left|\tilde{\delta}\sigma^2\int_\Omega x_2\mathbf{v}\cdot\mathbf{w}\,dx\right|\le\mathfrak{C}|\tilde{\delta}|\sigma^2E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2,$$

and

$$\left|\tilde{\delta}Re\sigma C_F\int_\Omega x_2|\mathbf{v}|\mathbf{v}\cdot\mathbf{w}\,dx\right|\le\mathfrak{C}|\tilde{\delta}|Re\sigma C_FE_{\mathbf{v}}^2\|\nabla\mathbf{w}\|_2,$$

where $H^0_{\delta_1}:=\inf_\Omega H_{\delta_1}$ and $H^0_{\delta_2}:=\inf_\Omega H_{\delta_2}$.
In view of these, the inequality (31) becomes

$$\begin{aligned}(1+m)H^0_{\delta_1}\|\nabla\mathbf{w}\|_2^2\le\mathfrak{C}\Bigg\{&Re(1+\delta_1)E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2^2+|\tilde{\delta}|\|\nabla\mathbf{w}\|_2\Bigg(Re(1+\sigma C_F)E_{\mathbf{v}}^2\\&+ReE_q+mE_{\boldsymbol{\eta}}+(1+m)\left(1+\frac{\delta_1+\delta_2}{H^0_{\delta_1}}+\frac{\delta_2^2}{H^0_{\delta_1}H^0_{\delta_2}}\right)E_{\mathbf{v}}+\sigma^2E_{\mathbf{v}}\Bigg)\\&+|\tilde{\delta}|Re(E_p+E_q)E_{\mathbf{v}}+m(1+\delta_1)\|\nabla\boldsymbol{\phi}\|_2\|\nabla\mathbf{w}\|_2\Bigg\}.\end{aligned}\tag{32}$$

Taking the $L^2(\Omega)$-inner product on both sides of (28) with $\boldsymbol{\phi}$, we get

$$
\begin{aligned}
&ReJ\int_\Omega H_{\delta_1}\Big((\mathbf{u}\cdot\nabla)\boldsymbol{\phi}\Big)\cdot\boldsymbol{\phi}\,dx + ReJ\int_\Omega H_{\delta_1}\Big((\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\Big)\cdot\boldsymbol{\phi}\,dx\\
&+\tilde{\delta}ReJ\int_\Omega x_2\Big((\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\Big)\cdot\boldsymbol{\phi}\,dx - ReJ\delta_1\int_\Omega w_3\eta_3\phi_2\,dx - \tilde{\delta}ReJ\int_\Omega v_3\eta_3\phi_2\,dx\\
&+ReJ\delta_1\int_\Omega w_3\eta_2\phi_3\,dx + \tilde{\delta}ReJ\int_\Omega v_3\eta_2\phi_3\,dx\\
&= -2m\int_\Omega H_{\delta_1}\boldsymbol{\phi}\cdot\boldsymbol{\phi}\,dx - 2\tilde{\delta}m\int_\Omega x_2\boldsymbol{\eta}\cdot\boldsymbol{\phi}\,dx + m\int_\Omega H_{\delta_1}(\nabla\times\mathbf{w})\cdot\boldsymbol{\phi}\,dx\\
&+\tilde{\delta}m\int_\Omega x_2(\nabla\times\mathbf{v})\cdot\boldsymbol{\phi}\,dx + m\delta_1\int_\Omega w_3\phi_1\,dx + \tilde{\delta}m\int_\Omega v_3\phi_1\,dx\\
&+n_1\int_\Omega \nabla\cdot(H_{\delta_1}\nabla\boldsymbol{\phi})\cdot\boldsymbol{\phi}\,dx + \tilde{\delta}n_1\int_\Omega \nabla\cdot(x_2\nabla\boldsymbol{\eta})\cdot\boldsymbol{\phi}\,dx\\
&-n_1\delta_1^2\int_\Omega\frac{1}{H_{\delta_1}}(\phi_2^2+\phi_3^2)\,dx - \tilde{\delta}n_1(\delta_1+\delta_2)\int_\Omega\frac{1}{H_{\delta_1}}(\eta_2\phi_2+\eta_3\phi_3)\,dx\\
&+\tilde{\delta}n_1\delta_2^2\int_\Omega\frac{x_2}{H_{\delta_1}H_{\delta_2}}(\eta_2\phi_2+\eta_3\phi_3)\,dx + (n_2+n_3)\int_\Omega H_{\delta_1}\nabla(\nabla\cdot\boldsymbol{\phi})\cdot\boldsymbol{\phi}\,dx\\
&+\tilde{\delta}(n_2+n_3)\int_\Omega x_2\nabla(\nabla\cdot\boldsymbol{\eta})\cdot\boldsymbol{\phi}\,dx + (n_2+n_3)\delta_1\int_\Omega\nabla\phi_2\cdot\boldsymbol{\phi}\,dx\\
&+\tilde{\delta}(n_2+n_3)\int_\Omega\nabla\eta_2\cdot\boldsymbol{\phi}\,dx - (n_2+n_3)\delta_1^2\int_\Omega\frac{1}{H_{\delta_1}}\phi_2^2\,dx\\
&-\tilde{\delta}(n_2+n_3)(\delta_1+\delta_2)\int_\Omega\frac{1}{H_{\delta_1}}\eta_2\phi_2\,dx + \tilde{\delta}(n_2+n_3)\delta_2^2\int_\Omega\frac{x_2}{H_{\delta_1}H_{\delta_2}}\eta_2\phi_2\,dx.
\end{aligned}
\tag{33}
$$

After simplification, (33) turns out to be

$$
\begin{aligned}
n_1H^0_{\delta_1}\|\nabla\boldsymbol{\phi}\|_2^2 + 2mH^0_{\delta_1}\|\boldsymbol{\phi}\|_2^2 &\le \left|ReJ\int_\Omega H_{\delta_1}\Big((\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\Big)\cdot\boldsymbol{\phi}\,dx\right|\\
&+\left|\tilde{\delta}ReJ\int_\Omega x_2\Big((\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\Big)\cdot\boldsymbol{\phi}\,dx\right| + \left|ReJ\delta_1\int_\Omega w_3\eta_3\phi_2\,dx\right|\\
&+\left|\tilde{\delta}ReJ\int_\Omega v_3\eta_3\phi_2\,dx\right| + \left|ReJ\delta_1\int_\Omega w_3\eta_2\phi_3\,dx\right| + \left|\tilde{\delta}ReJ\int_\Omega v_3\eta_2\phi_3\,dx\right|\\
&+\left|2\tilde{\delta}m\int_\Omega x_2\boldsymbol{\eta}\cdot\boldsymbol{\phi}\,dx\right| + \left|m\int_\Omega H_{\delta_1}(\nabla\times\mathbf{w})\cdot\boldsymbol{\phi}\,dx\right| + \left|\tilde{\delta}m\int_\Omega x_2(\nabla\times\mathbf{v})\cdot\boldsymbol{\phi}\,dx\right|\\
&+\left|m\delta_1\int_\Omega w_3\phi_1\,dx\right| + \left|\tilde{\delta}m\int_\Omega v_3\phi_1\,dx\right| + \left|\tilde{\delta}n_1\int_\Omega\nabla\cdot(x_2\nabla\boldsymbol{\eta})\cdot\boldsymbol{\phi}\,dx\right|
\end{aligned}
$$

$$+\left|\tilde{\delta}n_1(\delta_1+\delta_2)\int_\Omega \frac{1}{H_{\delta_1}}(\eta_2\phi_2+\eta_3\phi_3)\,dx\right|+\left|\tilde{\delta}n_1\delta_2^2\int_\Omega \frac{x_2}{H_{\delta_1}H_{\delta_2}}(\eta_2\phi_2+\eta_3\phi_3)\,dx\right|$$
$$+\left|(n_2+n_3)\delta_1\int_\Omega \phi_2(\nabla\cdot\boldsymbol{\phi})\,dx\right|+\left|\tilde{\delta}(n_2+n_3)\int_\Omega x_2(\nabla\cdot\boldsymbol{\eta})(\nabla\cdot\boldsymbol{\phi})\,dx\right|$$
$$+\left|\tilde{\delta}(n_2+n_3)\int_\Omega \phi_2(\nabla\cdot\boldsymbol{\eta})\,dx\right|+\left|(n_2+n_3)\delta_1\int_\Omega \nabla\phi_2\cdot\boldsymbol{\phi}\,dx\right|$$
$$+\left|\tilde{\delta}(n_2+n_3)\int_\Omega \nabla\eta_2\cdot\boldsymbol{\phi}\,dx\right|+\left|\tilde{\delta}(n_2+n_3)(\delta_1+\delta_2)\int_\Omega \frac{1}{H_{\delta_1}}\eta_2\phi_2\,dx\right|$$
$$+\left|\tilde{\delta}(n_2+n_3)\delta_2^2\int_\Omega \frac{x_2}{H_{\delta_1}H_{\delta_2}}\eta_2\phi_2\,dx\right|. \tag{34}$$

The terms appearing on the right-hand side of (34) admit the following bounds.

$$\left|ReJ\int_\Omega H_{\delta_1}\Big((\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\Big)\cdot\boldsymbol{\phi}\,dx\right|\le \mathfrak{C}ReJE_{\boldsymbol{\eta}}\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2,$$

$$\left|\tilde{\delta}ReJ\int_\Omega x_2\Big((\mathbf{v}\cdot\nabla)\boldsymbol{\eta}\Big)\cdot\boldsymbol{\phi}\,dx\right|\le \mathfrak{C}|\tilde{\delta}|ReJE_{\mathbf{v}}E_{\boldsymbol{\eta}}\|\nabla\boldsymbol{\phi}\|_2,$$

$$\left|ReJ\delta_1\int_\Omega w_3\eta_3\phi_2\,dx\right|\le \mathfrak{C}ReJ\delta_1E_{\boldsymbol{\eta}}\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2,$$

$$\left|\tilde{\delta}ReJ\int_\Omega v_3\eta_3\phi_2\,dx\right|\le \mathfrak{C}|\tilde{\delta}|ReJE_{\mathbf{v}}E_{\boldsymbol{\eta}}\|\nabla\boldsymbol{\phi}\|_2,$$

$$\left|2\tilde{\delta}m\int_\Omega x_2\boldsymbol{\eta}\cdot\boldsymbol{\phi}\,dx\right|\le \mathfrak{C}|\tilde{\delta}|mE_{\boldsymbol{\eta}}\|\nabla\boldsymbol{\phi}\|_2,$$

$$\left|m\int_\Omega H_{\delta_1}(\nabla\times\mathbf{w})\cdot\boldsymbol{\phi}\,dx\right|\le \mathfrak{C}m\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2,$$

$$\left|\tilde{\delta}m\int_\Omega x_2(\nabla\times\mathbf{v})\cdot\boldsymbol{\phi}\,dx\right|\le \mathfrak{C}|\tilde{\delta}|mE_{\mathbf{v}}\|\nabla\boldsymbol{\phi}\|_2,$$

$$\left|m\delta_1\int_\Omega w_3\phi_1\,dx\right|\le \mathfrak{C}m\delta_1\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2,$$

$$\left|\tilde{\delta} m \int_\Omega v_3 \phi_1 \, dx\right| \le \mathfrak{C}|\tilde{\delta}| m E_{\mathbf{v}} \|\nabla \boldsymbol{\phi}\|_2,$$

$$\left|\tilde{\delta} n_1 \int_\Omega \nabla \cdot (x_2 \nabla \boldsymbol{\eta}) \cdot \boldsymbol{\phi} \, dx\right| = \left|\tilde{\delta} n_1 \int_\Omega x_2 \nabla \boldsymbol{\eta} : \nabla \boldsymbol{\phi} \, dx\right| \le \mathfrak{C}|\tilde{\delta}| n_1 E_{\boldsymbol{\eta}} \|\nabla \boldsymbol{\phi}\|_2,$$

$$\left|\tilde{\delta} n_1 (\delta_1 + \delta_2) \int_\Omega \frac{1}{H_{\delta_1}} (\eta_2 \phi_2 + \eta_3 \phi_3) \, dx\right| \le \mathfrak{C}|\tilde{\delta}| \frac{n_1 (\delta_1 + \delta_2)}{H^0_{\delta_1}} E_{\boldsymbol{\eta}} \|\nabla \boldsymbol{\phi}\|_2,$$

$$\left|\tilde{\delta} n_1 \delta_2^2 \int_\Omega \frac{x_2}{H_{\delta_1} H_{\delta_2}} (\eta_2 \phi_2 + \eta_3 \phi_3) \, dx\right| \le \mathfrak{C}|\tilde{\delta}| \frac{n_1 \delta_2^2}{H^0_{\delta_1} H^0_{\delta_2}} E_{\boldsymbol{\eta}} \|\nabla \boldsymbol{\phi}\|_2,$$

$$\left|(n_2 + n_3) \delta_1 \int_\Omega \phi_2 (\nabla \cdot \boldsymbol{\phi}) \, dx\right| \le \mathfrak{C}(n_2 + n_3) \delta_1 \|\nabla \boldsymbol{\phi}\|_2^2,$$

$$\left|\tilde{\delta} (n_2 + n_3) \int_\Omega x_2 (\nabla \cdot \boldsymbol{\eta})(\nabla \cdot \boldsymbol{\phi}) \, dx\right| \le |\tilde{\delta}| (n_2 + n_3) E_{\boldsymbol{\eta}} \|\nabla \boldsymbol{\phi}\|_2,$$

$$\left|\tilde{\delta} (n_2 + n_3) \int_\Omega \phi_2 (\nabla \cdot \boldsymbol{\eta}) \, dx\right| \le |\tilde{\delta}| (n_2 + n_3) E_{\boldsymbol{\eta}} \|\nabla \boldsymbol{\phi}\|_2,$$

$$\left|(n_2 + n_3) \delta_1 \int_\Omega \nabla \phi_2 \cdot \boldsymbol{\phi} \, dx\right| \le \mathfrak{C}(n_2 + n_3) \delta_1 \|\nabla \boldsymbol{\phi}\|_2^2,$$

$$\left|\tilde{\delta} (n_2 + n_3) \int_\Omega \nabla \eta_2 \cdot \boldsymbol{\phi} \, dx\right| \le \mathfrak{C}|\tilde{\delta}| (n_2 + n_3) E_{\boldsymbol{\eta}} \|\nabla \boldsymbol{\phi}\|_2,$$

$$\left|\tilde{\delta} (n_2 + n_3)(\delta_1 + \delta_2) \int_\Omega \frac{1}{H_{\delta_1}} \eta_2 \phi_2 \, dx\right| \le \mathfrak{C}|\tilde{\delta}| \frac{(n_2 + n_3)(\delta_1 + \delta_2)}{H^0_{\delta_1}} E_{\boldsymbol{\eta}} \|\nabla \boldsymbol{\phi}\|_2,$$

and

$$\left|\tilde{\delta} (n_2 + n_3) \delta_2^2 \int_\Omega \frac{x_2}{H_{\delta_1} H_{\delta_2}} \eta_2 \phi_2 \, dx\right| \le \mathfrak{C}|\tilde{\delta}| \frac{(n_2 + n_3) \delta_2^2}{H^0_{\delta_1} H^0_{\delta_2}} E_{\boldsymbol{\eta}} \|\nabla \boldsymbol{\phi}\|_2.$$

Taking into account the preceding estimates, (34) gives

$$n_1 H^0_{\delta_1}\|\nabla\boldsymbol{\phi}\|_2^2 \le \mathfrak{C}\Bigg\{\delta_1(n_2+n_3)\|\nabla\boldsymbol{\phi}\|_2^2 + (1+\delta_1)\big(m+ReJE_{\boldsymbol{\eta}}\big)\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2 + |\tilde{\delta}|\|\nabla\boldsymbol{\phi}\|_2\Bigg((n_1+n_2+n_3)\Bigg(1+\frac{\delta_1+\delta_2}{H^0_{\delta_1}}+\frac{\delta_2^2}{H^0_{\delta_1}H^0_{\delta_2}}\Bigg)E_{\boldsymbol{\eta}} + mE_{\boldsymbol{\eta}} + \big(m+ReJE_{\boldsymbol{\eta}}\big)E_{\mathbf{v}}\Bigg)\Bigg\}. \tag{35}$$

Adding (32) and (35), we get

$$(1+m)H^0_{\delta_1}\|\nabla\mathbf{w}\|_2^2 + n_1H^0_{\delta_1}\|\nabla\boldsymbol{\phi}\|_2^2 \le \mathfrak{C}\big\{Re(1+\delta_1)E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2^2 + \delta_1(n_2+n_3)\|\nabla\boldsymbol{\phi}\|_2^2 + (1+\delta_1)\big(m+ReJE_{\boldsymbol{\eta}}\big)\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2 + |\tilde{\delta}|\mathfrak{D}_1\|\nabla\mathbf{w}\|_2 + |\tilde{\delta}|\mathfrak{D}_2\|\nabla\boldsymbol{\phi}\|_2 + |\tilde{\delta}|Re(E_p+E_q)E_{\mathbf{v}}\big\}, \tag{36}$$

where

$$\mathfrak{D}_1 = Re(1+\sigma C_F)E_{\mathbf{v}}^2 + ReE_q + mE_{\boldsymbol{\eta}} + (1+m)\left(1+\frac{\delta_1+\delta_2}{H^0_{\delta_1}}+\frac{\delta_2^2}{H^0_{\delta_1}H^0_{\delta_2}}\right)E_{\mathbf{v}} + \sigma^2E_{\mathbf{v}}$$

and

$$\mathfrak{D}_2 = (n_1+n_2+n_3)\left(1+\frac{\delta_1+\delta_2}{H^0_{\delta_1}}+\frac{\delta_2^2}{H^0_{\delta_1}H^0_{\delta_2}}\right)E_{\boldsymbol{\eta}} + mE_{\boldsymbol{\eta}} + \big(m+ReJE_{\boldsymbol{\eta}}\big)E_{\mathbf{v}}.$$

Applying Young's inequality, (36) yields

$$\frac{(1+m)H^0_{\delta_1}}{2}\|\nabla\mathbf{w}\|_2^2 + \frac{n_1H^0_{\delta_1}}{2}\|\nabla\boldsymbol{\phi}\|_2^2 \le \mathfrak{C}\Big\{Re(1+\delta_1)E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2^2 + \delta_1(n_2+n_3)\|\nabla\boldsymbol{\phi}\|_2^2 + \frac{(1+\delta_1)^2\big(m+ReJE_{\boldsymbol{\eta}}\big)^2}{(1+m)H^0_{\delta_1}}\|\nabla\boldsymbol{\phi}\|_2^2 + \left(\frac{\mathfrak{D}_1^2}{(1+m)H^0_{\delta_1}}+\frac{\mathfrak{D}_2^2}{n_1H^0_{\delta_1}}\right)\tilde{\delta}^2 + Re(E_p+E_q)E_{\mathbf{v}}|\tilde{\delta}|\Big\}$$

which implies

$$\left(\frac{(1+m)H^0_{\delta_1}}{2} - \mathfrak{C}Re(1+\delta_1)E_{\mathbf{v}}\right)\|\nabla\mathbf{w}\|_2^2$$
$$+\left(\frac{n_1 H^0_{\delta_1}}{2} - \mathfrak{C}\delta_1(n_2+n_3) - \mathfrak{C}\frac{(1+\delta_1)^2\left(m+ReJE_{\boldsymbol{\eta}}\right)^2}{(1+m)H^0_{\delta_1}}\right)\|\nabla\boldsymbol{\phi}\|_2^2$$
$$\leq \mathfrak{C}\left(\frac{\mathfrak{D}_1^2}{(1+m)H^0_{\delta_1}} + \frac{\mathfrak{D}_2^2}{n_1 H^0_{\delta_1}}\right)\tilde{\delta}^2 + \mathfrak{C}Re(E_p+E_q)E_{\mathbf{v}}|\tilde{\delta}|. \tag{37}$$

The desired result follows from (37) for sufficiently small values of $Re$, $m$, and $\delta_1$. □

**Remark 2.** *In the particular case when $\delta_1 = 0$ and $\delta_2 \to 0$, inequality (37) implies that the solution $\mathbf{v}$ converges to $\mathbf{u}$ in $\mathcal{X}$ and $\boldsymbol{\eta}$ converges to $\boldsymbol{\nu}$ in $\left(H_0^1(\Omega)\right)^3$. Hence, we conclude that the solution of the curved pipe flow converges to that of the straight pipe flow when $\delta \to 0$.*

### 3.3 Continuous dependence on the Forchheimer coefficient ($C_F$)

Let $(\mathbf{u}, p, \boldsymbol{\nu})$ and $(\mathbf{v}, q, \boldsymbol{\eta})$ be the solutions of the BVP (1)-(4) corresponding to the Forchheimer coefficients $C_{F_1}$ and $C_{F_2}$, respectively. Then the differences $(\mathbf{w}, \pi, \boldsymbol{\phi}) := (\mathbf{u}-\mathbf{v}, p-q, \boldsymbol{\nu}-\boldsymbol{\eta})$ solve the problem

$$\nabla\cdot(H\mathbf{w}) = 0, \tag{38}$$

$$ReH\Big((\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v}\Big) - Re\delta u_3 w_3\hat{a}_2 - Re\delta v_3 w_3\hat{a}_2 + Re\delta u_3 w_2\hat{a}_3$$
$$+Re\delta w_3 v_2\hat{a}_3$$
$$= -ReH\nabla\pi + mH\nabla\times\boldsymbol{\phi} + m\delta\phi_3\hat{a}_1 + (1+m)\nabla\cdot(H\nabla\mathbf{w})$$
$$-(1+m)\frac{\delta^2}{H}(w_2\hat{a}_2 + w_3\hat{a}_3) - \sigma^2 H\mathbf{w} - Re\sigma C_{F_1}H(|\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v})$$
$$-\widetilde{C_F}Re\sigma H|\mathbf{v}|\mathbf{v}, \tag{39}$$

$$ReHJ\Big((\mathbf{u}\cdot\nabla)\boldsymbol{\phi}+(\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\Big)-ReJ\delta u_3\phi_3\hat{a}_2-ReJ\delta w_3\eta_3\hat{a}_2+ReJ\delta u_3\phi_2\hat{a}_3$$
$$+ReJ\delta w_3\eta_2\hat{a}_3$$
$$=-2mH\boldsymbol{\phi}+mH\nabla\times\mathbf{w}+m\delta w_3\hat{a}_1+n_1\nabla\cdot(H\nabla\boldsymbol{\phi})-n_1\frac{\delta^2}{H}(\phi_2\hat{a}_2$$
$$+\phi_3\hat{a}_3)+(n_2+n_3)H\nabla(\nabla\cdot\boldsymbol{\phi})+(n_2+n_3)\delta\nabla\phi_2-(n_2+n_3)\frac{\delta^2}{H}\phi_2\hat{a}_2 \quad (40)$$
$$\text{in } \Omega,$$

and

$$\mathbf{w}=\mathbf{0}=\boldsymbol{\phi} \text{ on } \partial\Omega, \quad (41)$$

where $\mathbf{w}=(w_1,w_2,w_3)$, $\boldsymbol{\phi}=(\phi_1,\phi_2,\phi_3)$, $w_i=u_i-v_i$, $\phi_i=\nu_i-\eta_i$ $(i=1,2,3)$ and $\widetilde{C_F}=C_{F_1}-C_{F_2}$.

The following theorem establishes the continuous dependence of the solution on the Forchheimer coefficient $(C_F)$.

**Theorem 3.** *Suppose that* $(\mathbf{w},\pi,\boldsymbol{\phi})$ *solves the BVP* (38)-(41). *Then, for sufficiently small values of* $Re$, $m$ *and* $\delta$, $\mathbf{w}\to\mathbf{0}$ *in* $\mathcal{X}$ *and* $\boldsymbol{\phi}\to\mathbf{0}$ *in* $(H_0^1(\Omega))^3$ *as* $C_{F_2}\to C_{F_1}$.

*Proof.* Taking the $L^2(\Omega)$-inner product of (39) with $\mathbf{w}$ yields the estimate

$$(1+m)H_0\|\nabla\mathbf{w}\|_2^2$$
$$\le\mathfrak{C}\Big(Re(1+\delta)E_{\mathbf{v}}\|\nabla\mathbf{w}\|_2^2+m(1+\delta)\|\nabla\boldsymbol{\phi}\|_2\|\nabla\mathbf{w}\|_2+\widetilde{C_F}Re\sigma E_{\mathbf{v}}^2\|\nabla\mathbf{w}\|_2\Big). \quad (42)$$

Taking the $L^2(\Omega)$-inner product of (40) with $\boldsymbol{\phi}$, we obtain the estimate

$$n_1H_0\|\nabla\boldsymbol{\phi}\|_2^2\le\mathfrak{C}\Big((1+\delta)\big(m+ReJE_{\boldsymbol{\eta}}\big)\|\nabla\mathbf{w}\|_2\|\nabla\boldsymbol{\phi}\|_2+\delta(n_2+n_3)\|\nabla\boldsymbol{\phi}\|_2^2\Big). \quad (43)$$

Adding (42) and (43) and using the Young's inequality, we get

$$\left(\frac{(1+m)H_0}{2}-\mathfrak{C}Re(1+\delta)E_{\mathbf{v}}\right)\|\nabla\mathbf{w}\|_2^2$$
$$+\left(n_1H_0-\mathfrak{C}\delta(n_2+n_3)-\mathfrak{C}\frac{(1+\delta)^2\big(m+ReJE_{\boldsymbol{\eta}}\big)^2}{(1+m)H_0}\right)\|\nabla\boldsymbol{\phi}\|_2^2$$
$$\le\mathfrak{C}\frac{Re^2\sigma^2E_{\mathbf{v}}^4}{(1+m)H_0}\widetilde{C_F}^2. \quad (44)$$

The inequality (44) shows that $\|\nabla \mathbf{w}\|_2 \to 0$ and $\|\nabla \boldsymbol{\phi}\|_2 \to 0$ as $\widetilde{C_F} \to 0$ for sufficiently small values of $Re$, $m$ and $\delta$. Hence, the required result is proved. □

**Remark 3.** *For a particular case $C_{F_1} = 0$, when $C_{F_2} \to 0$, the inequality (44) implies that the solution to the Darcy–Lapwood –Forchheimer–Brinkman model for micropolar fluid flow in a curved pipe converges to that of the Darcy–Lapwood–Brinkman model.*

### 3.4 Continuous dependence on the micropolarity parameter $(m)$

Let $(\mathbf{u}, p, \boldsymbol{\nu})$ and $(\mathbf{v}, q, \boldsymbol{\eta})$ be the solutions of the BVP (1)-(4) with micropolarity parameters $m_1$ and $m_2$, respectively. The difference fields $(\mathbf{w}, \pi, \boldsymbol{\phi}) := (\mathbf{u} - \mathbf{v}, p - q, \boldsymbol{\nu} - \boldsymbol{\eta})$, then satisfy the problem

$$\nabla \cdot (H\mathbf{w}) = 0, \tag{45}$$

$$\begin{aligned} &ReH\Big((\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v}\Big) - Re\delta u_3 w_3 \hat{a}_2 - Re\delta v_3 w_3 \hat{a}_2 + Re\delta u_3 w_2 \hat{a}_3 \\ &\qquad + Re\delta w_3 v_2 \hat{a}_3 \\ &= -ReH\nabla\pi + m_1 H\nabla\times\boldsymbol{\phi} + \widetilde{m}H\nabla\times\boldsymbol{\eta} + m_1\delta\phi_3\hat{a}_1 + \widetilde{m}\delta\eta_3\hat{a}_1 \\ &+(1+m_1)\nabla\cdot(H\nabla\mathbf{w}) + \widetilde{m}\nabla\cdot(H\nabla\mathbf{v}) - (1+m_1)\frac{\delta^2}{H}(w_2\hat{a}_2 + w_3\hat{a}_3) \\ &\qquad -\widetilde{m}\frac{\delta^2}{H}(v_2\hat{a}_2 + v_3\hat{a}_3) - \sigma^2 H\mathbf{w} - Re\sigma C_F H(|\mathbf{u}|\mathbf{u} - |\mathbf{v}|\mathbf{v}), \end{aligned} \tag{46}$$

$$\begin{aligned} &ReHJ\Big((\mathbf{u}\cdot\nabla)\boldsymbol{\phi} + (\mathbf{w}\cdot\nabla)\boldsymbol{\eta}\Big) - ReJ\delta u_3\phi_3\hat{a}_2 - ReJ\delta w_3\eta_3\hat{a}_2 + ReJ\delta u_3\phi_2\hat{a}_3 \\ &\qquad + ReJ\delta w_3\eta_2\hat{a}_3 \\ &= -2m_1 H\boldsymbol{\phi} - 2\widetilde{m}H\boldsymbol{\eta} + m_1 H\nabla\times\mathbf{w} + \widetilde{m}H\nabla\times\mathbf{v} + m_1\delta w_3\hat{a}_1 + \widetilde{m}\delta v_3\hat{a}_1 \\ &+n_1\nabla\cdot(H\nabla\boldsymbol{\phi}) - n_1\frac{\delta^2}{H}(\phi_2\hat{a}_2 + \phi_3\hat{a}_3) + (n_2+n_3)H\nabla(\nabla\cdot\boldsymbol{\phi}) \\ &\qquad +(n_2+n_3)\delta\nabla\phi_2 - (n_2+n_3)\frac{\delta^2}{H}\phi_2\hat{a}_2 \end{aligned} \tag{47}$$

in $\Omega$,

and

$$\mathbf{w} = \mathbf{0} = \boldsymbol{\phi} \text{ on } \partial\Omega, \tag{48}$$

where $\mathbf{w} = (w_1, w_2, w_3)$, $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, $w_i = u_i - v_i$, $\phi_i = \nu_i - \eta_i$ $(i = 1, 2, 3)$ and $\widetilde{m} = m_1 - m_2$.

Our main result on the continuous dependence on the micropolarity parameter $(m)$ is formulated below.

**Theorem 4.** *If $(\mathbf{w}, \pi, \boldsymbol{\phi})$ is the solution of (45)-(48), then $\mathbf{w} \to \mathbf{0}$ in $\mathcal{X}$ and $\boldsymbol{\phi} \to \mathbf{0}$ in $(H_0^1(\Omega))^3$ as $m_2 \to m_1$, provided the values of $Re$, $m_1$ and $\delta$ are sufficiently small.*

*Proof.* Adopting the same line of reasoning from the preceding results, we infer that

$$\begin{aligned}
&\left(\frac{(1+m_1)H_0}{2} - \mathfrak{C}Re(1+\delta)E_{\mathbf{v}}\right)\|\nabla\mathbf{w}\|_2^2 \\
&+\left(\frac{n_1H_0}{2} - \mathfrak{C}\delta(n_2+n_3) - \mathfrak{C}\frac{(1+\delta)^2\left(m_1 + ReJE_{\boldsymbol{\eta}}\right)^2}{(1+m_1)H_0}\right)\|\nabla\boldsymbol{\phi}\|_2^2 \\
&\leq \mathfrak{C}\left(\frac{\left(E_{\boldsymbol{\eta}}(1+\delta) + E_{\mathbf{v}}\left(1+\frac{\delta^2}{H_0}\right)\right)^2}{(1+m_1)H_0} + \frac{\left(E_{\boldsymbol{\eta}} + E_{\mathbf{v}}(1+\delta)\right)^2}{n_1H_0}\right)\widetilde{m}^2. \qquad (49)
\end{aligned}$$

It is clear from (49) that, for sufficiently small values of $Re$, $m_1$ and $\delta$, the required result holds. □

**Remark 4.** *It is well known that in the absence of microrotational effects the system of micropolar equations reduced to the system of Navier-Stokes equations; it is also known that when the microrotation viscosity equals to zero, the linear velocity vector and the microrotation vector are uncoupled and the global motion is unaffected by the microrotation vector [8]. The same can be achieved from (49) by taking $m_1 = 0$ and allowing $m_2 \to 0$.*

## References

[1] K.A. Ames and B. Straughan, *Non-Standard and Improperly Posed Problems*, Academic Press, San Diego, 1997.

[2] K.A. Ames and L.E. Payne, On stabilizing against modeling errors in a penetrative convection problem for a porous medium, *Math. Models Methods Appl. Sci.* 4 (1994), 733-740.

[3] N. Arada, On generalized Newtonian fluids in curved pipes, *SIAM J. Math. Anal.* 48 (2016), 1210-1249.

[4] F. Boyer and P. Fabrie, *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models*, Springer, New York, 2012.

[5] H. Brezis, *Functional Analysis, Sobolev Space and Partial Differential Equations*, Springer, New York, 2011.

[6] A.O. Çelebi, V.K. Kalantarov and D. Uğurlu, Continuous dependence for the convective Brinkman–Forchheimer equations, *Appl. Anal.* 84 (2005), 877-888.

[7] A.O. Çelebi, V.K. Kalantarov and D. Uğurlu, On continuous dependence on coefficients of the Brinkman–Forchheimer equations, *Appl. Math. Lett.* 19 (2006), 801-807.

[8] A.C. Eringen, Theory of micropolar fluids, *J. Math. Mech.* 16 (1966), 1-18.

[9] M.H. Hamdan, Single-phase flow through porous channels a review of flow models and channel entry conditions, *Appl. Math. Comput.* 62 (1994), 203-222.

[10] C.T. Hsu and P. Cheng, Thermal dispersion in a porous medium, *Int. J. Heat Mass Transfer* 33 (1990), 1587-1597.

[11] Y. Liu, Convergence and continuous dependence for the Brinkman–Forchheimer equations, *Math. Comput. Modell.* 49 (2009), 1401-1415.

[12] G. Łukaszewicz, Long time behavior of 2D micropolar fluid flows, *Math. Comput. Modell.* 34 (2001), 487-509.

[13] A. Morro and B. Straughan, Continuous dependence on the source parameters for convective motion in porous media, *Nonlinear Anal.* 18 (1992), 307-315.

[14] L.E. Payne and B. Straughan, Structural stability for the Darcy equations of flow in porous media, *Proc. R. Soc. A* 454 (1974), 1691-1698.

[15] L.E. Payne and B. Straughan, Stability in the initial-time geometry problem for the Brinkman and Darcy equations of flow in porous media, *J. Math. Pures Appl.* 75 (1996), 225-271.

[16] L.E. Payne and B. Straughan, Analysis of the boundary condition at the interface between a viscous fluid and a porous medium and related modelling questions, *J. Math. Pures Appl.* 77 (1998), 317-354.

[17] L.E. Payne and B. Straughan, Convergence and continuous dependence for the Brinkman–Forchheimer equations, *Stud. Appl. Math.* 102 (1999), 419-439.

[18] A. Prabal and M. Devakar, On the existence, uniqueness and regularity of solutions of micropolar fluid flow through porous medium in a curved pipe, *Nonlinear Anal. Real World Appl.* 91 (2026), 104616.

[19] B. Straughan, *Stability and Wave Motion in Porous Media*, Applied Mathematical Sciences, Springer, New York, 2008.