

ON UNIFORM POLYNOMIAL TRISPLITTING OF EVOLUTION OPERATORS*

Claudia Luminița Mihit[†] Ghiocel Mot[‡]

Dedicated to the memory of Professor Mihail Megan

DOI 10.56082/annalsarscimath.2026.3.145

Abstract

The article explores the concept of uniform polynomial trisplitting, as generalization of the uniform polynomial trichotomy for the case of evolution operators in Banach spaces. We prove that the uniform polynomial trisplitting in continuous-time is equivalent to the uniform polynomial trisplitting in discrete-time (assuming the existence of the uniform polynomial growth), using compatible families of projections, respectively strongly compatible families of projections. Also, discrete criteria for this property are given and in particular, new results for uniform polynomial trichotomy are obtained.

Keywords: evolution operators, polynomial trisplitting, polynomial trichotomy.

MSC: 34D05, 34D09.

1 Introduction

The property of trichotomy played a central role in the qualitative theory of dynamical systems and was intensively approached in the last years from

*Accepted for publication on July 28, 2026

[†]claudia.mihit@uav.ro, Department of Mathematics and Computer Science, *Aurel Vlaicu* University of Arad, Elena Drăgoi Street no. 2, 310330 Arad, Romania

[‡]ghiocel.mot@uav.ro, Department of Mathematics and Computer Science, *Aurel Vlaicu* University of Arad, Elena Drăgoi Street no. 2, 310330 Arad, Romania

different points of view (see [1, 4, 25, 27]). R.J. Sacker and G.R. Sell [24] introduced for the first time the notion of an exponential trichotomy for skew-product semiflows and the studies were further developed by B. Aulbach [2] in the case of differential systems. For nonautonomous dynamics, notable results are obtained by K.J. Palmer [22], where the author showed that the hyperbolicity properties could be described through trichotomies. This trichotomy concept represents a direct generalization of the classical exponential dichotomy notion - we refer here to the dichotomy notion explored in recent years via different methods in [17, 18, 26] and the references therein. Nevertheless, in some situations the concept of exponential dichotomy is considered restrictive and therefore it is important to consider other asymptotic behaviors, such as those of polynomial type.

The nonuniform polynomial dichotomy was introduced by L. Barreira and C. Valls in [5] and an interesting study in the nonuniform case was made in [6]. A slightly modified concept of nonuniform polynomial dichotomy was studied in [8], where the authors proved characterizations using some families of norms compatible with the projection families. In the case of discrete nonautonomous systems, we remark the accomplishments from [10] for the polynomial dichotomy with respect to a sequence of norms. Regarding the polynomial trichotomic behavior, two concepts of uniform polynomial trichotomy were treated in [21] for the skew-evolution semiflows and a sufficient criteria for the uniform polynomial trichotomy was proved, based on a result from [9].

Another step in the trichotomies topics was made by S. Elaydi and K. Janglajew in [16], where the authors approached for the first time the exponential trichotomy in discrete-time for difference equations

$$x(n+1) = A(n)x(n), \quad n \in \mathbb{Z},$$

where $A(n)$ was a $k \times k$ invertible matrix defined on $\mathbb{Z}$ and then the results were extended to nonlinear difference equations. Previously, interesting concepts of trichotomy were studied by S. Elaydi and O. Hájek (see [14, 15]) for linear, respectively nonlinear differential systems. We emphasize that in [14–16], the authors treated versions of trichotomies that are reduced to double dichotomies which is in a sharp contrast with the approaches in [2, 22, 24]. Recently, important results for the property of the robustness of exponential trichotomy in the original sense of Sacker and Sell have been obtained in uniform and nonuniform context, for nonautonomous and variational systems (see D. Dragičević, A. Sasu and B. Sasu [11, 12]).

The exponential splitting property was introduced by B. Aulbach, J. Kalkbrenner in [3] for difference equations, to extend the framework of

dichotomy to a more general setting. In [19], the properties of exponential splitting, respectively strong exponential splitting are treated for linear discrete-time systems, as generalization of exponential dichotomy, respectively strong exponential dichotomy. In a natural manner, the uniform exponential trichotomy was extended to uniform exponential trisplitting in [7] for discrete skew-product semiflows.

In 1984, K.M. Przyluski and S. Rolewicz [23] proved the discrete-time version of the well-known stability theorem of R. Datko [9]: a linear discrete-time system

$$x(n+1) = A(n)x(n), \quad n \geq n_0,$$

on a Banach space X is uniformly exponentially stable if and only if there exists $p \in [1, +\infty)$ such that:

$$\sup_{n \geq n_0} \sum_{j=n+1}^{+\infty} \left\| \left(\prod_{i=n}^{j-1} A_i \right) x \right\|^p < +\infty.$$

Various methods, which led to the generalizations of the above result, have been presented in [13] and [20]. Furthermore, as shown in [13] this type of results have substantial applications in exploring robustness for stability and instability.

An important and difficult problem in the asymptotic theory of dynamical systems is the connection between discrete-time and continuous time dynamics. In this setting, the full recovery of the exponential trichotomy in continuous-time from discrete-time exponential trichotomy was proved for the first time in A. Sasu and B. Sasu [25], through admissibility techniques. In a distinct direction, the analysis of continuous and discrete exponential trichotomy for skew-evolution semiflows is made in [28] and discrete characterizations were given.

In the context of current developments and new trends in the qualitative theory of dynamical systems, we approach in this paper the property of uniform polynomial trisplitting of evolution operators in terms of compatible and strongly compatible families of projections. We prove the equivalence between uniform polynomial trisplitting in continuous-time and the uniform polynomial trisplitting in discrete-time and we deduce the equivalence between uniform polynomial trichotomy in continuous-time and the uniform polynomial trichotomy in discrete-time. Discrete results of Przyluski-Rolewicz type are established and also, the connection between uniform polynomial trisplitting and uniform polynomial trichotomy are obtained.

2 Evolution operators. Families of projections

In this section, we introduce the main concepts of evolution operators, projections and compatibility properties between them. We consider X a Banach space, $\mathcal{B}(X)$ will denote the Banach algebra of all bounded linear operators on X and the norms on X , respectively on $\mathcal{B}(X)$ will be denoted by $\|\cdot\|$. Further, we will use the sets

$$\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\}, \quad \Delta_d = \{(m, n) \in \mathbb{N}^2 : m \geq n\},$$

$$T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}, \quad T_d = \{(m, n, p) \in \mathbb{N}^3 : m \geq n \geq p\}.$$

Definition 1. We say that $U : \Delta \rightarrow \mathcal{B}(X)$ is an *evolution operator* on X if the following conditions hold:

(eo₁) $U(s, s) = I$ (the identity operator on X), for every $s \geq 0$;

(eo₂) $U(t, s)U(s, t_0) = U(t, t_0)$, for all $(t, s, t_0) \in T$.

Definition 2. A mapping $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ is a *family of projections* if

$$P(s)^2 = P(s), \quad \text{for every } s \geq 0.$$

Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and let $P_1, P_2, P_3 : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ be three families of projections.

Definition 3. We say that $\mathcal{P} = \{P_1, P_2, P_3\}$ is a family of *compatible* projections with U if:

(c₁) $P_1(s) + P_2(s) + P_3(s) = I$, for all $s \geq 0$;

(c₂) $P_i(s)P_j(s) = 0$, for all $i, j \in \overline{1, 3}$, $i \neq j$, $s \geq 0$;

(c₃) $P_i(t)U(t, s) = U(t, s)P_i(s)$, for all $(t, s) \in \Delta$, $i \in \overline{1, 3}$.

Definition 4. The family of projections $\mathcal{P} = \{P_1, P_2, P_3\}$ is said to be *strongly compatible* with U if it is compatible with U and for all $(t, s) \in \Delta$, the mapping $U(t, s)$ is an isomorphism from $\text{Range } P_i(s)$ to $\text{Range } P_i(t)$, $i = 2, 3$.

Proposition 1. If $\mathcal{P} = \{P_1, P_2, P_3\}$ is strongly compatible with U , then there exist $V_i : \Delta \rightarrow \mathcal{B}(X)$, such that $V_i(t, s)$ is an isomorphism from $\text{Range } P_i(t)$ to $\text{Range } P_i(s)$, $i = 2, 3$, with the following properties:

(v₁) $U(t, s)V_i(t, s)P_i(t) = P_i(s)$;

$$(v_2) \quad V_i(t, s)U(t, s)P_i(s) = P_i(s);$$

$$(v_3) \quad V_i(t, s)P_i(t) = P_i(s)V_i(t, s)P_i(t);$$

$$(v_4) \quad P_i(t) = V_i(t, t)P_i(t) = P_i(t)V_i(t, t)P_i(t);$$

$$(v_5) \quad V_i(t, t_0)P_i(t) = V_i(s, t_0)V_i(t, s)P_i(t),$$

for all $(t, s, t_0) \in T$.

Proof. See [18]. □

In what follows, we will denote by

$$U_i(t, s) = U(t, s)P_i(s), \quad \text{for all } (t, s) \in \Delta, \quad i = \overline{1, 3},$$

respectively

$$V_j^j(t, s) = V_j(t, s)P_j(t), \quad \text{for all } (t, s) \in \Delta, \quad j = 2, 3.$$

3 Uniform polynomial trisplitting with compatible families of projections

In this section, we prove the equivalence between the uniform polynomial trisplitting in continuous-time and the uniform polynomial trisplitting in discrete-time, using the uniform polynomial growth property and compatible families of projections. Next, we give Przyluski-Rolewicz type criteria for the uniform polynomial trisplitting and as consequences, we obtain the results for the uniform polynomial trichotomy.

We consider $U : \Delta \rightarrow \mathcal{B}(X)$ an evolution operator and $\mathcal{P} = \{P_1, P_2, P_3\}$ a compatible family of projections with U .

Definition 5. We say that the pair $(U, \mathcal{P})$ has an *uniform polynomial trisplitting* if there exist some real constants $N \geq 1$ and $\alpha < \beta$, $\gamma < \delta$ such that the following relations are satisfied:

$$(ups_1) \quad (s+1)^\alpha \|U_1(t, t_0)x_0\| \leq N(t+1)^\alpha \|U_1(s, t_0)x_0\|;$$

$$(ups_2) \quad (t+1)^\beta \|U_2(s, t_0)x_0\| \leq N(s+1)^\beta \|U_2(t, t_0)x_0\|;$$

$$(ups_3) \quad (t+1)^\gamma \|U_3(t, t_0)x_0\| \leq N(s+1)^\gamma \|U_3(s, t_0)x_0\|;$$

$$(ups_4) \quad (s+1)^\delta \|U_3(s, t_0)x_0\| \leq N(t+1)^\delta \|U_3(t, t_0)x_0\|,$$

for all $(t, s, t_0, x_0) \in T \times X$.

As a particular case, if $\alpha < 0 < \beta$, $\gamma < 0 < \delta$, then we say that $(U, \mathcal{P})$ is *uniformly polynomially trichotomic*.

Remark 1. The pair $(U, \mathcal{P})$ is uniformly polynomially trichotomic if and only if there exist some constants $N \geq 1$, $\nu_1, \nu_2 > 0$ such that:

$$\begin{aligned} (upt_1) \quad & (t+1)^{\nu_1} \|U_1(t, t_0)x_0\| \leq N(s+1)^{\nu_1} \|U_1(s, t_0)x_0\|; \\ (upt_2) \quad & (t+1)^{\nu_1} \|U_2(s, t_0)x_0\| \leq N(s+1)^{\nu_1} \|U_2(t, t_0)x_0\|; \\ (upt_3) \quad & (s+1)^{\nu_2} \|U_3(t, t_0)x_0\| \leq N(t+1)^{\nu_2} \|U_3(s, t_0)x_0\|; \\ (upt_4) \quad & (s+1)^{\nu_2} \|U_3(s, t_0)x_0\| \leq N(t+1)^{\nu_2} \|U_3(t, t_0)x_0\|, \end{aligned}$$

for all $(t, s, t_0, x_0) \in T \times X$.

In what follows, we show that the concepts of uniform polynomial trichotomy and uniform polynomial trisplitting are distinct, more precisely uniform polynomial trichotomy implies uniform polynomial trisplitting, but, for all that, the converse implication is false.

Example 1. On $X = \mathbb{R}^3$, endowed with the norm

$$\|(x_1, x_2, x_3)\| = |x_1| + |x_2| + |x_3|, \quad x = (x_1, x_2, x_3) \in X,$$

we consider the evolution operator $U : \Delta \rightarrow \mathcal{B}(X)$, defined by

$$U(t, t_0)x = \left(\left(\frac{t+1}{t_0+1} \right)^{-e^2} x_1, \left(\frac{t+1}{t_0+1} \right)^{-e} x_2, \left(\frac{t_0+1}{t+1} \right)^e x_3 \right)$$

and $P_i : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$, $i = \overline{1, 3}$ the canonical families of projections.

We immediately see that $(U, \mathcal{P})$ has a uniform polynomial trisplitting with $N = 1$, $\alpha = -e^2$, $\beta = -e$, $\gamma = e$ and $\delta = 2e$.

Let us suppose that $(U, \mathcal{P})$ is uniformly polynomially trichotomic. It follows that there are $N \geq 1$, $\nu_1 > 0$ with

$$(t+1)^{\nu_1} \|U_2(s, t_0)x_0\| \leq N(s+1)^{\nu_1} \|U_2(t, t_0)x_0\|, \quad \text{for all } (t, s, t_0, x_0) \in T \times X.$$

Thus, we obtain

$$\left(\frac{t+1}{s+1} \right)^{\nu_1+e} \leq N, \quad \text{for all } (t, s) \in \Delta$$

and for $s = e - 1$, $t \rightarrow +\infty$, it yields a contradiction.

In conclusion, $(U, \mathcal{P})$ is not uniformly polynomially trichotomic.

Definition 6. We say that $(U, \mathcal{P})$ has *uniform polynomial growth* if there exist $M \geq 1, \omega_1, \omega_2 > 0$ with:

$$\begin{aligned} (upg_1) \quad & (s+1)^{\omega_1} \|U_1(t, t_0)x_0\| \leq M(t+1)^{\omega_1} \|U_1(s, t_0)x_0\|; \\ (upg_2) \quad & (s+1)^{\omega_1} \|U_2(s, t_0)x_0\| \leq M(t+1)^{\omega_1} \|U_2(t, t_0)x_0\|; \\ (upg_3) \quad & (s+1)^{\omega_2} \|U_3(t, t_0)x_0\| \leq M(t+1)^{\omega_2} \|U_3(s, t_0)x_0\|; \\ (upg_4) \quad & (s+1)^{\omega_2} \|U_3(s, t_0)x_0\| \leq M(t+1)^{\omega_2} \|U_3(t, t_0)x_0\|, \end{aligned}$$

for all $(t, s, t_0, x_0) \in T \times X$.

Remark 2. If $(U, \mathcal{P})$ has a uniform polynomial trisplitting, then $(U, \mathcal{P})$ admits uniform polynomial growth.

Theorem 1. *The pair $(U, \mathcal{P})$ admits a uniform polynomial trisplitting if and only if $(U, \mathcal{P})$ has uniform polynomial growth and there exist some real constants $S \geq 1, a < b, c < d$ such that:*

$$\begin{aligned} (dups_1) \quad & (n+1)^a \|U_1(m, p)x_0\| \leq S(m+1)^a \|U_1(n, p)x_0\|; \\ (dups_2) \quad & (m+1)^b \|U_2(n, p)x_0\| \leq S(n+1)^b \|U_2(m, p)x_0\|; \\ (dups_3) \quad & (m+1)^c \|U_3(m, p)x_0\| \leq S(n+1)^c \|U_3(n, p)x_0\|; \\ (dups_4) \quad & (n+1)^d \|U_3(n, p)x_0\| \leq S(m+1)^d \|U_3(m, p)x_0\|, \end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$.

Proof. Necessity. It is obvious.

Sufficiency. We prove that the inequalities $(ups_i), i = \overline{1, 4}$ from Definition 5 are satisfied.

(ups_1) Let $(t, s, t_0) \in T, x_0 \in X$ and we denote $m = [t], n = [s], p = [t_0]$.

Case 1. If $m \geq n+1 \geq p+1$, then we have

$$\begin{aligned} \|U_1(t, t_0)x_0\| &\leq M \left(\frac{t+1}{m+1} \right)^{\omega_1} \|U_1(m, n+1)U_1(n+1, t_0)x_0\| \\ &\leq MS2^{\omega_1} \left(\frac{m+1}{n+2} \right)^a \|U_1(n+1, s)U_1(s, t_0)x_0\| \\ &\leq M^2 S2^{\omega_1} \left(\frac{m+1}{n+2} \right)^a \left(\frac{n+2}{s+1} \right)^{\omega_1} \|U_1(s, t_0)x_0\| \\ &\leq M^2 S2^{2\omega_1} \left(\frac{m+1}{n+2} \right)^a \|U_1(s, t_0)x_0\| \end{aligned}$$

$$\begin{aligned}
&\leq M^2 S 2^{2\omega_1} \max\{1, 2^{-2a}\} \left(\frac{t+1}{s+1}\right)^a \|U_1(s, t_0)x_0\| \\
&\leq N \left(\frac{t+1}{s+1}\right)^a \|U_1(s, t_0)x_0\|,
\end{aligned}$$

where $N = M^2 S 2^{2(\omega_1+\omega_2)} \max\{1, 2^{-2a}, 2^{2b}, 2^{2c}, 2^{-2d}\}$.

Case 2. For $p+1 \leq m < n+1$, or $m < p+1 \leq n+1$, we obtain

$$\begin{aligned}
&\|U_1(t, t_0)x_0\| \leq M \left(\frac{t+1}{s+1}\right)^{\omega_1} \|U_1(s, t_0)x_0\| \\
&\leq M 2^{2\omega_1} \max\{1, 2^{-2a}\} \left(\frac{t+1}{s+1}\right)^a \|U_1(s, t_0)x_0\| \leq N \left(\frac{t+1}{s+1}\right)^a \|U_1(s, t_0)x_0\|.
\end{aligned}$$

It yields that

$$(s+1)^a \|U_1(t, t_0)x_0\| \leq N(t+1)^a \|U_1(s, t_0)x_0\|,$$

for all $(t, s, t_0, x_0) \in T \times X$.

(*ups*₂) We consider $(t, s, t_0, x_0) \in T \times X$ and $m = [t]$, $n = [s]$, $p = [t_0]$.

Case 1. For $m \geq n+1 \geq p+1$, we deduce

$$\begin{aligned}
\|U_2(t, t_0)x_0\| &\geq \frac{1}{M} \left(\frac{m+1}{t+1}\right)^{\omega_1} \|U_2(m, n+1)U_2(n+1, t_0)x_0\| \\
&\geq \frac{1}{MS} 2^{-\omega_1} \left(\frac{m+1}{n+2}\right)^b \|U_2(n+1, t_0)x_0\| \\
&\geq \frac{1}{M^2 S} 2^{-\omega_1} \left(\frac{m+1}{n+2}\right)^b \left(\frac{n+2}{s+1}\right)^{-\omega_1} \|U_2(s, t_0)x_0\| \\
&\geq \frac{1}{M^2 S} 2^{-2\omega_1} \left(\frac{m+1}{n+2}\right)^b \|U_2(s, t_0)x_0\| \\
&\geq \frac{1}{M^2 S} 2^{-2\omega_1} \frac{1}{\max\{1, 2^{2b}\}} \left(\frac{t+1}{s+1}\right)^b \|U_2(s, t_0)x_0\|,
\end{aligned}$$

which implies that

$$(t+1)^b \|U_2(s, t_0)x_0\| \leq N(s+1)^b \|U_2(t, t_0)x_0\|.$$

Case 2. If $p+1 \leq m < n+1$, or $m < p+1 \leq n+1$, then

$$\|U_2(t, t_0)x_0\| \geq \frac{1}{M} \left(\frac{t+1}{s+1}\right)^{-\omega_1} \|U_2(s, t_0)x_0\|$$

$$\geq \frac{1}{M} 2^{-2\omega_1} \frac{1}{\max\{1, 2^{2b}\}} \left(\frac{t+1}{s+1} \right)^b \|U_2(s, t_0)x_0\|.$$

Thus,

$$(t+1)^b \|U_2(s, t_0)x_0\| \leq N(s+1)^b \|U_2(t, t_0)x_0\|,$$

for all $(t, s, t_0, x_0) \in T \times X$.

(*ups*₃) Let $(t, s, t_0) \in T$, $x_0 \in X$ and we denote $m = [t]$, $n = [s]$, $p = [t_0]$.

Case 1. If $m \geq n+1 \geq p+1$, then

$$\begin{aligned} \|U_3(t, t_0)x_0\| &\leq M \left(\frac{t+1}{m+1} \right)^{\omega_2} \|U_3(m, n+1)U_3(n+1, t_0)x_0\| \\ &\leq M^2 S 2^{\omega_2} \left(\frac{n+2}{m+1} \right)^c \left(\frac{n+2}{s+1} \right)^{\omega_2} \|U_3(s, t_0)x_0\| \\ &\leq M^2 S 2^{2\omega_2} \max\{1, 2^{2c}\} \left(\frac{s+1}{t+1} \right)^c \|U_3(s, t_0)x_0\| \\ &\leq N \left(\frac{s+1}{t+1} \right)^c \|U_3(s, t_0)x_0\|. \end{aligned}$$

Case 2. For $p+1 \leq m < n+1$, or $m < p+1 \leq n+1$, it follows that

$$\begin{aligned} \|U_3(t, t_0)x_0\| &\leq M \left(\frac{t+1}{s+1} \right)^{\omega_2} \|U_3(s, t_0)x_0\| \\ &\leq M 2^{2\omega_2} \max\{1, 2^{2c}\} \left(\frac{s+1}{t+1} \right)^c \|U_3(s, t_0)x_0\| \leq N \left(\frac{s+1}{t+1} \right)^c \|U_3(s, t_0)x_0\|. \end{aligned}$$

We obtain

$$(t+1)^c \|U_3(t, t_0)x_0\| \leq N(s+1)^c \|U_3(s, t_0)x_0\|,$$

for all $(t, s, t_0, x_0) \in T \times X$.

(*ups*₄) Based on an analogous argument as in (*ups*₂), we deduce

$$(s+1)^d \|U_3(s, t_0)x_0\| \leq M^2 S 2^{2\omega_2} \max\{1, 2^{-2d}\} (t+1)^d \|U_3(t, t_0)x_0\|$$

and then

$$(s+1)^d \|U_3(s, t_0)x_0\| \leq N(t+1)^d \|U_3(t, t_0)x_0\|,$$

for all $(t, s, t_0, x_0) \in T \times X$.

We conclude that $(U, \mathcal{P})$ has a uniform polynomial trisplitting. $\square$

Corollary 1. *The pair $(U, \mathcal{P})$ is uniformly polynomially trichotomic if and only if $(U, \mathcal{P})$ has uniform polynomial growth and there are some constants $T \geq 1$, $a < 0 < b$, $c < 0 < d$ with:*

$$\begin{aligned} (dupt_1) \quad & (n+1)^a \|U_1(m, p)x_0\| \leq T(m+1)^a \|U_1(n, p)x_0\|; \\ (dupt_2) \quad & (m+1)^b \|U_2(n, p)x_0\| \leq T(n+1)^b \|U_2(m, p)x_0\|; \\ (dupt_3) \quad & (m+1)^c \|U_3(m, p)x_0\| \leq T(n+1)^c \|U_3(n, p)x_0\|; \\ (dupt_4) \quad & (n+1)^d \|U_3(n, p)x_0\| \leq T(m+1)^d \|U_3(m, p)x_0\|, \end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$.

Proof. It is immediate from Theorem 1. $\square$

A necessary condition for the uniform polynomial trisplitting is given by:

Theorem 2. *If the pair $(U, \mathcal{P})$ has a uniform polynomial trisplitting, then there exist some real constants $D \geq 1$, $d_1 < d_2$, $d_3 < d_4$ such that:*

$$\begin{aligned} (Dps_1) \quad & \sum_{j=n}^{+\infty} (j+1)^{-d_1-1} \|U_1(j, p)x_0\| \leq D(n+1)^{-d_1} \|U_1(n, p)x_0\|; \\ (Dps_2) \quad & \sum_{j=n}^m (j+1)^{-d_2} \|U_2(j, p)x_0\| \leq D(m+1)^{1-d_2} \|U_2(m, p)x_0\|; \\ (Dps_3) \quad & \sum_{j=n}^{+\infty} (j+1)^{d_3-1} \|U_3(j, p)x_0\| \leq D(n+1)^{d_3} \|U_3(n, p)x_0\|; \\ (Dps_4) \quad & \sum_{j=n}^m (j+1)^{d_4} \|U_3(j, p)x_0\| \leq D(m+1)^{d_4+1} \|U_3(m, p)x_0\|, \end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$.

Proof. Let $\alpha < d_1 < d_2 < \beta$, $d_3 < \gamma < \delta < d_4$ and

$$D = N \max \left\{ \frac{d_1 - \alpha + 1}{d_1 - \alpha}, \frac{\gamma - d_3 + 1}{\gamma - d_3} \right\}.$$

Thus, for all $(m, n, p, x_0) \in T_d \times X$ the following relations hold:

(Dps_1)

$$\sum_{j=n}^{+\infty} (j+1)^{-d_1-1} \|U_1(j, p)x_0\| \leq N(n+1)^{-\alpha} \sum_{j=n}^{+\infty} (j+1)^{\alpha-d_1-1} \|U_1(n, p)x_0\|$$

$$\begin{aligned}
&\leq N(n+1)^{-\alpha} \left[(n+1)^{\alpha-d_1-1} + \frac{1}{d_1-\alpha} (n+1)^{\alpha-d_1} \right] \|U_1(n, p)x_0\| \\
&\leq N \cdot \frac{d_1-\alpha+1}{d_1-\alpha} (n+1)^{-d_1} \|U_1(n, p)x_0\| \leq D(n+1)^{-d_1} \|U_1(n, p)x_0\|;
\end{aligned}$$

(Dps₂)

$$\begin{aligned}
\sum_{j=n}^m (j+1)^{-d_2} \|U_2(j, p)x_0\| &\leq N(m+1)^{-\beta} \sum_{j=n}^m (j+1)^{\beta-d_2} \|U_2(m, p)x_0\| \\
&\leq N(m+1)^{1-d_2} \|U_2(m, p)x_0\| \leq D(m+1)^{1-d_2} \|U_2(m, p)x_0\|;
\end{aligned}$$

(Dps₃)

$$\begin{aligned}
\sum_{j=n}^{+\infty} (j+1)^{d_3-1} \|U_3(j, p)x_0\| &\leq N(n+1)^\gamma \sum_{j=n}^{+\infty} (j+1)^{d_3-\gamma-1} \|U_3(n, p)x_0\| \\
&\leq N(n+1)^\gamma \left[(n+1)^{d_3-\gamma-1} + \frac{1}{\gamma-d_3} (n+1)^{d_3-\gamma} \right] \|U_3(n, p)x_0\| \\
&\leq D(n+1)^{d_3} \|U_3(n, p)x_0\|;
\end{aligned}$$

(Dps₄)

$$\begin{aligned}
\sum_{j=n}^m (j+1)^{d_4} \|U_3(j, p)x_0\| &\leq N(m+1)^\delta \sum_{j=n}^m (j+1)^{d_4-\delta} \|U_3(m, p)x_0\| \\
&\leq N(m+1)^{d_4+1} \|U_3(m, p)x_0\| \leq D(m+1)^{d_4+1} \|U_3(m, p)x_0\|.
\end{aligned}$$

So, the inequalities are verified. $\square$

Corollary 2. *If $(U, \mathcal{P})$ is uniformly polynomially trichotomic, then there are some constants $D \geq 1$, $d_1 < 0 < d_2$, $d_3 < 0 < d_4$ with:*

$$\begin{aligned}
(Dpt_1) \quad &\sum_{j=n}^{+\infty} (j+1)^{-d_1-1} \|U_1(j, p)x_0\| \leq D(n+1)^{-d_1} \|U_1(n, p)x_0\|; \\
(Dpt_2) \quad &\sum_{j=n}^m (j+1)^{-d_2} \|U_2(j, p)x_0\| \leq D(m+1)^{1-d_2} \|U_2(m, p)x_0\|; \\
(Dpt_3) \quad &\sum_{j=n}^{+\infty} (j+1)^{d_3-1} \|U_3(j, p)x_0\| \leq D(n+1)^{d_3} \|U_3(n, p)x_0\|; \\
(Dpt_4) \quad &\sum_{j=n}^m (j+1)^{d_4} \|U_3(j, p)x_0\| \leq D(m+1)^{d_4+1} \|U_3(m, p)x_0\|,
\end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$.

Proof. It is a consequence of Theorem 2. $\square$

In what follows, we present a sufficient condition for the uniform polynomial trisplitting.

Theorem 3. *Let $(U, \mathcal{P})$ be a pair with uniform polynomial growth. If there are some real constants $D \geq 1$, $d_1 < d_2$, $d_3 < d_4$ such that:*

$$\begin{aligned} (Dps'_1) \quad & \sum_{j=n}^{+\infty} (j+1)^{-d_1} \|U_1(j, p)x_0\| \leq D(n+1)^{-d_1} \|U_1(n, p)x_0\|; \\ (Dps'_2) \quad & \sum_{j=n}^m (j+1)^{-d_2} \|U_2(j, p)x_0\| \leq D(m+1)^{-d_2} \|U_2(m, p)x_0\|; \\ (Dps'_3) \quad & \sum_{j=n}^{+\infty} (j+1)^{d_3} \|U_3(j, p)x_0\| \leq D(n+1)^{d_3} \|U_3(n, p)x_0\|; \\ (Dps'_4) \quad & \sum_{j=n}^m (j+1)^{d_4} \|U_3(j, p)x_0\| \leq D(m+1)^{d_4} \|U_3(m, p)x_0\|, \end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$, then $(U, \mathcal{P})$ admits a uniform polynomial trisplitting.

Proof. Considering $j = m$ in (Dps'_1) , respectively (Dps'_3) , we deduce:

$$(n+1)^{d_1} \|U_1(m, p)x_0\| \leq D(m+1)^{d_1} \|U_1(n, p)x_0\|,$$

respectively

$$(m+1)^{d_3} \|U_3(m, p)x_0\| \leq D(n+1)^{d_3} \|U_3(n, p)x_0\|,$$

for all $(m, n, p, x_0) \in T_d \times X$.

Similarly, taking $j = n$ in (Dps'_2) , respectively (Dps'_4) , we obtain:

$$(m+1)^{d_2} \|U_2(n, p)x_0\| \leq D(n+1)^{d_2} \|U_2(m, p)x_0\|,$$

respectively,

$$(n+1)^{d_4} \|U_3(n, p)x_0\| \leq D(m+1)^{d_4} \|U_3(m, p)x_0\|,$$

for all $(m, n, p, x_0) \in T_d \times X$.

From Theorem 1, we conclude that $(U, \mathcal{P})$ has a uniform polynomial trisplitting. $\square$

In particular, we obtain:

Corollary 3. *Let $(U, \mathcal{P})$ be a pair with uniform polynomial growth. If there exist $D \geq 1$, $d_1 < 0 < d_2$, $d_3 < 0 < d_4$ with:*

$$\begin{aligned} (Dpt'_1) \quad & \sum_{j=n}^{+\infty} (j+1)^{-d_1} \|U_1(j, p)x_0\| \leq D(n+1)^{-d_1} \|U_1(n, p)x_0\|; \\ (Dpt'_2) \quad & \sum_{j=n}^m (j+1)^{-d_2} \|U_2(j, p)x_0\| \leq D(m+1)^{-d_2} \|U_2(m, p)x_0\|; \\ (Dpt'_3) \quad & \sum_{j=n}^{+\infty} (j+1)^{d_3} \|U_3(j, p)x_0\| \leq D(n+1)^{d_3} \|U_3(n, p)x_0\|; \\ (Dpt'_4) \quad & \sum_{j=n}^m (j+1)^{d_4} \|U_3(j, p)x_0\| \leq D(m+1)^{d_4} \|U_3(m, p)x_0\|, \end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$, then $(U, \mathcal{P})$ is uniformly polynomially trichotomic.

4 Uniform polynomial trisplitting with strongly compatible families of projections

In what follows, we demonstrate the equivalence between the uniform polynomial trisplitting in continuous-time and the uniform polynomial trisplitting in discrete-time, using the uniform polynomial growth, strongly compatible families of projections and the properties from Proposition 1. First, we prove two auxiliary results for the uniform polynomial trisplitting, respectively uniform polynomial growth in terms of strongly compatible families of projections (Proposition 2, respectively Proposition 3). Next, we obtain Przyłuski-Rolewicz type conditions for the uniform polynomial trisplitting and in particular, Przyłuski-Rolewicz type criteria for the uniform polynomial trichotomy.

Let $U : \Delta \rightarrow \mathcal{B}(X)$ be an evolution operator and let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a strongly compatible family of projections with U .

A first characterization for the uniform polynomial trisplitting, using strongly compatible families of projections is given by:

Proposition 2. *The pair $(U, \mathcal{P})$ admits a uniform polynomial trisplitting if and only if there are some real constants $N \geq 1$, $\alpha < \beta$ and $\gamma < \delta$ such that:*

$$\begin{aligned} (ups_1) \quad & (s+1)^\alpha \|U_1(t, t_0)x_0\| \leq N(t+1)^\alpha \|U_1(s, t_0)x_0\|; \\ (sups_2) \quad & (s+1)^\beta \|V_2^2(t, t_0)x_0\| \leq N(t_0+1)^\beta \|V_2^2(t, s)x_0\|; \\ (ups_3) \quad & (t+1)^\gamma \|U_3(t, t_0)x_0\| \leq N(s+1)^\gamma \|U_3(s, t_0)x_0\|; \\ (sups_4) \quad & (t_0+1)^\delta \|V_3^3(t, t_0)x_0\| \leq N(s+1)^\delta \|V_3^3(t, s)x_0\|, \end{aligned}$$

for all $(t, s, t_0, x_0) \in T \times X$.

Proof. We demonstrate $(sups_2) \Leftrightarrow (ups_2)$ and $(sups_4) \Leftrightarrow (ups_4)$.

For the implication $(sups_2) \Rightarrow (ups_2)$, we observe that:

$$\begin{aligned} (t+1)^\beta \|U_2(s, t_0)x_0\| &= (t+1)^\beta \|V_2^2(t, s)V_2^2(t, t)U_2(t, s)U_2(s, t_0)x_0\| \\ &\leq N(s+1)^\beta \|V_2^2(t, t)U_2(t, t_0)x_0\| = N(s+1)^\beta \|U_2(t, t_0)x_0\|, \end{aligned}$$

for all $(t, s, t_0, x_0) \in T \times X$.

For the converse implication $(ups_2) \Rightarrow (sups_2)$, we have:

$$\begin{aligned} (s+1)^\beta \|V_2^2(t, t_0)x_0\| &= (s+1)^\beta \|U_2(t, t_0)V_2^2(t, t_0)x_0\| \\ &\leq N(t_0+1)^\beta \|U_2(s, t_0)V_2^2(s, t_0)V_2^2(t, s)x_0\| = N(t_0+1)^\beta \|V_2^2(t, s)x_0\|, \end{aligned}$$

for all $(t, s, t_0, x_0) \in T \times X$.

Similarly, it yields that $(sups_4) \Leftrightarrow (ups_4)$. $\square$

Concerning the uniform polynomial growth with strongly compatible families of projections, we obtain:

Proposition 3. *The pair $(U, \mathcal{P})$ admits a uniform polynomial growth if and only if there exist $M \geq 1$, $\omega_1, \omega_2 > 0$ with:*

$$\begin{aligned} (upg_1) \quad & (s+1)^{\omega_1} \|U_1(t, t_0)x_0\| \leq M(t+1)^{\omega_1} \|U_1(s, t_0)x_0\|; \\ (supg_2) \quad & (t_0+1)^{\omega_1} \|V_2^2(t, t_0)x_0\| \leq M(s+1)^{\omega_1} \|V_2^2(t, s)x_0\|; \\ (upg_3) \quad & (s+1)^{\omega_2} \|U_3(t, t_0)x_0\| \leq M(t+1)^{\omega_2} \|U_3(s, t_0)x_0\|; \\ (supg_4) \quad & (t_0+1)^{\omega_2} \|V_3^3(t, t_0)x_0\| \leq M(s+1)^{\omega_2} \|V_3^3(t, s)x_0\|, \end{aligned}$$

for all $(t, s, t_0, x_0) \in T \times X$.

Proof. It follows based on an analogous technique as in the proof of Proposition 2. $\square$

Theorem 4. *The pair $(U, \mathcal{P})$ has a uniform polynomial trisplitting if and only if $(U, \mathcal{P})$ has uniform polynomial growth and there exist some real constants $S \geq 1$, $a < b$, $c < d$ such that:*

$$\begin{aligned} (dups_1) \quad & (n+1)^a \|U_1(m, p)x_0\| \leq S(m+1)^a \|U_1(n, p)x_0\|; \\ (sdups_2) \quad & (n+1)^b \|V_2^2(m, p)x_0\| \leq S(p+1)^b \|V_2^2(m, n)x_0\|; \\ (dups_3) \quad & (m+1)^c \|U_3(m, p)x_0\| \leq S(n+1)^c \|U_3(n, p)x_0\|; \\ (sdups_4) \quad & (p+1)^d \|V_3^3(m, p)x_0\| \leq S(n+1)^d \|V_3^3(m, n)x_0\|, \end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$.

Proof. Necessity. It is obvious from Proposition 2.

Sufficiency. It is sufficient to prove the implications $(sdups_2) \Rightarrow (sups_2)$ and $(sdups_4) \Rightarrow (sups_4)$, where $(sups_2)$ and $(sups_4)$ are the inequalities from Proposition 2.

$(sdups_2) \Rightarrow (sups_2)$. Let $(t, s, t_0, x_0) \in T \times X$ and we denote $m = [t]$, $n = [s]$, $p = [t_0]$.

Case 1. If $m \geq n \geq p+1$, then

$$\begin{aligned} \|V_2^2(t, t_0)x_0\| &= \|V_2^2(n, t_0)V_2^2(t, n)x_0\| \\ &\leq M \left(\frac{p+2}{t_0+1} \right)^{\omega_1} \|V_2^2(n, p+1)V_2^2(t, n)x_0\| \\ &\leq MS2^{\omega_1} \left(\frac{p+2}{n+1} \right)^b \|V_2^2(s, n)V_2^2(t, s)x_0\| \\ &\leq M^2S2^{2\omega_1} \left(\frac{p+2}{n+1} \right)^b \|V_2^2(t, s)x_0\| \\ &\leq M^2S2^{2\omega_1} \max\{1, 2^{2b}\} \left(\frac{t_0+1}{s+1} \right)^b \|V_2^2(t, s)x_0\|, \end{aligned}$$

which implies that

$$(s+1)^b \|V_2^2(t, t_0)x_0\| \leq N(t_0+1)^b \|V_2^2(t, s)x_0\|,$$

where $N = M^2S2^{2(\omega_1+\omega_2)} \max\{1, 2^{-2a}, 2^{2b}, 2^{2c}, 2^{-2d}\}$.

Case 2. If $n \leq m < p+1$, we deduce

$$\|V_2^2(t, t_0)x_0\| \leq M \left(\frac{s+1}{t_0+1} \right)^{\omega_1} \|V_2^2(t, s)x_0\|$$

$$\leq M2^{2\omega_1} \max\{1, 2^{2b}\} \left(\frac{t_0 + 1}{s + 1}\right)^b \|V_2^2(t, s)x_0\|$$

and then

$$(s + 1)^b \|V_2^2(t, t_0)x_0\| \leq N(t_0 + 1)^b \|V_2^2(t, s)x_0\|,$$

for all $(t, s, t_0, x_0) \in T \times X$.

In a similar manner we obtain $(sdups_4) \Rightarrow (sups_4)$, more precisely, for all $(t, s, t_0, x_0) \in T \times X$ it yields that

$$(t_0 + 1)^d \|V_3^3(t, t_0)x_0\| \leq M^2 S 2^{2\omega_2} \max\{1, 2^{-2d}\} (s + 1)^d \|V_3^3(t, s)x_0\|$$

and then

$$(t_0 + 1)^d \|V_3^3(t, t_0)x_0\| \leq N(s + 1)^d \|V_3^3(t, s)x_0\|.$$

Using Proposition 2, we deduce that $(U, \mathcal{P})$ has a uniform polynomial trisplitting. $\square$

Corollary 4. *The pair $(U, \mathcal{P})$ has a uniform polynomial trichotomy if and only if $(U, \mathcal{P})$ has uniform polynomial growth and there exist some constants $T \geq 1$, $a < 0 < b$, $c < 0 < d$ with*

$$\begin{aligned} (dupt_1) \quad & (n + 1)^a \|U_1(m, p)x_0\| \leq T(m + 1)^a \|U_1(n, p)x_0\|; \\ (sdupt_2) \quad & (n + 1)^b \|V_2^2(m, p)x_0\| \leq T(p + 1)^b \|V_2^2(m, n)x_0\|; \\ (dupt_3) \quad & (m + 1)^c \|U_3(m, p)x_0\| \leq T(n + 1)^c \|U_3(n, p)x_0\|; \\ (sdupt_4) \quad & (p + 1)^d \|V_3^3(m, p)x_0\| \leq T(n + 1)^d \|V_3^3(m, n)x_0\|, \end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$.

Proof. It follows from Theorem 4. $\square$

Next, we give a necessary criterion for the uniform polynomial trisplitting.

Theorem 5. *If the pair $(U, \mathcal{P})$ has a uniform polynomial trisplitting, then there are some real constants $D \geq 1$, $d_1 < d_2$, $d_3 < d_4$ with:*

$$\begin{aligned} (Dps_1) \quad & \sum_{j=n}^{+\infty} (j + 1)^{-d_1-1} \|U_1(j, p)x_0\| \leq D(n + 1)^{-d_1} \|U_1(n, p)x_0\|; \\ (sDps_2) \quad & \sum_{j=p}^n (j + 1)^{-d_2} \|V_2^2(m, j)x_0\| \leq D(n + 1)^{1-d_2} \|V_2^2(m, n)x_0\|; \end{aligned}$$

$$\begin{aligned}
(Dps_3) \quad & \sum_{j=n}^{+\infty} (j+1)^{d_3-1} \|U_3(j, p)x_0\| \leq D(n+1)^{d_3} \|U_3(n, p)x_0\|; \\
(sDps_4) \quad & \sum_{j=p}^n (j+1)^{d_4} \|V_3^3(m, j)x_0\| \leq D(n+1)^{d_4+1} \|V_3^3(m, n)x_0\|,
\end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$.

Proof. Let $\alpha < d_1 < d_2 < \beta$, $d_3 < \gamma < \delta < d_4$ and

$$D = N \max \left\{ \frac{d_1 - \alpha + 1}{d_1 - \alpha}, \frac{\gamma - d_3 + 1}{\gamma - d_3} \right\}.$$

As in Theorem 2, we obtain that (Dps_1) and (Dps_3) hold.

For all $(m, n, p, x_0) \in T_d \times X$, we deduce:

$$\begin{aligned}
(sDps_2) \quad & \sum_{j=p}^n (j+1)^{-d_2} \|V_2^2(m, j)x_0\| \leq N(n+1)^{-\beta} \sum_{j=p}^n (j+1)^{\beta-d_2} \|V_2^2(m, n)x_0\| \\
& \leq N(n+1)^{1-d_2} \|V_2^2(m, n)x_0\| \leq D(n+1)^{1-d_2} \|V_2^2(m, n)x_0\|; \\
(sDps_4) \quad & \sum_{j=p}^n (j+1)^{d_4} \|V_3^3(m, j)x_0\| \leq N(n+1)^\delta \sum_{j=p}^n (j+1)^{d_4-\delta} \|V_3^3(m, n)x_0\| \\
& \leq N(n+1)^{d_4+1} \|V_3^3(m, n)x_0\| \leq D(n+1)^{d_4+1} \|V_3^3(m, n)x_0\|.
\end{aligned}$$

□

As a consequence, we deduce:

Corollary 5. *If $(U, \mathcal{P})$ is uniformly polynomially trichotomic, then there exist some constants $D \geq 1$, $d_1 < 0 < d_2$, $d_3 < 0 < d_4$ such that:*

$$\begin{aligned}
(Dpt_1) \quad & \sum_{j=n}^{+\infty} (j+1)^{-d_1-1} \|U_1(j, p)x_0\| \leq D(n+1)^{-d_1} \|U_1(n, p)x_0\|; \\
(sDpt_2) \quad & \sum_{j=p}^n (j+1)^{-d_2} \|V_2^2(m, j)x_0\| \leq D(n+1)^{1-d_2} \|V_2^2(m, n)x_0\|;
\end{aligned}$$

$$\begin{aligned}
(Dpt_3) \quad & \sum_{j=n}^{+\infty} (j+1)^{d_3-1} \|U_3(j, p)x_0\| \leq D(n+1)^{d_3} \|U_3(n, p)x_0\|; \\
(sDpt_4) \quad & \sum_{j=p}^n (j+1)^{d_4} \|V_3^3(m, j)x_0\| \leq D(n+1)^{d_4+1} \|V_3^3(m, n)x_0\|,
\end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$.

Proof. It follows from Theorem 5. $\square$

A sufficient condition for the uniform polynomial trisplitting is the following:

Theorem 6. *Let $(U, \mathcal{P})$ be a pair with uniform polynomial growth. If there are some real constants $D \geq 1$, $d_1 < d_2$, $d_3 < d_4$ such that:*

$$\begin{aligned}
(Dps'_1) \quad & \sum_{j=n}^{+\infty} (j+1)^{-d_1} \|U_1(j, p)x_0\| \leq D(n+1)^{-d_1} \|U_1(n, p)x_0\|; \\
(sDps'_2) \quad & \sum_{j=p}^n (j+1)^{-d_2} \|V_2^2(m, j)x_0\| \leq D(n+1)^{-d_2} \|V_2^2(m, n)x_0\|; \\
(Dps'_3) \quad & \sum_{j=n}^{+\infty} (j+1)^{d_3} \|U_3(j, p)x_0\| \leq D(n+1)^{d_3} \|U_3(n, p)x_0\|; \\
(sDps'_4) \quad & \sum_{j=p}^n (j+1)^{d_4} \|V_3^3(m, j)x_0\| \leq D(n+1)^{d_4} \|V_3^3(m, n)x_0\|,
\end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$, then $(U, \mathcal{P})$ has a uniform polynomial trisplitting.

Proof. For $j = m$ in (Dps'_1) , respectively (Dps'_3) , we obtain the conditions $(dups_1)$, $(dups_3)$ from Theorem 4.

Similarly, taking $j = p$ in $(sDps'_2)$, respectively $(sDps'_4)$, we deduce that $(sdups_2)$ and $(sdups_4)$ from Theorem 4 hold.

Thus, $(U, \mathcal{P})$ has a uniform polynomial trisplitting. $\square$

Corollary 6. *Let $(U, \mathcal{P})$ be with uniform polynomial growth. If there are $D \geq 1$, $d_1 < 0 < d_2$, $d_3 < 0 < d_4$ with:*

$$\begin{aligned}
(Dpt'_1) \quad & \sum_{j=n}^{+\infty} (j+1)^{-d_1} \|U_1(j, p)x_0\| \leq D(n+1)^{-d_1} \|U_1(n, p)x_0\|; \\
(sDpt'_2) \quad & \sum_{j=p}^n (j+1)^{-d_2} \|V_2^2(m, j)x_0\| \leq D(n+1)^{-d_2} \|V_2^2(m, n)x_0\|; \\
(Dpt'_3) \quad & \sum_{j=n}^{+\infty} (j+1)^{d_3} \|U_3(j, p)x_0\| \leq D(n+1)^{d_3} \|U_3(n, p)x_0\|; \\
(sDpt'_4) \quad & \sum_{j=p}^n (j+1)^{d_4} \|V_3^3(m, j)x_0\| \leq D(n+1)^{d_4} \|V_3^3(m, n)x_0\|,
\end{aligned}$$

for all $(m, n, p, x_0) \in T_d \times X$, then $(U, \mathcal{P})$ is uniformly polynomially trichotomic.

Proof. It is a consequence of Theorem 6. □

Acknowledgements. We are grateful to the reviewer for the careful evaluation and detailed reading of our work.

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