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OPTIMAL DISTRIBUTED CONTROL OF SECOND-GRADE FLUID FLOWS THROUGH A POROUS MEDIUM*

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With profound respect, this work is dedicated to Professor Constantin Fetecau for his 80th birthday

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Abstract

This paper deals with optimal distributed control of a strongly nonlinear system modeling the steady-state flow of a variable-viscosity second-grade fluid through a porous medium. We consider the internal flow within a bounded Lipschitz domain (of dimension two or three) under the no-slip boundary condition, assuming that the external force serves as the control parameter. The problem is rigorously formulated in both classical and weak settings. Our main result establishes the existence of an optimal weak solution by applying a known solvability principle for degenerate nonlinear operator equations involving weakly and completely continuous mappings in Sobolev spaces. In addition, we analyze the associated value function, proving its lower semicontinuity with respect to the one-sided Hausdorff distance, and present a necessary condition for the existence of a classical solution to the considered flow problem.

Keywords: second-grade fluid, porous media, optimization problem, distributed control, classical solution, weak solution, existence theorem, value function, energy discrepancy functional.

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1 Introduction and problem statement

As is known, the classical Navier–Stokes equations do not account for the shear-dependent viscosity and viscoelastic effects observed in many industrial and biological fluids, which are commonly referred to as *complex* or *non-Newtonian* fluids [3, 16, 40]. Given the wide variety of microscopic mechanisms responsible for non-Newtonian behavior, numerous constitutive models have been proposed in the scientific literature. Among them, *second-grade fluids* stand out as a prominent family. They belong to the class of differential-type fluids of complexity 2 and play an important role in the theory of viscoelastic media [19, 21–23, 29, 38].

For the mathematical description of steady-state flows of a second-grade fluid with a constant density through a porous medium, the following system can be used:

$$\begin{cases} (\vec{y}\cdot\nabla)\vec{y} - \nabla\cdot\mathbb{S} + \nabla p + \kappa\vec{y} + \beta|\vec{y}|\vec{y} = \vec{f}, \\ \nabla\cdot\vec{y} = 0, \\ \mathbb{S} = \eta\mathbb{D}(\vec{y}) + \alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y}) + \alpha\mathbb{D}(\vec{y})\mathbb{W}(\vec{y}) - \alpha\mathbb{W}(\vec{y})\mathbb{D}(\vec{y}), \end{cases} \tag{1}$$

where

- $\vec{y} = \vec{y}(\vec{x})$ is the velocity field at a point $\vec{x} \in \mathbb{R}^n$, $n = 2, 3$;
- $\mathbb{S} = \mathbb{S}(\vec{x})$ is the extra stress tensor;
- $p = p(\vec{x})$ is the pressure;
- κ is the Darcy resistance coefficient, $\kappa \geq 0$;
- β is the Forchheimer resistance coefficient, $\beta \geq 0$;
- $\vec{f} = \vec{f}(\vec{x})$ is the external force density;
- $\mathbb{D}(\vec{y})$ is the rate-of-strain tensor, $\mathbb{D}(\vec{y}) = \big(\nabla\vec{y} + (\nabla\vec{y})^\top\big)/2$;
- $\mathbb{W}(\vec{y})$ is the vorticity tensor, $\mathbb{W}(\vec{y}) = \big(\nabla\vec{y} - (\nabla\vec{y})^\top\big)/2$;
- ∇ stands for the gradient with respect to the spatial variables $x_1, \ldots, x_n$;
- η is the viscosity coefficient, $\eta \geq 0$;
- α is a material modulus characterizing the elastic properties of the fluid, $\alpha \geq 0$.

System (1) is normalized with respect to the constant fluid density, so that the density does not appear explicitly. Moreover, in what follows, we suppose that the contributions of $\alpha\mathbb{D}(\vec{y})\mathbb{W}(\vec{y})$ and $\alpha\mathbb{W}(\vec{y})\mathbb{D}(\vec{y})$, which involve products of first-order spatial derivatives of the velocity, are negligible relative to the other terms appearing in the third equation of (1), and may consequently be omitted. This reduction is analogous to a first-order linearization when all terms of order higher than one are discarded. Substitution of the simplified extra stress tensor into the momentum equation leads to the following system:

$$\begin{cases} (\vec{y}\cdot\nabla)\vec{y} - \nabla\cdot\left[\eta\mathbb{D}(\vec{y}) + \alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\right] + \nabla p + \kappa\vec{y} + \beta|\vec{y}|\vec{y} = \vec{f}, \\ \nabla\cdot\vec{y} = 0, \end{cases} \tag{2}$$

which is commonly regarded as a suitable model for the flow of viscous fluids containing polymer additives, in particular aqueous solutions of polyacrylamide and polyethylene oxide [1, 34].

It should be noted that even for this simplified formulation, the issue of existence and uniqueness of solutions is nontrivial. The principal difficulty arises from the term $\nabla\cdot[\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})]$, which is a third-order differential operator, whereas the viscous term $\nabla\cdot[\eta\mathbb{D}(\vec{y})]$ is merely the Laplacian. A possible approach to overcoming this obstacle is based on a modified Faedo–Galerkin scheme involving a specially constructed basis of eigenfunctions [6].

The mathematical analysis of the governing equations is of fundamental importance for predicting flow behavior and for the design of optimal flow configurations in various technological contexts. Following the pioneering works of Oskolkov [31,32], a substantial number of studies have been devoted to diverse aspects of the well-posedness of the second-grade fluid model and its modifications [9, 13, 15, 17, 18, 24, 33, 35]. A discussion of some open problems can be found in the monograph [16].

The present paper is concerned with an optimal control problem for the equations describing the internal flow of second-grade fluid through a porous medium in a bounded domain $\Omega \subset \mathbb{R}^n$.

In contrast to the majority of earlier contributions, which treat the viscosity as constant, we consider a *nonlocal variable-viscosity formulation*. More precisely, the viscosity is assumed to depend on the shear rate according to $\eta = \eta(|\mathbf{M}\mathbb{D}(\vec{y})|)$, where $\mathbf{M}$ is a prescribed linear operator such that

$\mathbf{M}\mathbb{D}$ is frame-indifferent [37]. Under this hypothesis, (2) assumes the form

$$\begin{cases} (\vec{y}\cdot\nabla)\vec{y} - \nabla\cdot\left[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y}) + \alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\right] \\ \qquad\qquad +\nabla p + \kappa\vec{y} + \beta|\vec{y}|\vec{y} = \vec{f}, \\ \nabla\cdot\vec{y} = 0. \end{cases} \tag{3}$$

We highlight five important particular cases of the above system in Table 1.

Table 1: Some particular cases of system (3)

Special case	**Model for fluid flow**
$\mathbf{M}$ is the zero operator, $\alpha = 0$, $\kappa = 0$, and $\beta = 0$	The standard Navier–Stokes system for Newtonian fluids in the absence of a porous matrix [8, 14, 27, 36]
$\mathbf{M}$ is the identity operator, $\alpha = 0$, $\kappa = 0$, and $\beta = 0$	The flow model for incompressible nonlinearly viscous fluid in the absence of a porous matrix [10, 28]
$\mathbf{M}$ is the zero operator, $\alpha > 0$, $\kappa = 0$, and $\beta = 0$	The flow model for an incompressible fluid with polymer additives in the absence of a porous matrix [1, 34]
$\mathbf{M}$ is the zero operator, $\alpha > 0$, $\kappa > 0$, and $\beta > 0$	The model for flows of an incompressible fluid with polymer additives through a porous medium [11]
$\mathbf{M}$ is an averaging operator (e.g., a convolution with a smooth kernel), $\alpha = 0$, $\kappa = 0$, and $\beta = 0$	The nonlocal model for fluid flow, where the viscosity at a material point $\vec{x}$ is influenced by the shear rate in a neighborhood of that point [20]

For the purpose of the subsequent analysis, it is convenient to eliminate the pressure p from (3). To this end, we employ the Leray projector $\boldsymbol{\pi}_{\mathrm{L}}$, which is defined via the Leray–Helmholtz decomposition of a vector field into its divergence-free part and its gradient part (see, e.g., [14, Chap. IV]). Application of $\boldsymbol{\pi}_{\mathrm{L}}$ to both sides of the first equation in (3) yields

$$\begin{cases} \boldsymbol{\pi}_{\mathrm{L}}(\vec{y}\cdot\nabla)\vec{y} - \boldsymbol{\pi}_{\mathrm{L}}\nabla\cdot\left[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y}) + \alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\right] \\ \qquad\qquad + \kappa\boldsymbol{\pi}_{\mathrm{L}}\vec{y} + \beta\pi_{\mathrm{L}}(|\vec{y}|\vec{y}) = \boldsymbol{\pi}_{\mathrm{L}}\vec{f}, \\ \nabla\cdot\vec{y} = 0. \end{cases} \tag{4}$$

Because $\boldsymbol{\pi}_{\mathrm{L}}\vec{y} = \vec{y}$ in the case of an incompressible fluid flow with the

no-penetration boundary condition, system (4) reduces to

$$\begin{cases} \boldsymbol{\pi}_{\mathrm{L}}(\vec{y}\cdot\nabla)\vec{y} - \boldsymbol{\pi}_{\mathrm{L}}\nabla\cdot\left[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y}) + \alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\right] \\ \qquad\qquad + \kappa\vec{y} + \beta\pi_{\mathrm{L}}(|\vec{y}|\vec{y}) = \boldsymbol{\pi}_{\mathrm{L}}\vec{f}, \\ \nabla\cdot\vec{y} = 0. \end{cases} \tag{5}$$

Note that if a vector function $\vec{y}$ satisfies (5), then there exists a pressure field p such that the pair $(\vec{y}, p)$ satisfies (3). Therefore the subsequent analysis will be focused on (5).

Since our primary objective is to investigate the problem of optimal distributed control, we introduce into the first equation of (5) a *control operator* $\mathbf{C}$, which maps the control input $\vec{u}$ into the corresponding external force term. As a result, we have

$$\begin{cases} \boldsymbol{\pi}_{\mathrm{L}}(\vec{y}\cdot\nabla)\vec{y} - \boldsymbol{\pi}_{\mathrm{L}}\nabla\cdot\left[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y}) + \alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\right] \\ \qquad\qquad + \kappa\vec{y} + \beta\pi_{\mathrm{L}}(|\vec{y}|\vec{y}) = \boldsymbol{\pi}_{\mathrm{L}}\mathbf{C}\vec{u}, \\ \nabla\cdot\vec{y} = 0. \end{cases} \tag{6}$$

It is assumed that the operator $\mathbf{C}$ is linear; for example, it may be the identity or a smoothing operator.

We impose the no-slip condition on the boundary of the flow domain:

$$\vec{y} = \vec{0} \text{ on } \partial\Omega \tag{7}$$

and suppose that the control function $\vec{u}$ satisfies the following constraint:

$$\vec{u} \in \mathcal{U}, \tag{8}$$

where $\mathcal{U}$ is a specified set of admissible controls.

The optimal control problem is stated as follows:

Find a velocity field $\vec{y}$ and a control $\vec{u}$ that minimize a given cost functional J subject to the constraints (6)–(8).

Symbolically, we write

$$J(\vec{y}, \vec{u}) \to \min \text{ subject to (6)–(8).} \tag{9}$$

A rigorous formulation of problem (9) in an appropriate functional-analytic framework, including the notions of classical and weak solutions, will be provided in Section 4.

This paper continues the work [12], where the special case of model (6) without a porous matrix (i.e., $\kappa = 0$ and $\beta = 0$) and $\mathbf{C} = \mathbf{Id}$ is studied under the Navier slip boundary condition. Our main aim is to establish the

solvability of problem (9) in the class of weak solutions without imposing a monotonicity assumption on the viscosity function η (cf. [10]). The proof relies on a solvability result (see Proposition 2) for a class of degenerate nonlinear operator equations involving weakly continuous and completely continuous mappings in Sobolev spaces.

A further contribution of this work, inspired by the approaches developed in [10], is the introduction of the value function $V_J = V_J(\mathcal{U})$, which captures the dependence of the optimal value of the cost functional J on the admissible control set. We prove that V_J is lower semicontinuous with respect to the one-sided Hausdorff distance. Moreover, a necessary condition for the existence of classical solutions to the control system (6)–(8) is derived in terms of the value function associated with the energy discrepancy functional.

The rest of the paper is organized as follows. In Section 2, we present the notation, function spaces, and auxiliary results that will be used throughout. Section 3 describes the main assumptions on the model data and discusses several examples. Section 4 provides the functional setting of the problem and introduces the value function. Finally, Section 5 states the main results of the work (Theorem 11) and contains their proof.

2 Preliminaries

2.1 Basic notations and function spaces

By $\mathbb{Z}$ we denote the set of all integers, and let

$$\mathbb{Z}_+ := \{m \in \mathbb{Z} : \ m > 0\}.$$

For vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$ and matrices $\mathbb{F}, \mathbb{G} \in \mathbb{R}^{n\times n}$, we define the inner products:

$$\vec{a} \cdot \vec{b} := \sum_{i=1}^{n} a_i b_i, \quad \mathbb{F} : \mathbb{G} := \sum_{i,j=1}^{n} \mathrm{F}_{ij}\mathrm{G}_{ij}.$$

The corresponding Euclidean norms are given by

$$|\vec{a}| := \sqrt{\vec{a} \cdot \vec{a}}, \quad |\mathbb{F}| := \sqrt{\mathbb{F} : \mathbb{F}}.$$

For Banach spaces E_1 and E_2, the space of all bounded linear operators from E_1 into E_2 is denoted by $\mathscr{L}(E_1, E_2)$. For brevity, $\mathscr{L}(E) := \mathscr{L}(E, E)$.

Given subsets U and W of a Banach space E, the one-sided Hausdorff distance from U to W is defined as

$$d_E(U, W) := \sup_{u\in U} \inf_{w\in W} \|u - w\|_E.$$

Strong and weak convergence in a Banach space are denoted by $\to$ and $\xrightarrow{\text{w}}$, respectively. We will write $\xrightarrow{\text{unif.}}$ to indicate uniform convergence.

Continuous and compact embeddings are denoted by $\hookrightarrow$ and $\hookrightarrow\hookrightarrow$, respectively.

Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, with locally Lipschitz boundary $\partial\Omega$. We shall employ the standard Lebesgue spaces $L^q(\Omega)$, for $q \geq 1$, and the Sobolev spaces $H^m(\Omega) := W^{m,2}(\Omega)$ with $m \in \mathbb{Z}_+$. For their definitions and fundamental properties, we refer to [16, 30].

For vector- and matrix-valued functions, we introduce the following notation:

$$\begin{aligned} \boldsymbol{L}^q(\Omega) &:= L^q(\Omega)^n, & \mathbb{L}^q(\Omega) &:= L^q(\Omega)^{n\times n},\\ \boldsymbol{H}^m(\Omega) &:= H^m(\Omega)^n, & \mathbb{H}^m(\Omega) &:= H^m(\Omega)^{n\times n},\\ \boldsymbol{C}(\overline{\Omega}) &:= C(\overline{\Omega})^n, & \mathbb{C}(\overline{\Omega}) &:= C(\overline{\Omega})^{n\times n}. \end{aligned}$$

Let $m \in \mathbb{Z}_+$. We define the following spaces:

$$\begin{aligned} \mathcal{V}_0(\Omega) &:= \{\vec{\phi}\colon \Omega \to \mathbb{R}^n :\ \vec{\phi} \in \boldsymbol{C}^\infty(\Omega) \text{ and } \operatorname{supp}\vec{\phi} \subset \Omega\},\\ \mathcal{V}_\sigma(\Omega) &:= \{\vec{\phi}\colon \Omega \to \mathbb{R}^n :\ \vec{\phi} \in \boldsymbol{C}^\infty(\Omega) \text{ and } \nabla\cdot\vec{\phi} = 0 \text{ in } \Omega\},\\ \mathcal{V}_{0,\sigma}(\Omega) &:= \mathcal{V}_0(\Omega) \cap \mathcal{V}_\sigma(\Omega),\\ \boldsymbol{H}^m_{0,\sigma}(\Omega) &:= \text{the closure of the set } \mathcal{V}_{0,\sigma}(\Omega) \text{ in } \boldsymbol{H}^m(\Omega),\\ \boldsymbol{H}^{-m}_{0,\sigma}(\Omega) &:= \text{the dual space of } \boldsymbol{H}^m_{0,\sigma}(\Omega). \end{aligned}$$

The space $\boldsymbol{H}^m_{0,\sigma}(\Omega)$ is a Hilbert space when equipped with the standard inner product $(\cdot,\cdot)_{\boldsymbol{H}^m(\Omega)}$. However, for the study of problem (9), it is preferable to use the following inner product and norm on $\boldsymbol{H}^1_{0,\sigma}(\Omega)$:

$$(\vec{y}_1, \vec{y}_2)_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} := \int_\Omega \mathbb{D}(\vec{y}_1) : \mathbb{D}(\vec{y}_2)\,\mathrm{d}\vec{x}, \quad \|\vec{y}\|_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} := \sqrt{(\vec{y},\vec{y})_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}}.$$

In view of Korn's inequality, this inner product $(\cdot,\cdot)_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}$ is well defined and the associated Euclidean norm $\|\cdot\|_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}$ is equivalent to the standard $\boldsymbol{H}^1$-norm (see, e.g., [28, Chap. I]).

We also recall the compact embedding

$$\boldsymbol{H}^1(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^p(\Omega), \quad \forall p \in [1, 6),$$

which, in particular, implies

$$\boldsymbol{H}^1_{0,\sigma}(\Omega) \hookrightarrow\hookrightarrow \boldsymbol{L}^p(\Omega), \quad \forall p \in [1, 6). \tag{10}$$

2.2 Auxiliary results

Proposition 1 (Continuity of the Nemytskii operator in Lebesgue spaces)**.** *Let $\mathscr{U}$ be a bounded domain in the space $\mathbb{R}^n$. Suppose $\varphi\colon \mathscr{U} \times \mathbb{R} \to \mathbb{R}$ is a function such that*

- *there exist constants $p_1 \geq 1$, $p_2 \geq 1$, $b > 0$, and a function $a \in L^{p_2}(\mathscr{U})$ such that*
$$|\varphi(\vec{x}, s)| \leq a(\vec{x}) + b|s|^{p_1/p_2},$$
for any $s \in \mathbb{R}$ and almost every $\vec{x} \in \mathscr{U}$;
- *the function $\varphi(\cdot, s)\colon \mathscr{U} \to \mathbb{R}$ is measurable for any $s \in \mathbb{R}$;*
- *the function $\varphi(\vec{x}, \cdot)\colon \mathbb{R} \to \mathbb{R}$ is continuous for almost every $\vec{x} \in \mathscr{U}$.*

Then the superposition operator Λ_φ defined by

$$\Lambda_\varphi[w](\cdot) := \varphi(\cdot, w(\cdot)), \quad \forall w \in L^{p_1}(\mathscr{U}),$$

is continuous and bounded as a mapping from $L^{p_1}(\mathscr{U})$ into $L^{p_2}(\mathscr{U})$.

A proof of this statement can be found in the book [26, Chap. I, § 2].

The following abstract existence result, which is analogous to Proposition 1 of [12], will play a key role in the present paper. It is designed for a degenerate operator equation in which the principal (dissipative) part $\mathbf{A}$ and its degenerate perturbation $\mathbf{Q}$ act in different Sobolev spaces, reflecting the mismatch between the regularity levels of the terms in this equation.

Proposition 2 (Abstract existence result)**.** *Let $\mathbf{A}$ be an operator from $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ into $\boldsymbol{H}^{-1}_{0,\sigma}(\Omega)$, let $\mathcal{E}_{\mathrm{A}}$ be the functional defined by*

$$\mathcal{E}_{\mathrm{A}}\colon \boldsymbol{H}^1_{0,\sigma}(\Omega) \to \mathbb{R}, \quad \mathcal{E}_{\mathrm{A}}(\vec{y}) := \langle \mathbf{A}(\vec{y}), \vec{y} \rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega) \times \boldsymbol{H}^1_{0,\sigma}(\Omega)},$$

and let $\mathbf{Q}$ be an operator from $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ into $\boldsymbol{H}^{-3}_{0,\sigma}(\Omega)$.

Suppose that

- *the operator $\mathbf{A}$ is weakly continuous, i.e., the convergence $\vec{y}_m \xrightarrow{\mathrm{w}} \vec{y}$ in $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ implies that $\mathbf{A}(\vec{y}_m) \xrightarrow{\mathrm{w}} \mathbf{A}(\vec{y})$ in $\boldsymbol{H}^{-1}_{0,\sigma}(\Omega)$ as $m \to \infty$;*
- *the inequality $\mathcal{E}_{\mathrm{A}}(\vec{y}) \geq c_{\mathrm{A}} \|\vec{y}\|^2_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}$ holds for all $\vec{y} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ with a positive constant c_{A};*
- *the functional $\mathcal{E}_{\mathrm{A}}$ is lower semicontinuous with respect to weak convergence in $\boldsymbol{H}^1_{0,\sigma}(\Omega)$;*

- *the operator* $\mathbf{Q}$ *is completely continuous, i.e., the convergence* $\vec{y}_m \xrightarrow{\mathrm{w}} \vec{y}$ *in* $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ *implies that* $\mathbf{Q}(\vec{y}_m) \to \mathbf{Q}(\vec{y})$ *in* $\boldsymbol{H}^{-3}_{0,\sigma}(\Omega)$ *as* $m \to \infty$;
- *the equality* $\langle \mathbf{Q}(\vec{v}), \vec{v} \rangle_{\boldsymbol{H}^{-3}_{0,\sigma}(\Omega) \times \boldsymbol{H}^{3}_{0,\sigma}(\Omega)} = 0$ *holds for all* $\vec{v} \in \boldsymbol{H}^3_{0,\sigma}(\Omega)$.

Then, for any $h \in \boldsymbol{H}^{-1}_{0,\sigma}(\Omega)$, *the following equation*

$$\begin{aligned} \langle \mathbf{A}(\vec{y}), \vec{v} \rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega) \times \boldsymbol{H}^{1}_{0,\sigma}(\Omega)} &+ \langle \mathbf{Q}(\vec{y}), \vec{v} \rangle_{\boldsymbol{H}^{-3}_{0,\sigma}(\Omega) \times \boldsymbol{H}^{3}_{0,\sigma}(\Omega)} \\ &= \langle h, \vec{v} \rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega) \times \boldsymbol{H}^{1}_{0,\sigma}(\Omega)}, \quad \forall \vec{v} \in \boldsymbol{H}^3_{0,\sigma}(\Omega), \end{aligned}$$

has at least one solution $\vec{y}_h \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ *such that*

$$\langle \mathbf{A}(\vec{y}_h), \vec{y}_h \rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega) \times \boldsymbol{H}^{1}_{0,\sigma}(\Omega)} \le \langle h, \vec{y}_h \rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega) \times \boldsymbol{H}^{1}_{0,\sigma}(\Omega)}.$$

Notably, this existence result can be extended to the case of models incorporating various slip boundary conditions [2, 4, 5, 7].

3 Main assumptions and motivating examples

3.1 Assumptions on the model data

Throughout the paper, we assume that the following seven hypotheses are satisfied:

(H.1) Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n = 2, 3$;

(H.2) $\alpha \ge 0$, $\kappa \ge 0$, and $\beta \ge 0$;

(H.3) $\eta\colon [0, +\infty) \to [0, +\infty)$ is a continuous function and there exist constants $\eta_{\min}$ and $\eta_{\max}$ such that $0 < \eta_{\min} \le \eta(s) \le \eta_{\max}$ for every $s \in [0, +\infty)$;

(H.4) $\mathbf{M}$ is a linear operator from $\mathbb{L}^2(\Omega)$ to itself such that, for any sequence $\{\vec{y}_m\}_{m=1}^{\infty} \subset \boldsymbol{H}^1_{0,\sigma}(\Omega)$, the convergence $\vec{y}_m \xrightarrow{\mathrm{w}} \vec{y}$ in $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ implies $\mathbf{M}\mathbb{D}(\vec{y}_m) \to \mathbf{M}\mathbb{D}(\vec{y})$ in $\mathbb{L}^2(\Omega)$ as $m \to \infty$;

(H.5) $\mathcal{U}$ is a bounded, sequentially weakly closed subset of $\boldsymbol{L}^2(\Omega)$;

(H.6) $J\colon \boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{L}^2(\Omega) \to \mathbb{R}$ is a lower semicontinuous functional with respect to weak convergence in $\boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{L}^2(\Omega)$;

(H.7) $\mathbf{C}$ is a bounded linear operator from $\boldsymbol{L}^2(\Omega)$ to itself.

3.2 Examples of cost functionals

Here, we give two examples of cost functionals satisfying condition (H.6):

1. The functional $J_1(\vec{y},\vec{u}) := (1-\chi)\|\vec{y}-\vec{y}_0\|^2_{\boldsymbol{L}^2(\Omega)} + \chi\|\vec{u}\|^2_{\boldsymbol{L}^2(\Omega)}$, where $\vec{y}_0$ is a given vector function describing the desired velocity field in the domain Ω, while χ is a numerical parameter, $0 \le \chi \le 1$.

2. The functional $J_2(\vec{y},\vec{u}) := -\|\vec{y}-\vec{y}_1\|^2_{\boldsymbol{L}^2(\Omega)} + \|\vec{u}\|^2_{\boldsymbol{L}^2(\Omega)}$, where $\vec{y}_1$ is an unfavourable velocity field, i.e., a velocity field whose occurrence in the domain Ω is undesirable.

In the first example, we consider a "compromise" cost functional, where the cost of the control $\vec{u}$ is penalized. In some applications, however, the cost of control is immaterial, whereas the state $\vec{y}$ is of primary importance. Mathematically, this means that the functional either does not depend explicitly on $\vec{u}$, or the parameter χ tends to zero. Such problems are referred to as *rigid* control problems (see [25, Chap. 1, Sect. 5]).

Note that, although the first term of J_2 has a negative sign, it is actually continuous with respect to weak convergence in $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ due to the compact embedding (10) with $p = 2$. Therefore, J_2 satisfies condition (H.6).

3.3 The averaging operator as an exemplary case of M

A natural example of an operator satisfying condition (H.4) is the averaging operator $\mathbf{M}_\rho$ defined by

$$\mathbf{M}_\rho\mathbb{K}(\vec{x}) := \int_{\mathbb{R}^n} \rho(\vec{x}-\vec{x}')\widetilde{\mathbb{K}}(\vec{x}')\,\mathrm{d}\vec{x}', \quad \forall\mathbb{K}\in\mathbb{L}^2(\Omega),$$

where

$$\widetilde{\mathbb{K}}(\vec{x}) := \begin{cases} \mathbb{K}(\vec{x}) & \text{if } \vec{x}\in\overline{\Omega}, \\ \mathbb{O} & \text{if } \vec{x}\in\mathbb{R}^n\setminus\overline{\Omega}, \end{cases}$$

the symbol $\mathbb{O}$ denotes the zero matrix, and $\rho\colon \mathbb{R}^n \to \mathbb{R}$ is a radially symmetric smooth function with compact support such that

$$\int_{\mathbb{R}^n} \rho(\vec{x})\,\mathrm{d}\vec{x} = 1.$$

From a physical perspective, the function ρ may be interpreted as an isotropic *influence function* [20]. As established by Vorotnikov [39], $\mathbf{M}_\rho\mathbb{D}$ is frame-indifferent, which confirms the physical consistency of the nonlocal rheological model with $\mathbf{M} := \mathbf{M}_\rho$.

Lemma 3 (Uniform convergence property)**.** *Let* $\{\vec{y}_m\}_{m=1}^{\infty}$ *be a sequence such that* $\vec{y}_m \overset{\text{w}}{\to} \vec{y}_0$ *in* $\boldsymbol{H}^1_{0,\sigma}(\Omega)$*. Then*

$$\mathbf{M}_\rho \mathbb{D}(\vec{y}_m) \xrightarrow{\text{unif.}} \mathbf{M}_\rho \mathbb{D}(\vec{y}_0) \text{ on } \overline{\Omega} \text{ as } m \to \infty. \tag{11}$$

Proof. First, we observe that

$$\vec{y}_m \to \vec{y}_0 \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } m \to \infty,$$

due to (10) with $p = 2$. Therefore, we have

$$\lim_{m\to\infty} \|\vec{y}_m - \vec{y}_0\|_{\boldsymbol{L}^2(\Omega)} = 0. \tag{12}$$

Further, using integration by parts, we obtain

$$\begin{aligned}
&[\mathbf{M}_\rho \mathbb{D}(\vec{\psi})]_{ij}(\vec{x}) \\
&= \frac{1}{2}\int_\Omega \rho(\vec{x}-\vec{x}')\left(\frac{\partial \psi_i(\vec{x}')}{\partial x_j'} + \frac{\partial \psi_j(\vec{x}')}{\partial x_i'}\right) \mathrm{d}\vec{x}' \\
&= -\frac{1}{2}\int_\Omega \left(\frac{\partial [\rho(\vec{x}-\vec{x}')]}{\partial x_j'}\psi_i(\vec{x}') + \frac{\partial [\rho(\vec{x}-\vec{x}')]}{\partial x_i'}\psi_j(\vec{x}')\right) \mathrm{d}\vec{x}',
\end{aligned} \tag{13}$$

for any $\vec{x} \in \overline{\Omega}$ and $\vec{\psi} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$.

Using the Cauchy–Schwarz inequality, we derive from (13) the following estimate

$$\|\mathbf{M}_\rho \mathbb{D}(\vec{\psi})\|_{\mathbb{C}(\overline{\Omega})} \le \sup\left\{\left|\frac{\partial \rho}{\partial x_s}\right| : \vec{x} \in \mathbb{R}^n,\, s = 1, \dots, n\right\} \sqrt{\text{meas}(\Omega)}\, \|\vec{\psi}\|_{\boldsymbol{L}^2(\Omega)}.$$

Setting $\vec{\psi} = \vec{y}_m - \vec{y}_0$ in the above estimate, we get

$$\begin{aligned}
\|\mathbf{M}_\rho \mathbb{D}(\vec{y}_m - \vec{y}_0)\|_{\mathbb{C}(\overline{\Omega})} \le \sup\left\{\left|\frac{\partial \rho}{\partial x_s}\right| : \vec{x} \in \mathbb{R}^n,\, s = 1, \dots, n\right\} \\
\times \sqrt{\text{meas}(\Omega)}\, \|\vec{y}_m - \vec{y}_0\|_{\boldsymbol{L}^2(\Omega)}.
\end{aligned}$$

The last inequality, together with (12), yields that

$$\lim_{m\to\infty} \|\mathbf{M}_\rho \mathbb{D}(\vec{y}_m - \vec{y}_0)\|_{\mathbb{C}(\overline{\Omega})} = 0,$$

and hence (11) holds. Thus, Lemma 3 is proved. □

The above lemma immediately yields the following proposition.

Proposition 4 (Weak-to-strong continuity property)**.** *Condition* (H.4) *holds for the operator* $\mathbf{M} := \mathbf{M}_\rho$.

4 Classical and weak solutions and the value function

We now proceed to formulate problem (9) within the framework of Sobolev spaces. As a preliminary step, we introduce the concept of a classical solution, which requires the existence of all derivatives appearing in the governing equations.

Definition 5 (Classical solution)**.** We shall say that $(\vec{y},\vec{u})$ is a *classical solution* of the control system (6)–(8) if $\vec{y}\in \boldsymbol{C}^3(\Omega)\cap\boldsymbol{C}(\overline{\Omega})$, $\vec{u}\in\mathcal{U}$, and the pair $(\vec{y},\vec{u})$ satisfies (6) and (7).

The set of all classical solutions to (6)–(8) will be denoted by $\mathcal{S}_{\mathrm{cl}}(\mathcal{U})$.

Proposition 6 (Integral identity for classical solutions)**.** *If a pair $(\vec{y},\vec{u})$ is a classical solution of* (6)–(8)*, then*

$$\begin{aligned}
&-\sum_{i=1}^{n}\int_\Omega y_i\vec{y}\cdot\frac{\partial\vec{v}}{\partial x_i}\,\mathrm{d}\vec{x}+\int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y}):\mathbb{D}(\vec{v})\,\mathrm{d}\vec{x}\\
&\quad-\alpha\sum_{i=1}^{n}\int_\Omega y_i\mathbb{D}(\vec{y}):\frac{\partial\mathbb{D}(\vec{v})}{\partial x_i}\,\mathrm{d}\vec{x}+\kappa\int_\Omega\vec{y}\cdot\vec{v}\,\mathrm{d}\vec{x}\\
&\quad+\beta\int_\Omega\vec{y}|\vec{y}|\cdot\vec{v}\,\mathrm{d}\vec{x}=\int_\Omega\mathbf{C}\vec{u}\cdot\vec{v}\,\mathrm{d}\vec{x},
\end{aligned}\tag{14}$$

for every $\vec{v}\in\boldsymbol{H}^2_{0,\sigma}(\Omega)$.

Proof. Let $\vec{v}$ be an arbitrary function from $\boldsymbol{H}^2_{0,\sigma}(\Omega)$. Taking the inner product of the first equation in (6) with $\vec{v}$ and integrating over Ω, we obtain

$$\begin{aligned}
&\underbrace{\int_\Omega\big(\boldsymbol{\pi}_{\mathrm{L}}(\vec{y}\cdot\nabla)\vec{y}\big)\cdot\vec{v}\,\mathrm{d}\vec{x}}_{=:\mathrm{Int}_1}-\underbrace{\int_\Omega\big(\boldsymbol{\pi}_{\mathrm{L}}\nabla\cdot\big[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\big]\big)\cdot\vec{v}\,\mathrm{d}\vec{x}}_{=:\mathrm{Int}_2}\\
&\quad+\kappa\int_\Omega\vec{y}\cdot\vec{v}\,\mathrm{d}\vec{x}+\beta\underbrace{\int_\Omega\boldsymbol{\pi}_{\mathrm{L}}(|\vec{y}|\vec{y})\cdot\vec{v}\,\mathrm{d}\vec{x}}_{=:\mathrm{Int}_3}=\underbrace{\int_\Omega(\boldsymbol{\pi}_{\mathrm{L}}\mathbf{C}\vec{u})\cdot\vec{v}\,\mathrm{d}\vec{x}}_{=:\mathrm{Int}_4}.
\end{aligned}\tag{15}$$

Since $(\vec{a},\vec{v})_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{\pi}_{\mathrm{L}}\vec{a},\vec{v})_{\boldsymbol{L}^2(\Omega)}$ for any $\vec{a}\in\boldsymbol{L}^2(\Omega)$, we have

$$\begin{aligned}
\mathrm{Int}_1 &= \int_\Omega \big((\vec{y}\cdot\nabla)\vec{y}\big)\cdot\vec{v}\,\mathrm{d}\vec{x},\\
\mathrm{Int}_2 &= \int_\Omega \big(\nabla\cdot\big[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\big]\big)\cdot\vec{v}\,\mathrm{d}\vec{x},\\
\mathrm{Int}_3 &= \int_\Omega |\vec{y}|\vec{y}\cdot\vec{v}\,\mathrm{d}\vec{x}, \qquad (16)\\
\mathrm{Int}_4 &= \int_\Omega \mathbf{C}\vec{u}\cdot\vec{v}\,\mathrm{d}\vec{x}. \qquad (17)
\end{aligned}$$

Applying integration by parts, using the incompressibility condition and the no-slip boundary condition, we find

$$\mathrm{Int}_1 = -\sum_{i=1}^{n}\int_\Omega y_i\vec{y}\cdot\frac{\partial\vec{v}}{\partial x_i}\,\mathrm{d}\vec{x} \qquad (18)$$

and

$$\mathrm{Int}_2 = -\int_\Omega \big[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\big] : \nabla\vec{v}\,\mathrm{d}\vec{x}. \qquad (19)$$

Taking into account the symmetry of the following matrix

$$\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y}),$$

one readily verifies that

$$\begin{aligned}
&\int_\Omega \big[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\big] : \nabla\vec{v}\,\mathrm{d}\vec{x}\\
&\quad= \frac{1}{2}\int_\Omega \big[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\big] : \nabla\vec{v}\,\mathrm{d}\vec{x}\\
&\qquad+ \frac{1}{2}\int_\Omega \big[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\big]^{\top} : (\nabla\vec{v})^{\top}\,\mathrm{d}\vec{x}\\
&\quad= \frac{1}{2}\int_\Omega \big[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\big] : \big(\nabla\vec{v}+(\nabla\vec{v})^{\top}\big)\,\mathrm{d}\vec{x}\\
&\quad= \int_\Omega \big[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\big] : \mathbb{D}(\vec{v})\,\mathrm{d}\vec{x}. \qquad (20)
\end{aligned}$$

Combining (19) and (20) yields

$$\mathrm{Int}_2 = -\int_\Omega \big[\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y})+\alpha(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y})\big] : \mathbb{D}(\vec{v})\,\mathrm{d}\vec{x}. \qquad (21)$$

Substitution of (16)–(18) and (21) into (15) leads to

$$
\begin{aligned}
&-\sum_{i=1}^{n}\int_\Omega y_i\vec{y}\cdot\frac{\partial\vec{v}}{\partial x_i}\,\mathrm{d}\vec{x}+\int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y}):\mathbb{D}(\vec{v})\,\mathrm{d}\vec{x}\\
&\quad+\alpha\underbrace{\int_\Omega(\vec{y}\cdot\nabla)\mathbb{D}(\vec{y}):\mathbb{D}(\vec{v})\,\mathrm{d}\vec{x}}_{=:\mathrm{Int}_5}+\kappa\int_\Omega\vec{y}\cdot\vec{v}\,\mathrm{d}\vec{x}\\
&\quad+\beta\int_\Omega|\vec{y}|\vec{y}\cdot\vec{v}\,\mathrm{d}\vec{x}=\int_\Omega\mathbf{C}\vec{u}\cdot\vec{v}\,\mathrm{d}\vec{x}.
\end{aligned}
\tag{22}
$$

Finally, integrating the term Int_5 by parts, we obtain (14) from (22). Thus, Proposition 6 is proved. □

Proposition 7 (Energy balance equation)**.** *If a pair* $(\vec{y},\vec{u})$ *is a classical solution of* (6)–(8)*, then*

$$
\int_\Omega\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)|\mathbb{D}(\vec{y})|^2\,\mathrm{d}\vec{x}+\kappa\int_\Omega|\vec{y}|^2\,\mathrm{d}\vec{x}+\beta\int_\Omega|\vec{y}|^3\,\mathrm{d}\vec{x}=\int_\Omega\mathbf{C}\vec{u}\cdot\vec{y}\,\mathrm{d}\vec{x}. \tag{23}
$$

Proof. Taking $\vec{v}=\vec{y}$ in (14), we get

$$
\begin{aligned}
&\underbrace{-\sum_{i=1}^{n}\int_\Omega y_i\vec{y}\cdot\frac{\partial\vec{y}}{\partial x_i}\,\mathrm{d}\vec{x}}_{=:\mathrm{Int}_6}+\int_\Omega\eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y}):\mathbb{D}(\vec{y})\,\mathrm{d}\vec{x}\\
&\quad-\alpha\underbrace{\sum_{i=1}^{n}\int_\Omega y_i\mathbb{D}(\vec{y}):\frac{\partial\mathbb{D}(\vec{y})}{\partial x_i}\,\mathrm{d}\vec{x}}_{=:\mathrm{Int}_7}+\kappa\int_\Omega\vec{y}\cdot\vec{y}\,\mathrm{d}\vec{x}\\
&\quad+\beta\int_\Omega|\vec{y}|\vec{y}\cdot\vec{y}\,\mathrm{d}\vec{x}=\int_\Omega\mathbf{C}\vec{u}\cdot\vec{y}\,\mathrm{d}\vec{x}.
\end{aligned}
\tag{24}
$$

A straightforward integration by parts gives

$$
\begin{aligned}
\mathrm{Int}_6&=\frac{1}{2}\sum_{i=1}^{n}\int_\Omega y_i\frac{\partial|\vec{y}|^2}{\partial x_i}\,\mathrm{d}\vec{x}\\
&=-\frac{1}{2}\int_\Omega(\nabla\cdot\vec{y})|\vec{y}|^2\,\mathrm{d}\vec{x}\\
&=0
\end{aligned}
$$

and

$$\begin{aligned}\mathrm{Int}_7 &= \frac{1}{2}\sum_{i=1}^{n}\int_\Omega y_i \frac{\partial|\mathbb{D}(\vec{y})|^2}{\partial x_i}\,\mathrm{d}\vec{x}\\ &= -\frac{1}{2}\int_\Omega (\nabla\cdot\vec{y})|\mathbb{D}(\vec{y})|^2\,\mathrm{d}\vec{x}\\ &= 0.\end{aligned}$$

Consequently, (24) reduces to (23), which completes the proof. □

Since classical solutions may not exist for arbitrary controls $\vec{u}\in\mathcal{U}$, we introduce the following weak formulation, which is directly motivated by Propositions 6 and 7.

Definition 8 (Weak solution)**.** We shall say that $(\vec{y},\vec{u})$ is a *weak solution* of the control system (6)–(8) if

- $(\vec{y},\vec{u})\in \boldsymbol{H}^1_{0,\sigma}(\Omega)\times\mathcal{U}$;
- equality (14) holds for any test function $\vec{v}\in\boldsymbol{H}^3_{0,\sigma}(\Omega)$;
- the energy inequality holds:

$$\int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y})|)|\mathbb{D}(\vec{y})|^2\,\mathrm{d}\vec{x}+\kappa\int_\Omega|\vec{y}|^2\,\mathrm{d}\vec{x}+\beta\int_\Omega|\vec{y}|^3\,\mathrm{d}\vec{x} \le \int_\Omega \mathbf{C}\vec{u}\cdot\vec{y}\,\mathrm{d}\vec{x}. \quad (25)$$

The set of all weak solutions to (6)–(8) will be denoted by $\mathcal{S}_{\mathrm{weak}}(\mathcal{U})$. From the definitions it follows that the following inclusion holds:

$$\mathcal{S}_{\mathrm{cl}}(\mathcal{U})\subset\mathcal{S}_{\mathrm{weak}}(\mathcal{U}).$$

Definition 9 (Optimal weak solution)**.** A pair $(\vec{y}_*,\vec{u}_*)$ is called an *optimal weak solution* to problem (9) if $(\vec{y}_*,\vec{u}_*)$ belongs to the set $\mathcal{S}_{\mathrm{weak}}(\mathcal{U})$ and

$$J(\vec{y}_*,\vec{u}_*)=\inf\big\{J(\vec{y},\vec{u}):\ (\vec{y},\vec{u})\in\mathcal{S}_{\mathrm{weak}}(\mathcal{U})\big\}. \quad (26)$$

The set of optimal weak solutions to (9) will be denoted by $\mathcal{S}^{\mathrm{opt}}_{\mathrm{weak}}(\mathcal{U})$.

The notion of an optimal classical solution can be defined analogously.

To address situations in which the admissible control set may vary, we introduce the function V_J. The class of all bounded, sequentially weakly closed subsets of $\boldsymbol{L}^2(\Omega)$ is denoted by $\mathfrak{P}_{\boldsymbol{L}^2(\Omega)}$. By definition, we set

$$V_J \colon \mathfrak{P}_{\boldsymbol{L}^2(\Omega)} \to \mathbb{R} \cup \{-\infty\},$$
$$V_J(\mathcal{U}) := \inf \big\{ J(\vec{y}, \vec{u}) : \ (\vec{y}, \vec{u}) \in \mathcal{S}_{\mathrm{weak}}(\mathcal{U}) \big\}.$$

Note that any closed, convex, bounded subset of $\boldsymbol{L}^2(\Omega)$ belongs to $\mathfrak{P}_{\boldsymbol{L}^2(\Omega)}$.

Definition 10 (Value function)**.** The function V_J is called the *value function* of problem (9).

Clearly, if $(\vec{y}_*, \vec{u}_*)$ is a pair belonging to $\mathcal{S}^{\mathrm{opt}}_{\mathrm{weak}}(\mathcal{U})$, then $V_J(\mathcal{U}) = J(\vec{y}_*, \vec{u}_*)$.

5 Main results and their proof

Let us introduce the *energy discrepancy functional* J_{d}, which measures the gap in the energy balance, by

$$J_{\mathrm{d}}(\vec{y}, \vec{u}) := \int_\Omega \mathbf{C}\vec{u} \cdot \vec{y} \, \mathrm{d}\vec{x} - \int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y})|)|\mathbb{D}(\vec{y})|^2 \, \mathrm{d}\vec{x}$$
$$- \kappa \int_\Omega |\vec{y}|^2 \, \mathrm{d}\vec{x} - \beta \int_\Omega |\vec{y}|^3 \, \mathrm{d}\vec{x}. \tag{27}$$

Indeed, for a classical solution, the energy balance (Proposition 7) implies the equality $J_{\mathrm{d}} = 0$. On the other hand, for a weak solution, $J_{\mathrm{d}} \geq 0$, which may be interpreted as the energy defect associated with the lack of regularity, analogous to the turbulent energy dissipation defect in non-smooth flows with small-scale motions.

The findings of this paper are summarized in the following theorem.

Theorem 11 (Main results)**.** *Under conditions* (H.1)–(H.7)*, the following statements hold.*

(a) *Problem* (9) *has at least one optimal weak solution.*

(b) *The value function V_J is lower semicontinuous in the following sense: if $\mathcal{U}_m \in \mathfrak{P}_{\boldsymbol{L}^2(\Omega)}$, for any $m \in \mathbb{Z}_+ \cup \{0\}$, and*

$$\lim_{m\to\infty} d_{\boldsymbol{L}^2(\Omega)}(\mathcal{U}_m, \mathcal{U}_0) = 0, \tag{28}$$

then

$$V_J(\mathcal{U}_0) \leq \liminf_{m\to\infty} V_J(\mathcal{U}_m).$$

(c) *If the set of classical solutions* $\mathcal{S}_{\mathrm{cl}}(\mathcal{U})$ *is nonempty, then* $V_{J_{\mathrm{d}}}(\mathcal{U}) = 0$.

Proof. We begin by proving assertion (a). The proof is divided into four steps.

Step 1: Weak solvability. We first establish that $\mathcal{S}_{\mathrm{weak}}(\mathcal{U}) \neq \emptyset$.

Fix an arbitrary vector function $\vec{u} \in \mathcal{U}$. To apply Proposition 2, we define the concrete realizations of the abstract operators $\mathbf{A}$ and $\mathbf{Q}$ as follows:

$$\begin{aligned}
&\mathbf{A}\colon \boldsymbol{H}^1_{0,\sigma}(\Omega) \to \boldsymbol{H}^{-1}_{0,\sigma}(\Omega),\\
&\langle \mathbf{A}(\vec{y}), \vec{\omega}\rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega)\times\boldsymbol{H}^1_{0,\sigma}(\Omega)} := \int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y})|)\mathbb{D}(\vec{y}) : \mathbb{D}(\vec{\omega})\,\mathrm{d}\vec{x}\\
&\qquad\qquad + \kappa\int_\Omega \vec{y}\cdot\vec{\omega}\,\mathrm{d}\vec{x} + \beta\int_\Omega |\vec{y}|\vec{y}\cdot\vec{\omega}\,\mathrm{d}\vec{x}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{Q}\colon \boldsymbol{H}^1_{0,\sigma}(\Omega) \to \boldsymbol{H}^{-3}_{0,\sigma}(\Omega),\\
&\langle \mathbf{Q}(\vec{y}), \vec{v}\rangle_{\boldsymbol{H}^{-3}_{0,\sigma}(\Omega)\times\boldsymbol{H}^3_{0,\sigma}(\Omega)} := -\sum_{i=1}^n \int_\Omega y_i\vec{y}\cdot\frac{\partial\vec{v}}{\partial x_i}\,\mathrm{d}\vec{x}\\
&\qquad\qquad - \alpha\sum_{i=1}^n \int_\Omega y_i\mathbb{D}(\vec{y}) : \frac{\partial\mathbb{D}(\vec{v})}{\partial x_i}\,\mathrm{d}\vec{x},
\end{aligned}$$

where $\vec{y} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$, $\vec{\omega} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$, and $\vec{v} \in \boldsymbol{H}^3_{0,\sigma}(\Omega)$.

Moreover, we define the functional $\widehat{u} \in \boldsymbol{H}^{-1}_{0,\sigma}(\Omega)$ by

$$\langle \widehat{u}, \vec{\omega}\rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega)\times\boldsymbol{H}^1_{0,\sigma}(\Omega)} := \int_\Omega \mathbf{C}\vec{u}\cdot\vec{\omega}\,\mathrm{d}\vec{x}.$$

With these operators at hand, equation (14) can be rewritten in the abstract form

$$\begin{aligned}
\langle \mathbf{A}(\vec{y}), \vec{v}\rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega)\times\boldsymbol{H}^1_{0,\sigma}(\Omega)} + \langle \mathbf{Q}(\vec{y}), \vec{v}\rangle_{\boldsymbol{H}^{-3}_{0,\sigma}(\Omega)\times\boldsymbol{H}^3_{0,\sigma}(\Omega)}&\\
= \langle \widehat{u}, \vec{v}\rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega)\times\boldsymbol{H}^1_{0,\sigma}(\Omega)}, \quad (29)&
\end{aligned}$$

to be satisfied for all $\vec{v} \in \boldsymbol{H}^3_{0,\sigma}(\Omega)$.

By using conditions (H.3) and (H.4), the compact embedding (10) with $p = 4$, and the following relations:

$$\begin{aligned}
&\langle \mathbf{A}(\vec{y}), \vec{y}\rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega)\times\boldsymbol{H}^1_{0,\sigma}(\Omega)} \geq \eta_{\min}\|\vec{y}\|^2_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}, \quad \forall \vec{y} \in \boldsymbol{H}^1_{0,\sigma}(\Omega),\\
&\langle \mathbf{Q}(\vec{v}), \vec{v}\rangle_{\boldsymbol{H}^{-3}_{0,\sigma}(\Omega)\times\boldsymbol{H}^3_{0,\sigma}(\Omega)} = 0, \quad \forall \vec{v} \in \boldsymbol{H}^3_{0,\sigma}(\Omega),
\end{aligned}$$

one verifies that all the assumptions of Proposition 2 are satisfied. Application of this proposition to problem (29) yields the existence of a solution $\vec{y}_{\vec{u}} \in \boldsymbol{H}^1_{0,\sigma}(\Omega)$ satisfying

$$\langle \mathbf{A}(\vec{y}_{\vec{u}}), \vec{y}_{\vec{u}} \rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega)\times\boldsymbol{H}^1_{0,\sigma}(\Omega)} \leq \langle \widehat{u}, \vec{y}_{\vec{u}} \rangle_{\boldsymbol{H}^{-1}_{0,\sigma}(\Omega)\times\boldsymbol{H}^1_{0,\sigma}(\Omega)}.$$

The pair $(\vec{y}_{\vec{u}}, \vec{u})$ is then a weak solution of (6)–(8), so that $\mathcal{S}_{\text{weak}}(\mathcal{U}) \neq \emptyset$.

Step 2: Weak closedness. We now prove that $\mathcal{S}_{\text{weak}}(\mathcal{U})$ is sequentially weakly closed in $\boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{L}^2(\Omega)$.

Consider a sequence $\{(\vec{y}_m, \vec{u}_m)\}_{m=1}^{\infty}$ such that $(\vec{y}_m, \vec{u}_m) \in \mathcal{S}_{\text{weak}}(\mathcal{U})$, for $m \in \mathbb{Z}_+$, and suppose that

$$\vec{y}_m \xrightarrow{\text{w}} \vec{y}_0 \text{ in } \boldsymbol{H}^1_{0,\sigma}(\Omega) \text{ as } m \to \infty, \tag{30}$$

$$\vec{u}_m \xrightarrow{\text{w}} \vec{u}_0 \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } m \to \infty. \tag{31}$$

We must show the inclusion $(\vec{y}_0, \vec{u}_0) \in \mathcal{S}_{\text{weak}}(\mathcal{U})$.

Since each $(\vec{y}_m, \vec{u}_m)$ is a weak solution, the identity

$$\begin{aligned} &-\sum_{i=1}^{n} \int_\Omega y_{mi}\vec{y}_m \cdot \frac{\partial \vec{\phi}}{\partial x_i}\,\mathrm{d}\vec{x} + \int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}_m)|)\mathbb{D}(\vec{y}_m) : \mathbb{D}(\vec{\phi})\,\mathrm{d}\vec{x} \\ &- \alpha \sum_{i=1}^{n} \int_\Omega y_{mi}\mathbb{D}(\vec{y}_m) : \frac{\partial \mathbb{D}(\vec{\phi})}{\partial x_i}\,\mathrm{d}\vec{x} + \kappa \int_\Omega \vec{y}_m \cdot \vec{\phi}\,\mathrm{d}\vec{x} \\ &+ \beta \int_\Omega \vec{y}_m|\vec{y}_m| \cdot \vec{\phi}\,\mathrm{d}\vec{x} = \int_\Omega \mathbf{C}\vec{u}_m \cdot \vec{\phi}\,\mathrm{d}\vec{x}, \quad \forall m \in \mathbb{Z}_+, \end{aligned} \tag{32}$$

holds for every $\vec{\phi} \in \mathcal{V}_{0,\sigma}(\Omega)$, together with the energy inequality

$$\begin{aligned} &\int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}_m)|)|\mathbb{D}(\vec{y}_m)|^2\,\mathrm{d}\vec{x} + \kappa \int_\Omega |\vec{y}_m|^2\,\mathrm{d}\vec{x} + \beta \int_\Omega |\vec{y}_m|^3\,\mathrm{d}\vec{x} \\ &\qquad \leq \int_\Omega \mathbf{C}\vec{u}_m \cdot \vec{y}_m\,\mathrm{d}\vec{x}, \quad \forall m \in \mathbb{Z}_+. \end{aligned} \tag{33}$$

From (30) and condition (H.4), we infer that

$$\mathbb{D}(\vec{y}_m) \xrightarrow{\text{w}} \mathbb{D}(\vec{y}_0) \text{ in } \mathbb{L}^2(\Omega) \text{ as } m \to \infty, \tag{34}$$

$$\mathbf{M}\mathbb{D}(\vec{y}_m) \to \mathbf{M}\mathbb{D}(\vec{y}_0) \text{ in } \mathbb{L}^2(\Omega) \text{ as } m \to \infty. \tag{35}$$

By virtue of Proposition 1 and condition (H.3), we deduce from (35) that

$$\eta(|\mathbf{M}\mathbb{D}(\vec{y}_m)|) \to \eta(|\mathbf{M}\mathbb{D}(\vec{y}_0)|) \text{ in } L^2(\Omega) \text{ as } m \to \infty, \tag{36}$$

$$\sqrt{\eta(|\mathbf{M}\mathbb{D}(\vec{y}_m)|)} \to \sqrt{\eta(|\mathbf{M}\mathbb{D}(\vec{y}_0)|)} \text{ in } L^2(\Omega) \text{ as } m \to \infty. \tag{37}$$

Furthermore, using the compact embedding (10) with $p = 4$, we obtain

$$\vec{y}_m \to \vec{y}_0 \text{ in } \boldsymbol{L}^4(\Omega) \text{ as } m \to \infty. \tag{38}$$

Passing to the limit $m \to \infty$ in (32) with the help of (31), (34), (36), and (38), we obtain

$$\begin{aligned}
&-\sum_{i=1}^{n} \int_\Omega y_{0i}\vec{y}_0 \cdot \frac{\partial \vec{\phi}}{\partial x_i}\,\mathrm{d}\vec{x} + \int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}_0)|)\mathbb{D}(\vec{y}_0) : \mathbb{D}(\vec{\phi})\,\mathrm{d}\vec{x} \\
&\quad - \alpha \sum_{i=1}^{n} \int_\Omega y_{0i}\mathbb{D}(\vec{y}_0) : \frac{\partial \mathbb{D}(\vec{\phi})}{\partial x_i}\,\mathrm{d}\vec{x} + \kappa \int_\Omega \vec{y}_0 \cdot \vec{\phi}\,\mathrm{d}\vec{x} \\
&\quad + \beta \int_\Omega |\vec{y}_0|\vec{y}_0 \cdot \vec{\phi}\,\mathrm{d}\vec{x} = \int_\Omega \mathbf{C}\vec{u}_0 \cdot \vec{\phi}\,\mathrm{d}\vec{x}.
\end{aligned}$$

Since $\mathcal{V}_{0,\sigma}(\Omega)$ is dense in $\boldsymbol{H}^3_{0,\sigma}(\Omega)$, the above equality holds for arbitrary test function $\vec{v} \in \boldsymbol{H}^3_{0,\sigma}(\Omega)$:

$$\begin{aligned}
&-\sum_{i=1}^{n} \int_\Omega y_{0i}\vec{y}_0 \cdot \frac{\partial \vec{v}}{\partial x_i}\,\mathrm{d}\vec{x} + \int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}_0)|)\mathbb{D}(\vec{y}_0) : \mathbb{D}(\vec{v})\,\mathrm{d}\vec{x} \\
&\quad - \alpha \sum_{i=1}^{n} \int_\Omega y_{0i}\mathbb{D}(\vec{y}_0) : \frac{\partial \mathbb{D}(\vec{v})}{\partial x_i}\,\mathrm{d}\vec{x} + \kappa \int_\Omega \vec{y}_0 \cdot \vec{v}\,\mathrm{d}\vec{x} \\
&\quad + \beta \int_\Omega |\vec{y}_0|\vec{y}_0 \cdot \vec{v}\,\mathrm{d}\vec{x} = \int_\Omega \mathbf{C}\vec{u}_0 \cdot \vec{v}\,\mathrm{d}\vec{x}.
\end{aligned} \tag{39}$$

Moreover, from (34), (37), and the boundedness of the function η (see condition (H.3)), we derive

$$\sqrt{\eta(|\mathbf{M}\mathbb{D}(\vec{y}_m)|)}\mathbb{D}(\vec{y}_m) \xrightarrow{\mathrm{w}} \sqrt{\eta(|\mathbf{M}\mathbb{D}(\vec{y}_0)|)}\mathbb{D}(\vec{y}_0) \text{ in } \mathbb{L}^2(\Omega),$$

which, combined with (33) and (38), implies

$$\begin{aligned}
&\int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}_0)|)|\mathbb{D}(\vec{y}_0)|^2\,\mathrm{d}\vec{x} \\
&\le \liminf_{m\to\infty} \int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}_m)|)|\mathbb{D}(\vec{y}_m)|^2\,\mathrm{d}\vec{x} \\
&\le \liminf_{m\to\infty} \int_\Omega \mathbf{C}\vec{u}_m \cdot \vec{y}_m\,\mathrm{d}\vec{x} - \kappa \int_\Omega |\vec{y}_m|^2\,\mathrm{d}\vec{x} - \beta \int_\Omega |\vec{y}_m|^3\,\mathrm{d}\vec{x} \\
&= \int_\Omega \mathbf{C}\vec{u}_0 \cdot \vec{y}_0\,\mathrm{d}\vec{x} - \kappa \int_\Omega |\vec{y}_0|^2\,\mathrm{d}\vec{x} - \beta \int_\Omega |\vec{y}_0|^3\,\mathrm{d}\vec{x}.
\end{aligned} \tag{40}$$

Because $\{\vec{u}_m\}_{m=1}^{\infty} \subset \mathcal{U}$ and $\mathcal{U}$ is sequentially weakly closed in $\boldsymbol{L}^2(\Omega)$, it follows from (31) that

$$\vec{u}_0 \in \mathcal{U}. \tag{41}$$

Combining (39)–(41), we conclude that $(\vec{y}_0, \vec{u}_0)$ is a weak solution of problem (6)–(8), i.e., $(\vec{y}_0, \vec{u}_0) \in \mathcal{S}_{\text{weak}}(\mathcal{U})$.

Step 3: Boundedness. We next establish that the set $\mathcal{S}_{\text{weak}}(\mathcal{U})$ is bounded in $\boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{L}^2(\Omega)$.

Consider the projection operator $\Pr_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}$ defined by

$$\Pr_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} \colon \boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{L}^2(\Omega) \to \boldsymbol{H}^1_{0,\sigma}(\Omega), \quad \Pr_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}(\vec{y}, \vec{u}) := \vec{y}.$$

Since $\mathcal{U}$ is bounded in $\boldsymbol{L}^2(\Omega)$, it suffices to prove that $\Pr_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} \mathcal{S}_{\text{weak}}(\mathcal{U})$ is a bounded set in $\boldsymbol{H}^1_{0,\sigma}(\Omega)$.

From the energy inequality (25), we deduce that

$$\begin{aligned}
&\int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y})|)|\mathbb{D}(\vec{y})|^2 \,\mathrm{d}\vec{x} + \kappa \int_\Omega |\vec{y}|^2 \,\mathrm{d}\vec{x} + \beta \int_\Omega |\vec{y}|^3 \,\mathrm{d}\vec{x} \\
&\leq \sup_{\vec{u} \in \mathcal{U}} \left| \int_\Omega \mathbf{C}\vec{u} \cdot \vec{y} \,\mathrm{d}\vec{x} \right| \\
&\leq \|\mathbf{C}\|_{\mathscr{L}(\boldsymbol{L}^2(\Omega))} \sup_{\vec{u} \in \mathcal{U}} \|\vec{u}\|_{\boldsymbol{L}^2(\Omega)} \|\vec{y}\|_{\boldsymbol{L}^2(\Omega)},
\end{aligned} \tag{42}$$

for any $\vec{y} \in \Pr_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} \mathcal{S}_{\text{weak}}(\mathcal{U})$.

On the other hand, condition (H.3) yields

$$\eta_{\min} \|\vec{y}\|^2_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} \leq \int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y})|)|\mathbb{D}(\vec{y})|^2 \,\mathrm{d}\vec{x}. \tag{43}$$

Combining (42) and (43), we obtain

$$\eta_{\min} \|\vec{y}\|^2_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} \leq \|\mathbf{C}\|_{\mathscr{L}(\boldsymbol{L}^2(\Omega))} \sup_{\vec{u} \in \mathcal{U}} \|\vec{u}\|_{\boldsymbol{L}^2(\Omega)} \|\vec{y}\|_{\boldsymbol{L}^2(\Omega)}. \tag{44}$$

Let $\mathbf{Id}$ denote the embedding operator of $\boldsymbol{H}^1_{0,\sigma}(\Omega)$ into $\boldsymbol{L}^2(\Omega)$. Then

$$\|\vec{y}\|_{\boldsymbol{L}^2(\Omega)} \leq \|\mathbf{Id}\|_{\mathscr{L}(\boldsymbol{H}^1_{0,\sigma}(\Omega), \boldsymbol{L}^2(\Omega))} \|\vec{y}\|_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}. \tag{45}$$

From (44) and (45) we derive the estimate

$$\|\vec{y}\|_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} \leq \frac{1}{\eta_{\min}} \|\mathbf{C}\|_{\mathscr{L}(\boldsymbol{L}^2(\Omega))} \|\mathbf{Id}\|_{\mathscr{L}(\boldsymbol{H}^1_{0,\sigma}(\Omega), \boldsymbol{L}^2(\Omega))} \sup_{\vec{u} \in \mathcal{U}} \|\vec{u}\|_{\boldsymbol{L}^2(\Omega)}. \tag{46}$$

Since $\vec{y}$ is arbitrary, this establishes the boundedness of $\mathrm{Pr}_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}\mathcal{S}_{\mathrm{weak}}(\mathcal{U})$ in $\boldsymbol{H}^1_{0,\sigma}(\Omega)$.

Step 4: Existence of optimal weak solutions. With the results of the previous three steps at hand, we may apply a generalized Weierstrass theorem (see [41, Sect. 38.3]) to conclude that there exists a pair $(\vec{y}_*, \vec{u}_*) \in \mathcal{S}_{\mathrm{weak}}(\mathcal{U})$ satisfying (26). This completes the proof of (a).

Further, we prove (b). Suppose, for contradiction, that there exists a subsequence $\{m_\ell\}_{\ell=1}^{\infty}$ such that

$$\lim_{\ell\to\infty} V_J(\mathcal{U}_{m_\ell}) < V_J(\mathcal{U}_0). \tag{47}$$

Let $\{(\vec{y}^*_{m_\ell}, \vec{u}^*_{m_\ell})\}_{\ell=1}^{\infty}$ be a sequence such that $(\vec{y}^*_{m_\ell}, \vec{u}^*_{m_\ell}) \in \mathcal{S}^{\mathrm{opt}}_{\mathrm{weak}}(\mathcal{U}_{m_\ell})$ for each ℓ. By Definition 8, we have

$$\begin{aligned}
&-\sum_{i=1}^{n}\int_\Omega y^*_{m_\ell i}\vec{y}^*_{m_\ell}\cdot\frac{\partial\vec{v}}{\partial x_i}\,\mathrm{d}\vec{x} + \int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}^*_{m_\ell})|)\mathbb{D}(\vec{y}^*_{m_\ell}) : \mathbb{D}(\vec{v})\,\mathrm{d}\vec{x}\\
&\quad - \alpha\sum_{i=1}^{n}\int_\Omega y^*_{m_\ell i}\mathbb{D}(\vec{y}^*_{m_\ell}) : \frac{\partial\mathbb{D}(\vec{v})}{\partial x_i}\,\mathrm{d}\vec{x} + \kappa\int_\Omega \vec{y}^*_{m_\ell}\cdot\vec{v}\,\mathrm{d}\vec{x}\\
&\quad + \beta\int_\Omega |\vec{y}^*_{m_\ell}|\vec{y}^*_{m_\ell}\cdot\vec{v}\,\mathrm{d}\vec{x} = \int_\Omega \mathbf{C}\vec{u}^*_{m_\ell}\cdot\vec{v}\,\mathrm{d}\vec{x}, \quad \forall\vec{v}\in\boldsymbol{H}^3_{0,\sigma}(\Omega),\ \ell\in\mathbb{Z}_+,
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
\int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}^*_{m_\ell})|)|\mathbb{D}(\vec{y}^*_{m_\ell})|^2\,\mathrm{d}\vec{x} + \kappa\int_\Omega |\vec{y}^*_{m_\ell}|^2\,\mathrm{d}\vec{x} + \beta\int_\Omega |\vec{y}^*_{m_\ell}|^3\,\mathrm{d}\vec{x}\\
\le \int_\Omega \mathbf{C}\vec{u}^*_{m_\ell}\cdot\vec{y}^*_{m_\ell}\,\mathrm{d}\vec{x}, \quad \forall\ell\in\mathbb{Z}_+.
\end{aligned} \tag{49}$$

Furthermore, by the definition of the value function V_J,

$$V_J(\mathcal{U}_{m_\ell}) = J(\vec{y}^*_{m_\ell}, \vec{u}^*_{m_\ell}), \quad \forall\ell\in\mathbb{Z}_+. \tag{50}$$

We claim that the number sets $\{\|\vec{y}^*_{m_\ell}\|_{\boldsymbol{H}^1_{0,\sigma}(\Omega)}\}_{\ell=1}^{\infty}$ and $\{\|\vec{u}^*_{m_\ell}\|_{\boldsymbol{L}^2(\Omega)}\}_{\ell=1}^{\infty}$ are bounded. Indeed, by the triangle inequality,

$$\sup_{\vec{u}\in\mathcal{U}_{m_\ell}}\|\vec{u}\|_{\boldsymbol{L}^2(\Omega)} \le \sup_{\vec{u}\in\mathcal{U}_0}\|\vec{u}\|_{\boldsymbol{L}^2(\Omega)} + \varepsilon_{m_\ell}, \tag{51}$$

where $\varepsilon_{m_\ell} := d_{\boldsymbol{L}^2(\Omega)}(\mathcal{U}_{m_\ell}, \mathcal{U}_0)$.

From (28), the sequence $\{\varepsilon_{m_\ell}\}_{\ell=1}^{\infty}$ converges and is therefore bounded.

Let $\varepsilon_* := \sup_{\ell\in\mathbb{Z}_+} \varepsilon_{m_\ell}$. Then (51) gives

$$\sup_{\ell\in\mathbb{Z}_+} \|\vec{u}^*_{m_\ell}\|_{\boldsymbol{L}^2(\Omega)} \le \sup_{\vec{u}\in\mathcal{U}_0} \|\vec{u}\|_{\boldsymbol{L}^2(\Omega)} + \varepsilon_*. \tag{52}$$

Applying (46) with $\vec{y} = \vec{y}^*_{m_\ell}$ and $\mathcal{U} = \mathcal{U}_{m_\ell}$, we get

$$\|\vec{y}^*_{m_\ell}\|_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} \le \frac{1}{\eta_{\min}} \|\mathbf{C}\|_{\mathscr{L}(\boldsymbol{L}^2(\Omega))} \|\mathbf{Id}\|_{\mathscr{L}(\boldsymbol{H}^1_{0,\sigma}(\Omega),\boldsymbol{L}^2(\Omega))} \sup_{\vec{u}\in\mathcal{U}_{m_\ell}} \|\vec{u}\|_{\boldsymbol{L}^2(\Omega)}.$$

Together with (51), this yields

$$\begin{aligned}\sup_{\ell\in\mathbb{Z}_+} \|\vec{y}^*_{m_\ell}\|_{\boldsymbol{H}^1_{0,\sigma}(\Omega)} \le{}& \frac{1}{\eta_{\min}} \|\mathbf{C}\|_{\mathscr{L}(\boldsymbol{L}^2(\Omega))} \|\mathbf{Id}\|_{\mathscr{L}(\boldsymbol{H}^1_{0,\sigma}(\Omega),\boldsymbol{L}^2(\Omega))} \\ &\times \Big(\sup_{\vec{u}\in\mathcal{U}_0} \|\vec{u}\|_{\boldsymbol{L}^2(\Omega)} + \varepsilon_* \Big).\end{aligned} \tag{53}$$

Since $\mathcal{U}_0$ is bounded in $\boldsymbol{L}^2(\Omega)$, (52) and (53) imply the boundedness of the sequence $\{(\vec{y}^*_{m_\ell}, \vec{u}^*_{m_\ell})\}_{\ell=1}^\infty$ in $\boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{L}^2(\Omega)$. Hence, passing to a subsequence if necessary, we may assume that

$$\vec{y}^*_{m_\ell} \xrightarrow{\mathrm{w}} \vec{y}^* \ \text{ in } \boldsymbol{H}^1_{0,\sigma}(\Omega) \text{ as } \ell\to\infty, \tag{54}$$

$$\vec{u}^*_{m_\ell} \xrightarrow{\mathrm{w}} \vec{u}^* \ \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } \ell\to\infty, \tag{55}$$

for some $(\vec{y}^*, \vec{u}^*) \in \boldsymbol{H}^1_{0,\sigma}(\Omega) \times \boldsymbol{L}^2(\Omega)$.

Arguing as in Step 2, we pass to the limit $\ell\to\infty$ in (48) and (49), obtaining

$$\begin{aligned}&-\sum_{i=1}^n \int_\Omega y_i^* \vec{y}^* \cdot \frac{\partial \vec{v}}{\partial x_i}\,\mathrm{d}\vec{x} + \int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}^*)|)\mathbb{D}(\vec{y}^*) : \mathbb{D}(\vec{v})\,\mathrm{d}\vec{x} \\ &\quad - \alpha \sum_{i=1}^n \int_\Omega y_i^* \mathbb{D}(\vec{y}^*) : \frac{\partial \mathbb{D}(\vec{v})}{\partial x_i}\,\mathrm{d}\vec{x} + \kappa \int_\Omega \vec{y}^* \cdot \vec{v}\,\mathrm{d}\vec{x} \\ &\quad + \beta \int_\Omega |\vec{y}^*| \vec{y}^* \cdot \vec{v}\,\mathrm{d}\vec{x} = \int_\Omega \mathbf{C}\vec{u}^* \cdot \vec{v}\,\mathrm{d}\vec{x}\end{aligned}$$

and

$$\int_\Omega \eta(|\mathbf{M}\mathbb{D}(\vec{y}^*)|)|\mathbb{D}(\vec{y}^*)|^2\,\mathrm{d}\vec{x} + \kappa \int_\Omega |\vec{y}^*|^2\,\mathrm{d}\vec{x} + \beta \int_\Omega |\vec{y}^*|^3\,\mathrm{d}\vec{x} \le \int_\Omega \mathbf{C}\vec{u}^* \cdot \vec{y}^*\,\mathrm{d}\vec{x}.$$

Now, for each $\ell \in \mathbb{Z}_+$, there exists $\underline{\vec{u}}_\ell \in \mathcal{U}_0$ such that

$$\|\vec{u}^*_{m_\ell} - \underline{\vec{u}}_\ell\|_{\boldsymbol{L}^2(\Omega)} \le \varepsilon_{m_\ell}.$$

Since $\lim_{\ell\to\infty}\varepsilon_{m_\ell}=0$, we have

$$\lim_{\ell\to\infty}\|\vec{u}^{\,*}_{m_\ell}-\underline{\vec{u}}_\ell\|_{\boldsymbol{L}^2(\Omega)}=0.$$

Combining this with (55), we get

$$\underline{\vec{u}}_\ell \xrightarrow{\mathrm{w}} \vec{u}^{\,*} \ \text{ in } \boldsymbol{L}^2(\Omega) \text{ as } \ell\to\infty. \tag{56}$$

Since $\mathcal{U}_0$ is sequentially weakly closed in $\boldsymbol{L}^2(\Omega)$ and $\{\underline{\vec{u}}_\ell\}_{\ell=1}^{\infty}\subset\mathcal{U}_0$, the limit $\vec{u}^{\,*}$ arising from (56) belongs to $\mathcal{U}_0$.

Thus, we have shown that

$$(\vec{y}^{\,*},\vec{u}^{\,*})\in\mathcal{S}_{\mathrm{weak}}(\mathcal{U}_0).$$

Using (50), (54), (55), and condition (H.6), we obtain

$$\begin{aligned}
V_J(\mathcal{U}_0) &= \inf\{J(\vec{y},\vec{u}):\ (\vec{y},\vec{u})\in\mathcal{S}_{\mathrm{weak}}(\mathcal{U}_0)\}\\
&\le J(\vec{y}^{\,*},\vec{u}^{\,*})\\
&\le \liminf_{\ell\to\infty} J(\vec{y}^{\,*}_{m_\ell},\vec{u}^{\,*}_{m_\ell})\\
&= \lim_{\ell\to\infty} V_J(\mathcal{U}_{m_\ell}),
\end{aligned}$$

which contradicts (47). This contradiction completes the proof of (b).

Assertion (c) follows immediately from Proposition 7 and (27). Indeed, if a classical solution exists, the energy balance holds as the equality $J_{\mathrm{d}}=0$, so the infimum of J_{d} over all weak solutions cannot exceed zero. Since $J_{\mathrm{d}}\ge 0$ for all weak solutions, the infimum is exactly zero.

Thus, Theorem 11 is completely proved. □

6 Conclusions

In this paper, we have developed a rigorous framework for optimal distributed control of steady-state flows of a second-grade fluid with variable viscosity through a porous medium. The main result of the work is the proof of the existence of an optimal weak solution for the strongly nonlinear system describing such flows. This result is obtained by applying an abstract solvability principle for degenerate nonlinear operator equations involving weakly continuous and completely continuous mappings in Sobolev spaces.

The key novelty of the present study is the combination of three essential features that have not been considered together in the existing literature:

- the second-grade rheological model, which accounts for viscoelastic effects;
- the presence of a porous medium, modeled by the linear Darcy term $\kappa\vec{y}$ and the nonlinear Forchheimer term $\beta|\vec{y}|\vec{y}$;
- the nonlocal dependence of the viscosity on the shear rate, which allows the material response at a point to be influenced by the deformation in a neighbourhood of that point.

These features lead to a strongly nonlinear system whose mathematical analysis requires careful compactness arguments and special energy estimates.

In addition to the existence result, we have introduced and analysed the value function associated with the admissible control set and established its lower semicontinuity with respect to the one-sided Hausdorff distance. This implies that optimal control is stable in the sense that small perturbations of the admissible control set cannot lead to a significant reduction of the optimal value of the cost functional. Furthermore, a necessary condition for the existence of classical solutions has been derived in terms of the value function and the energy discrepancy functional.

Several directions for future research naturally arise from the results of this work. Of particular interest is the extension to optimal boundary control problems, where the control action is applied through boundary conditions rather than through body forces. Such a formulation is more realistic for many applications but requires a different functional framework and new estimates for solutions. Another important direction is the analysis of non-stationary problems, which involve time derivatives and necessitate the use of Bochner spaces and compactness results for time-dependent functions. Finally, the development of efficient numerical schemes for solving optimal control problems remains an open challenge, especially given the nonlocal nature of the viscosity operator.

Thus, the results obtained in this paper provide a foundation for further developments in the theory of optimal control for complex rheological models in porous media and open up perspectives for applications in the petroleum industry, biomechanics, and chemical engineering.

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