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SOME TWO-FUNCTION TOPOLOGICAL MINIMAX THEOREMS*

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Dedicated to Professor Constantin Fetecau on the occasion of his 80th anniversary

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Abstract

In this note, we extend some topological minimax theorems of Ricceri (see [3]) to the case of two functions.

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1 Introduction and statements of the main results

Let X, Y be two non-empty sets and let φ be a real-valued function on $X \times Y$. Set

$$\varphi_* = \sup_{y \in Y} \inf_{x \in X} \varphi(x, y)$$

and

$$\varphi^* = \inf_{x \in X} \sup_{y \in Y} \varphi(x, y).$$

It is clear that

$$\varphi_* \leq \varphi^*.$$

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This is called the trivial minimax inequality. The opposite inequality

$$\varphi^* \leq \varphi_*$$

is called non-trivial minimax inequality and of course it is equivalent to the minimax equality

$$\varphi_* = \varphi^*. \tag{1}$$

Starting from the pioneristic work of von Neumann (see [10]), many results ensuring (see 1) were established. For an introductory bibliography see, for example, the classical survey of Simons (see [8]). In particular, in the last century many minimax theorems were established in which the function is defined in topological vector spaces with both vectorial and topological hypotheses. In other theorems the function is defined in topological spaces and the vectorial hypotheses are replaced by topological hypotheses: they are the so-called topological minimax theorems. Many topological minimax theorems have been established by Ricceri, (see, for example, [3], [5] and [6]). In particular, in [3], assuming that Y is a real interval, he proved some minimax theorems where the quasi-convexity of f in X (present in each of the most classical results) was replaced by inf-connectedness. This allowed one to obtain several consequences and applications in non-convex settings (see [4]).

Now, let $f, g : X \times Y \to \mathbb{R}$, with $f(x, y) \leq g(x, y)$ for every $x \in X, y \in Y$. We call non-trivial minimax inequality involving f, g the following

$$f^* \leq g_*. \tag{2}$$

So, if $f = g$, (2) is equivalent to (1). For a given minimax theorem for one function φ, it is an usual fact to see whether it is possible to find a two-function version of it. The most natural way to get this is to split the hypotheses on φ to f and g. For example, the two-function version of the most classical Fan-Sion's theorem ([9]) has been obtained by Simons ([7]). In [1] and [2] the two-function versions of the results of [5] and [6] have been established.

The aim of the present paper is to extend to two functions the above mentioned classical Ricceri's results of [3]. In particular, we will establish the Theorems 1, 2 and 3 below. To this end, if X is a topological space, I is a real interval and f, g are two real functions defined on $X \times I$, we will consider the following conditions:

H1) for every $(x, \lambda) \in X \times I$, one has $f(x, \lambda) \leq g(x, \lambda)$.

H2) for every $\lambda \in I$, $f(\cdot, \lambda)$ is lower semicontinuous.

H3) the functions f is upper semicontinuous in $X \times I$.

H4) for every $x \in X$ and for every $\alpha \in \mathbb{R}$ the set $\{\lambda \in I : f(x, \lambda) > \alpha\}$ is connected.

H5) for every $x \in X$, $g(x, \cdot)$ is upper semicontinuous.

H6) the functions g is lower semicontinuous in $X \times I$.

H7) there exists a dense subset $D \subseteq I$ such that, for every $\lambda \in D$, and for every $\alpha \in \mathbb{R}$, the set $\{x \in X : g(x, \lambda) < \alpha\}$ is connected.

H8) I is compact or there exists $\lambda_0 \in I$ such that, for every $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x, \lambda_0) \leq \alpha\}$ is compact.

H9) for every $\lambda \in I$ and for every $\alpha \in \mathbb{R}$ the set $\{x \in X : g(x, \lambda) \leq \alpha\}$ is connected.

Theorem 1. *Let X be a topological space, $I \subseteq \mathbb{R}$ an interval and $f, g : X \times I \to \mathbb{R}$ two functions satisfying conditions H1), H2, H4), H5), H7) and H8). Then, the relation (2) holds.*

Theorem 2. *Let X be a topological space, $I \subseteq \mathbb{R}$ a compact interval and $f, g : X \times I \to \mathbb{R}$ two functions satisfying conditions H1), H2), H3), H4), H5) and H7). Then, the relation (2) holds.*

Theorem 3. *Let X be a compact topological space, $I \subseteq \mathbb{R}$ an interval and $f, g : X \times I \to \mathbb{R}$ two functions satisfying conditions H1), H2), H4), H6) and H9). Then, the relation (2) holds.*

Observe that, if $f = g$, Theorems 1, 2 and 3 coincide with Theorem 1.1 of [3].

2 Proofs of the main results

The proofs of the Theorems 1, 2 and 3 are based on the Lemma 1 below (see [3], Theorem 2.5). Before stating it, we recall that, if X, Y are two non-empty sets and V is a subset of $X \times Y$, one has the following notations:

$* \; p_X(V) = \{x \in X : \exists \; y \in Y : (x, y) \in V\}$;
$* \; p_Y(V) = \{y \in Y : \exists \; x \in X : (x, y) \in V\}$;
$*$ if $x \in X$, $V_x = \{y \in Y : (x, y) \in V\}$;
$*$ if $y \in Y$, $V^y : \{x \in X : (x, y) \in V\}$.

Lemma 1. *Let X be a topological space, I a compact real interval, and let S, T two subsets of $X \times I$. Assume that at least one of the following sets of conditions are verified:*

(δ_1) *$p_I(S)$ is connected, S_x is open for every $x \in X$, the set $\{\lambda \in p_I(S) : S^\lambda$is connected $\}$ is dense in $p_I(S)$, T_x is non-empty and connected for every $x \in X$, T^λ is open for every $\lambda \in I$.*

(δ_2) $p_I(S)$ is connected, S_x is open for every $x \in X$, the set $\{\lambda \in p_I(S) : S^\lambda$is connected $\}$ is dense in $p_I(S)$, T is closed, T_x is non-empty and connected for every $x \in X$.

(δ_3) $p_I(S)$ is connected, X is compact, S is closed, $S^\lambda$is connected for every $\lambda \in I$, T is closed, T_x is non-empty and connected for every $x \in X$, T^λ is open for every $\lambda \in I$.

Then, at least one the following two assertions does hold:

i) $p_I(S) \neq I$;

ii) $S \cap T \neq \emptyset$.

We will also use the following

Proposition 1. *Let X be a topological space, Y a non-empty set, $f, g : X \times Y \to \mathbb{R}$ two functions satisfying the following conditions:*

h1) for every $(x, y) \in X \times Y$, one has $f(x, y) \leq g(x, y)$;

h2) for every $y \in Y$, $f(\cdot, y)$ is lower semicontinuous;

h3) there exists $y_0 \in Y$ such that, for every $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x, y_0) \leq \alpha\}$ is compact;

h4) there exists an increasing sequence $\{Y_n\}$ of subsets of Y such that $\cup_{n\in\mathbb{N}} Y_n = Y$ and $f^*(n) \leq g_*(n)$ for every $n \in \mathbb{N}$, where

$$f^*(n) = \inf_{x\in X} \sup_{y\in Y_n} f(x, y)$$

and

$$g_*(n) = \sup_{y\in Y_n} \inf_{x\in X} g(x, y).$$

Under such hypotheses, the relation (2) holds.

Proof. It is not restrictive to assume that $y_0 \in Y_1$. Arguing by contradiction, assume that $g_* < f^*$ and fix $h \in \mathbb{R}$ satisfying $g_* < h < f^*$.

For every $n \in \mathbb{N}$, put

$$C_n = \{x \in X : \sup_{y\in Y_n} f(x, y) \leq h\}.$$

Each set C_n is non-empty. Otherwise, one would have, taking into account h4)

$$h \leq \inf_{x\in X} \sup_{y\in Y_n} f(x, y) = f^*(n) \leq g_*(n) = \sup_{y\in Y_n} \inf_{x\in X} g(x, y) \leq g_* < h.$$

Since the sequence $\{Y_n\}$ is increasing, the sequence $\{C_n\}$ is decreasing. So, thanks to h2), $\{C_n\}$ is a decreasing sequence of non-empty closed and compact sets. So, there exists $x^* \in \cap_{n\in\mathbb{N}} C_n$. Then, for every $n \in \mathbb{N}$, one has $\sup_{y\in Y_n} f(x^*, y) \leq h$ and so

$$\sup_{y\in Y} f(x^*, y) = \sup_{n\in\mathbb{N}} \sup_{y\in Y_n} f(x^*, y) \leq h.$$

Hence $f^* = \inf_{x\in X} \sup_{y\in Y} f(x, y) \leq h$, absurd.

Proof of Theorem 1. Thanks to Proposition 1 and H2), we can limit ourselves to the case in which I is compact.

Arguing by contradiction, suppose that $g_* < f^*$. Fix $h \in \mathbb{R}$ satisfying $g_* < h < f^*$. Let us put

$$S = \{(x, \lambda) \in X \times I : g(x, \lambda) < h\},$$

$$T = \{(x, \lambda) \in X \times I : f(x, \lambda) > h\}.$$

Consider the set S. First of all, observe that $p_I(S) = I$. In fact, if $\lambda_0 \in I$, one has $h > g_* \geq \inf_{x\in X} g(x, \lambda_0)$, hence there exists $x \in X$ such that $g(x, \lambda_0) > h$. Moreover, if $x_0 \in X$, the set $S_{x_0} = \{\lambda \in I : g(x_0, \lambda) < h\}$ is open thanks to H5). Finally, from H7), the set $\{\lambda \in P_I(S) : S^\lambda$ is connected$\}$, since, as we have seen, $P_I(S) = I$, contains D, so it is dense in $p_I(S)$.

Now, consider the set T. Observe that, if $x_0 \in X$, the set $T_{x_0} = \{\lambda \in I : f(x_0, \lambda) > h\}$ is non-empty because, since $h < f^* \leq \sup_{\lambda\in I} f(x_0, \lambda)$, there exists $\lambda \in I$ such that $f(x_0, \lambda) > h$, so, T_{x_0} is non-empty. Moreover, it is connected thanks to H4).

Finally, if $\lambda_0 \in I$, the set $T^{\lambda_0} = \{x \in X : f(x, \lambda_0) > h\}$ is open thanks to H2).

So, the conditions (δ_1) of Lemma 1 are satisfied but this is absurd because both the assertions i) and ii) are clearly false. Hence, Theorem 1 is proved. □

Proof of Theorem 2. Arguing as in Theorem 1 by contradiction, let us fix $h \in \mathbb{R}$ satisfying $g_* < h < f^*$. Let us put

$$S = \{(x, \lambda) \in X \times I : g(x, \lambda) < h\},$$

$$T = \{(x, \lambda) \in X \times I : f(x, \lambda) \geq h\}.$$

Consider the set S. As in Theorem 1, it is possible to see that $p_I(S) = I$. Moreover, if $x_0 \in X$, the set $S_{x_0} = \{\lambda \in I : g(x_0, \lambda) < h\}$ is open thanks to H5). Finally, thanks to H7), the set $\{\lambda \in P_I(S) : S^\lambda \text{ is connected}\}$, since, as we have seen, $P_I(S) = I$, contains D, so it is dense in $p_I(S)$.

Now, consider the set T. Observe that it is closed, thanks to H3). Moreover, if $x_0 \in X$, the set $T_{x_0} = \{\lambda \in I : f(x_0, \lambda) \geq h\}$ is non-empty because $h < f^* \leq \sup_{\lambda \in I} f(x_0, \lambda)$ and it is connected thanks to H4).

So, the conditions (δ_2) of Lemma 1 are satisfied but this is absurd because both the assertions i) and ii) are clearly false. Hence, Theorem 2 is proved. □

Proof of Theorem 3. By H2) and the compactness of X, thanks to Proposition 1 we can limit ourselves to the case in which I is compact.

Arguing again by contradiction, let us fix $h \in \mathbb{R}$ satisfying $g_* < h < f^*$. Let us put

$$S = \{(x, \lambda) \in X \times I : g(x, \lambda) \leq h\},$$

$$T = \{(x, \lambda) \in X \times I : f(x, \lambda) > h\}.$$

Consider the set S. As in Theorem 1, it is possible to see that $p_I(S) = I$. Moreover, we can observe that S is closed thanks to H2). Finally, if $\lambda_0 \in I$, the set $S^{\lambda_0} = \{x \in X : g(x, \lambda_0) \leq h\}$ is connected thanks to H9).

Now, consider the set T. Observe that, if $x_0 \in X$, the set $T_{x_0} = \{\lambda \in I : f(x_0, \lambda) > h\}$ is non-empty because $h < f^* \leq \sup_{\lambda \in I} f(x_0, \lambda)$, and it is connected thanks to H4).

Moreover, if $\lambda_0 \in I$, the set $T^{\lambda_0} = \{x \in X : f(x, \lambda_0) > h\}$ is open thanks to H2).

So, the conditions (δ_3) of Lemma 1 are satisfied but this is absurd because both the assertions i) and ii) are clearly false. Hence, Theorem 3 is also proved. □

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