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ON EKELAND VARIATIONAL PRINCIPLE FOR PREORDERED UNIFORM SPACE*

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Abstract

We introduce a notion of preordered uniform space which extends the classical notion of uniform space and uniformizable topological space. We establish two versions of Ekeland variational principle for the present setting. The first version is for a real-valued submonotone map in a general preordered uniform space while the second version is for a vector-valued map but within an at most countable uniform structure.

Keywords: uniform space, preordered space, submonotone map, variational principle.

MSC: 58E30, 06A75, 54E15, 49J52.

1 Introduction

Uniform topological spaces appear in general topology in relation with Tychonoff's separation axiom ($T_{3\frac{1}{2}}$). Specifically, a topological space X is said to satisfy ($T_{3\frac{1}{2}}$) if for every $x \in X$ and every closed subset $C \subset X$ with $x \notin C$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and

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$f|_C \equiv 1$. It is shown in [1] that this axiom is equivalent to the property of X to be uniformizable, i.e., the topology of X can be induced by a so-called *uniform structure*. A uniform structure consists of a family of écarts, also called pseudometrics. Endowed with a uniform structure, X is called a *uniform space*. We refer to [1] for the precise definition of uniform structure and uniform space, or to Section 3.2 below.

The setting of the present paper is in fact more general: we consider a set P endowed with a *preorder* $\leq$ (a binary relation which is reflexive and transitive). We first extend the notion of uniform structure and uniform space to the case of a preordered set $(P, \leq)$, by defining a notion of *preordered uniform space* $(P, \leq, \mathcal{D})$ parametrized by a family $\mathcal{D}$ of $\leq$*-pseudometrics* (see Section 3.2). When the preorder $\leq$ is trivial (i.e., such that $x \leq y$ for all $x, y \in P$), we retrieve the classical notion of uniform space.

Instead of focussing on purely topological properties, our aim in this paper is to do variational analysis on preordered uniform spaces. Our main results are two versions of the classical Ekeland variational principle (which is usually formulated for metric spaces) valid for a preordered uniform space $(P, \leq, \mathcal{D})$:

- The first result (Theorem 1) is a version of Ekeland variational principle for an arbitrary complete preordered uniform space $(P, \leq, \mathcal{D})$ and concerns a map $f : P \to \mathbb{R} \cup \{+\infty\}$ with a submonotonicity property (it is not assumed continuous and not even lower semicontinuous).

- The second result (Theorem 2) is a version of Ekeland's principle for a family of maps $\mathcal{F} = \{f_d\}_{d \in \mathcal{D}}$ with $f_d : P \to \mathbb{R} \cup \{+\infty\}$, instead of just a map f, but with the assumption that the family $\mathcal{D}$ of $\leq$-pseudometrics is (at most) countable.

For showing these results, the main ingredient is a version of Ekeland variational principle over preordered monoids; see [3]. For the first theorem (Theorem 1), we apply this result to the monoid $\mathbb{K} = \mathbb{R} \cup \{+\infty\}$; for the second theorem (Theorem 2), we use the monoid $\mathbb{K} = (\mathbb{R} \cup \{+\infty\})^{\mathbb{N}}$. In order to be able to apply the general result in [3], we rely on a thorough analysis of preordered uniform spaces and their properties: key lemmas in this respect are Lemmas 1 (b), 4, and 5.

The paper is organized as follows. In Section 2, we introduce a general notion of preordered topological space. In Section 3, we first introduce the notions of $\leq$-pseudometric and preordered pseudometric space (Section 3.1); then we introduce the more general notion of preordered uniform space

(Section 3.2). Our main results are stated in Section 4 and the proofs are given in Section 5.

2 Preordered topological space

Throughout this paper, we fix a nonempty set P endowed with a preorder $\leq$, i.e., a binary relation on P which is reflexive and transitive (but not necessarily antisymmetric).

A sequence $(x_n) \subset P$ is said to be $\leq$-*ascending* if it satisfies $x_n \leq x_{n+1}$ for all n. Moreover, we say that x is an *upper bound* of a sequence $(x_n) \subset P$ if $x_n \leq x$ for all n.

Assume that the set P is equipped with a topology τ. We associate to this topology a second topology $\tilde{\tau}_{\leq}$ which takes the preorder $\leq$ into account and is defined in the following way. Hereafter, the notation $x_n \xrightarrow{\tau} x$ means that x is a limit of (x_n) with respect to τ (the limit is a priori not unique).

- A subset $A \subset P$ is said to be $(\leq, \tau)$-*closed* if

$$L_{(\leq,\tau)}(A) := \{x \in P : \exists (x_n) \subset A \leq\text{-ascending},\ x_n \xrightarrow{\tau} x\} = A$$

  (note that the inclusion $A \subset L_{(\leq,\tau)}(A)$ is always valid).

- We say that $U \subset P$ is $(\leq, \tau)$-*open* if $P \setminus U$ is $(\leq, \tau)$-closed.

Then it is straightforward to check that

$$\tilde{\tau}_{\leq} := \{U \subset P : U \text{ is } (\leq, \tau)\text{-open}\}$$

is indeed a topology on P. We call $\tilde{\tau}_{\leq}$ the $\leq$-*ascending extension* of τ.

Lemma 1. (a) $\tau \subset \tilde{\tau}_{\leq}$.

(b) *Let $(x_n) \subset P$ be a $\leq$-ascending sequence and let $x \in P$. Then*

$$x_n \xrightarrow{\tilde{\tau}_{\leq}} x \quad \Leftrightarrow \quad x_n \xrightarrow{\tau} x.$$

Proof. (a) Let $A \subset P$ be a closed subset for the topology τ and let us show that A is also closed for the topology $\tilde{\tau}_{\leq}$, i.e., that A is $(\leq, \tau)$-closed. So let $(x_n) \subset A$ be a $\leq$-ascending sequence and let $x \in P$ be such that $x_n \xrightarrow{\tau} x$. The latter fact, combined with the assumption that A is closed for the topology τ, already ensures that $x \in A$, and the claim is established.

(b) The implication $\Rightarrow$ easily follows from part (a). Let us prove the reversed implication $\Leftarrow$. Arguing by contradiction, assume that there exists $U \in \tilde{\tau}_{\leq}$ such that $x \in U$ and

$$\forall n_0 \in \mathbb{N},\ \exists n \geq n_0,\ x_n \notin U.$$

This yields a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \notin U$ for all k. Note that (x_{n_k}) is a $\leq$-ascending sequence (since (x_n) is $\leq$-ascending) and $x_{n_k} \xrightarrow{\tau} x$ (since $x_n \xrightarrow{\tau} x$). The subset $A := P \setminus U$ is $(\leq, \tau)$-closed (since $U \in \tilde{\tau}_{\leq}$) and $x_{n_k} \in A$ for all k. Whence $x \in A = P \setminus U$, a contradiction. □

Remark 1. (a) In Lemma 1 (b), the assumption that (x_n) is $\leq$-ascending is not needed for the implication $\Rightarrow$, but is required for the reversed implication $\Leftarrow$, as shown by the following example. Let $P = \mathbb{R}$ be endowed with the standard topology τ and standard order $\leq$. Let $(x_n)_{n\geq 1}$ be the sequence given by $x_n = \frac{1}{n}$ and let $x = 0$. Thus $x_n \xrightarrow{\tau} x$. However, (x_n) does not converge to x with respect to $\tilde{\tau}_{\leq}$. Indeed, it is easy to see that $(0, +\infty)$ is $(\leq, \tau)$-closed, hence $(-\infty, 0]$ is an open neighborhood of $x = 0$ in $\tilde{\tau}_{\leq}$ which contains no term of the sequence (x_n).

(b) If we define $\sigma = \tilde{\tau}_{\leq}$, then Lemma 1 implies that $\tilde{\sigma}_{\leq} = \sigma\,(= \tilde{\tau}_{\leq})$. Indeed, Lemma 1 (a) already yields the inclusion $\sigma \subset \tilde{\sigma}_{\leq}$. For the reversed inclusion, let $A \subset P$ be $(\leq, \sigma)$-closed and let us show that A is $(\leq, \tau)$-closed. So let $(x_n) \subset A$ be $\leq$-ascending and $x \in P$ such that $x_n \xrightarrow{\tau} x$. By Lemma 1 (b) we get $x_n \xrightarrow{\sigma} x$, whence $x \in A$ (since A is $(\leq, \sigma)$-closed), and the argument is complete.

(c) A subset $A \subset P$ is *sequentially closed* with respect to τ if for every sequence $(x_n) \subset A$ and $x \in P$ such that $x_n \xrightarrow{\tau} x$, we have $x \in A$. We say that $U \subset P$ is *sequentially open* with respect to τ if $P \setminus U$ is sequentially closed. The collection $\tilde{\tau}$ of all sequentially open subsets is a topology on P which contains τ and differs from τ in general. The equality $\tilde{\tau} = \tau$ holds if every $x \in P$ has a countable local basis in the topology τ, i.e., the topological space (P, τ) satisfies the axiom of countability (C1).

The inclusion of Lemma 1 (a) can be refined as the chain of inclusions $\tau \subset \tilde{\tau} \subset \tilde{\tau}_{\leq}$. If $\leq$ is the *trivial preorder* on P, i.e., such that $x \leq y$ for all $x, y \in P$, then $\tilde{\tau}_{\leq} = \tilde{\tau}$.

Remark 2. It can be noted that the $\leq$-ascending extension preserves inclusions between topologies, in the following sense. If τ and τ' are two topologies on P such that $\tau \subset \tau'$, then we obtain that $\tilde{\tau}_{\leq} \subset \tilde{\tau}'_{\leq}$. Indeed, if $A \subset P$ is $(\leq, \tau)$-closed, then whenever $(x_n) \subset A$ is a $\leq$-ascending sequence such that $x_n \xrightarrow{\tau'} x \in P$, then the inclusion $\tau \subset \tau'$ yields $x_n \xrightarrow{\tau} x$, whence

$x \in A$ (using that A is $(\leq, \tau)$-closed), and finally we have shown that A is $(\leq, \tau')$-closed.

We say that the preorder $\leq$ is *self-closed* for the topology τ if the set

$$P(\geq x) := \{y \in P : \ x \leq y\}$$

is closed (in τ) for all $x \in P$.

Lemma 2. (a) *If $\leq$ is self-closed for the topology τ, then it is self-closed for the topology $\tilde{\tau}_{\leq}$.*

(b) *The following conditions are equivalent:*

(i) *$\leq$ is self-closed for the topology $\tilde{\tau}_{\leq}$;*

(ii) *for every $\leq$-ascending sequence $(z_n) \subset P$ and every $z \in P$ such that $z_n \overset{\tau}{\to} z$ we have $z_n \leq z$ for all n (i.e., any limit of a $\leq$-ascending sequence is an upper bound of this sequence).*

Proof. (a) This follows from Lemma 1 (a).

(b) First we show the implication (ii) $\Rightarrow$ (i). Let $x \in P$ and let us show that $P(\geq x)$ is closed for the topology $\tilde{\tau}_{\leq}$, i.e., is $(\leq, \tau)$-closed. Let $(z_n) \subset P(\geq x)$ be a $\leq$-ascending sequence such that $z_n \overset{\tau}{\to} z \in P$. For any fixed n, condition (ii) yields $z_n \leq z$, while we know that $x \leq z_n$ (since $z_n \in P(\geq x)$), whence $z \in P(\geq x)$, which establishes (i).

Next let us show the implication (i) $\Rightarrow$ (ii). Let $(z_n) \subset P$ be a $\leq$-ascending sequence such that $z_n \overset{\tau}{\to} z$. For every n, the sequence $(z_m)_{m \geq n}$ is $\leq$-ascending, contained in $P(\geq z_n)$, and has limit z. Since $P(\geq z_n)$ is $(\leq, \tau)$-closed (by virtue of (i)), we obtain that $z \in P(\geq z_n)$; whence $z_n \leq z$ for all n, and the lemma is established. □

Remark 3. (a) Clearly, $\leq$ is self-closed for $\tilde{\tau}_{\leq}$ whenever the subset $P(\geq x)$ is sequentially closed with respect to the topology τ for all $x \in P$, i.e., $\leq$ is self-closed for the topology $\tilde{\tau}$ (see Remark 1 (c)). The latter condition is a priori weaker than requiring that $\leq$ is self-closed with respect to τ, though both notions coincide if for instance (P, τ) satisfies the axiom of countability (C1).

(b) Another sufficient condition for having that $\leq$ is self-closed with respect to $\tilde{\tau}_{\leq}$ is that $P(\leq x) := \{y \in P : \ y \leq x\} \in \tau$ for all $x \in P$. Indeed, if $(x_n) \subset P$ is a $\leq$-ascending sequence such that $x_n \overset{\tau}{\to} x$, then we obtain that $x_n \in P(\leq x)$ for all large enough n, and in fact $x_n \leq x$ for all n, since (x_n) is $\leq$-ascending. The desired conclusion now comes from Lemma 2 (b).

(c) The implication in Lemma 2 (a) is not an equivalence even if (P,τ) satisfies axiom (C1) (see part (a)). Take for instance $P=[0,1]\cup\{2\}$ endowed with the standard metric topology τ and the total order $\leq$ whose restriction to $(0,1]$ is the standard total order and such that $0<2<x$ for all $x\in(0,1]$. Then $\leq$ is not self-closed with respect to τ since $P(\geq 2)=(0,1]\cup\{2\}$ is not closed for the metric topology. However, $P(\geq 2)$ is $(\leq,\tau)$-closed, as well as $P(\geq x)$ for all $x\in[0,1]$, hence $\leq$ is self-closed with respect to $\tilde{\tau}_{\leq}$.

3 Preordered uniform space

In this section, we still consider a nonempty set P endowed with a preorder $\leq$. First, we apply the definitions and results of Section 2 to the case where the topology of P is induced by a pseudometric – in fact, a $\leq$-pseudometric. Then we replace the single $\leq$-pseudometric by a family of $\leq$-pseudometrics.

3.1 Preordered pseudometric space

We call *$\leq$-pseudometric on P* a map

$$d : P^2_{\leq} := \{(x,y)\in P\times P : x\leq y\} \to [0,+\infty]$$

such that

- $d(x,x)=0$ for all $x\in P$,
- $d(x,z)\leq d(x,y)+d(y,z)$ for all $x,y,z\in P$ such that $x\leq y\leq z$.

Then, we say that $(P,\leq,d)$ is a *preordered pseudometric space*.

For every $\eta\in\mathbb{R}^*_+$ and $x\in P$, we set

$$P^d_\eta(\leq x)=\{y\in P : y\leq x,\ d(y,x)<\eta\}.$$

The subsets $P^d_\eta(\leq x)$, for all η and x, form the basis of a topology τ_d on P.

Given a sequence $(x_n)\subset P$, we write $x_n \overset{d}{\to} x$ if x is a limit (possibly not unique) of (x_n) with respect to the topology τ_d. Note that this implies that $x_n\in P^d_{\eta_0}(\leq x)$ for all n large enough, for any fixed $\eta_0\in\mathbb{R}^*_+$, whence in particular $x_n\leq x$ for all n large. If (x_n) is $\leq$-ascending, then it follows that $x_n\leq x$ for all n, i.e., any limit of (x_n) in the topology τ_d is also an upper bound of this sequence.

The $\leq$-pseudometric d allows us to define two notions of distance from a point $x \in P$ to a subset $A \subset P$, namely

$$\begin{aligned} &\mathrm{dist}_d(\leq x, A) := \inf\{d(y,x) : y \in A,\ y \leq x\} \\ \text{and}\quad &\mathrm{dist}_d(\geq x, A) := \inf\{d(x,y) : y \in A,\ y \geq x\}. \end{aligned}$$

The former one is more closely related to the topology τ_d, as pointed out by the following lemma.

Lemma 3. *The closure of a subset $A \subset P$ with respect to τ_d is*

$$S_{(\leq,d)}(A) := \{x \in P : \mathrm{dist}_d(\leq x, A) = 0\}.$$

Proof. Let $\overline{A}^d$ denote the closure of A with respect to τ_d and first let us show that $\overline{A}^d \subset S_{(\leq,d)}(A)$. Let $x \in \overline{A}^d$. For every $\eta \in \mathbb{R}_+^*$, the set $P_\eta^d(\leq x)$ is an open neighborhood of x with respect to τ_d. Hence there exists $y \in A \cap P_\eta^d(\leq x)$, so $y \leq x$ and $d(y,x) < \eta$. We deduce that $\mathrm{dist}_d(\leq x, A) < \eta$ for all $\eta \in \mathbb{R}_+^*$, whence $\mathrm{dist}_d(\leq x, A) = 0$ and so $x \in S_{(\leq,d)}(A)$.

Conversely, let $x \in S_{(\leq,d)}(A)$. For showing that $x \in \overline{A}^d$, it is sufficient to show that any neighborhood V of x with respect to τ_d intersects A. By definition of τ_d, there is $\eta \in \mathbb{R}_+^*$ such that $P_\eta^d(\leq x) \subset V$. On the other hand, since $\mathrm{dist}_d(\leq x, A) = 0$, we can find $y \in A$ such that $y \leq x$ and $d(y,x) < \eta$. Thus $y \in A \cap P_\eta^d(\leq x) \subset A \cap V$, so that $A \cap V \neq \emptyset$, and the proof of the lemma is complete. □

Remark 4. (a) The topology τ_d satisfies the axiom (C1) (see Remark 1 (c)).

(b) In view of Lemma 3, the topology τ_d actually coincides with the $S_\leq$-topology considered in [2, §2] (with the slight difference that in [2], d is assumed to be defined on $P \times P$ instead of $P_\leq^2$, with values in $[0, +\infty)$ instead of $[0, +\infty]$).

Now let us denote by $\tilde{\tau}_{(\leq,d)}$ the $\leq$-ascending extension of τ_d, in the sense of Section 2. The closed subsets for the topology $\tilde{\tau}_{(\leq,d)}$ are the so-called $(\leq, \tau_d)$-closed subsets of Section 2; here, for simplicity, we will call them *$(\leq, d)$-closed subsets*. Also recall that $\tau_d \subset \tilde{\tau}_{(\leq,d)}$, hence $P_\eta^d(\leq x) \in \tilde{\tau}_{(\leq,d)}$ for all $\eta \in \mathbb{R}_+^*$, $x \in P$.

By $\overline{A}^{(\leq,d)}$ we denote the closure of A with respect to the topology $\tilde{\tau}_{(\leq,d)}$, i.e., $\overline{A}^{(\leq,d)}$ is the smallest $(\leq, d)$-closed subset in P which contains A, and A is itself $(\leq, d)$-closed if and only if $\overline{A}^{(\leq,d)} = A$.

In the next lemma, we write

$$L_{(\leq,d)}(A) := L_{(\leq,\tau_d)}(A) = \{x \in P : \exists (x_n) \subset A \leq\text{-ascending with } x_n \overset{d}{\to} x\}$$

(see, again, Section 2) while $S_{(\leq,d)}(A)$ comes from Lemma 3 above. Recall that $P(\leq x) = \{y \in P : y \leq x\}$ and $P(\geq x) = \{y \in P : x \leq y\}$.

Lemma 4. (a) *For every $x \in P$, the subset $P(\leq x)$ is open for the topology τ_d, thus also for $\tilde{\tau}_{(\leq,d)}$; the subset $P(\geq x)$ is closed for the topology τ_d, thus also for $\tilde{\tau}_{(\leq,d)}$. In particular, $\leq$ is self-closed with respect to τ_d and to $\tilde{\tau}_{(\leq,d)}$.*

(b) *For every $A \subset P$, we have the inclusions*

$$A \subset L_{(\leq,d)}(A) \subset \overline{A}^{(\leq,d)} \subset S_{(\leq,d)}(A).$$

(c) *The subset A is closed with respect to $\tilde{\tau}_{(\leq,d)}$ if and only if the first two inclusions of (b) are equalities. The subset A is closed with respect to τ_d if and only if all inclusions of (b) are equalities.*

Proof. (a) Let $x \in P$. Let $y \in P(\leq x)$ and choose any $\eta_0 \in \mathbb{R}_+^*$. Since $\leq$ is transitive, we easily have the inclusion $P^d_{\eta_0}(\leq y) \subset P(\leq x)$, which shows that y is an interior point of $P(\leq x)$, by the definition of the topology τ_d; hence $P(\leq x) \in \tau_d \subset \tilde{\tau}_{(\leq,d)}$.

Next let $y \in P \setminus P(\geq x)$ and, again, we choose any $\eta_0 \in \mathbb{R}_+^*$. For any $z \in P^d_{\eta_0}(\leq y)$, since $x \not\leq y$ and $z \leq y$, we must have $x \not\leq z$, hence $z \in P \setminus P(\geq x)$. This shows that $P(\geq x)$ is closed with respect to τ_d as well as with respect to $\tilde{\tau}_{(\leq,d)}$.

(b) The inclusion $A \subset L_{(\leq,d)}(A)$ is immediate. Since $A \subset \overline{A}^{(\leq,d)}$, we easily have $L_{(\leq,d)}(A) \subset L_{(\leq,d)}(\overline{A}^{(\leq,d)}) = \overline{A}^{(\leq,d)}$, where the last equality comes from the fact that $\overline{A}^{(\leq,d)}$ is $(\leq,d)$-closed. Finally, note that $S_{(\leq,d)}(A)$ contains A and is closed with respect to τ_d (due to Lemma 3) thus with respect to $\tilde{\tau}_{(\leq,d)}$ (since $\tau_d \subset \tilde{\tau}_{(\leq,d)}$); this yields $\overline{A}^{(\leq,d)} \subset S_{(\leq,d)}(A)$ as asserted.

(c) The first assertion comes from (b) and the fact that A is closed with respect to $\tilde{\tau}_{(\leq,d)}$ if and only if $A = \overline{A}^{(\leq,d)}$. The second assertion follows from (b) and the fact that A is closed with respect to τ_d if and only if $A = S_{(\leq,d)}(A)$ (due to Lemma 3). □

Remark 5. (a) All the inclusions in Lemma 4 (b) may be strict in general. Take for instance $P = [0,1] \times [0,1]$ equipped with the preorder $\leq$ given by

$$(x,y) \leq (x',y') \quad \text{if} \quad \begin{cases} x = x' < 1 \ \text{ and } \ y \leq y' < 1, & \text{or} \\ x \leq x' < 1 \ \text{ and } \ y' = 1, & \text{or} \\ x' = 1, \end{cases}$$

and the map $d : P_{\leq}^2 \to [0, +\infty)$ defined as the restriction of the standard metric on P. If $A = [0,1) \times [0,1)$, then it is easy to see that

$$A \subsetneq L_{(\leq,d)}(A) = [0,1) \times [0,1] \subsetneq \overline{A}^{(\leq,d)} = ([0,1) \times [0,1]) \cup \{(1,1)\} \subsetneq S_{(\leq,d)}(A) = P = [0,1] \times [0,1].$$

(b) For $x \in P$, note that

$$L_{(\leq,d)}(\{x\}) = \overline{\{x\}}^{(\leq,d)} = S_{(\leq,d)}(\{x\}) = \{y \in P : x \leq y,\ d(x,y) = 0\}.$$

In particular, a singleton $\{x\}$ is $(\leq, d)$-closed if and only if every $y \in P(\geq x) \setminus \{x\}$ is such that $d(x,y) \neq 0$. Thus the topology $\tilde{\tau}_{(\leq,d)}$ satisfies the separation axiom (T_1) (that is, singletons are $(\leq, d)$-closed in P) if and only if d is *nondegenerate*, that is, $(x,y) \in P_{\leq}^2$ and $x \neq y$ imply $d(x,y) \neq 0$.

Note also that if x is minimal with respect to the preorder $\leq$, which equivalently means that $P(\leq x) = \{x\}$, then Lemma 4 (a) implies that the singleton $\{x\}$ is $(\leq, d)$-open whatever d we consider.

3.2 Preordered uniform space

We call $\leq$-*uniform structure* on $(P, \leq)$ a (nonempty) family $\mathcal{D}$ of $\leq$-pseudometrics on P, and then we call the triple $(P, \leq, \mathcal{D})$ a *preordered uniform space*.

This gives rise to topologies on P:

- Let $\tau_{\mathcal{D}}$ be the coarsest topology on P which contains τ_d for all $d \in \mathcal{D}$. In other words, a basis for this topology is formed by finite intersections $U_1 \cap \ldots \cap U_k$ with $U_1, \ldots, U_k \in \bigcup_{d \in \mathcal{D}} \tau_d$. We will use the notation $x_n \xrightarrow{\mathcal{D}} x$ if $(x_n) \subset P$ converges to x with respect to the topology $\tau_{\mathcal{D}}$.
- Let $\tilde{\tau}_{(\leq,\mathcal{D})}$ denote the $\leq$-ascending extension of $\tau_{\mathcal{D}}$ in the sense of Section 2. In other words, the closed subsets for the topology $\tilde{\tau}_{(\leq,\mathcal{D})}$ are the $(\leq, \tau_{\mathcal{D}})$-closed subsets of P – which we will rather call $(\leq, \mathcal{D})$-closed for simplicity. Recall that $A \subset P$ is $(\leq, \mathcal{D})$-closed if, for every $\leq$-ascending sequence $(x_n) \subset A$ such that $x_n \xrightarrow{\mathcal{D}} x$, we get $x \in A$.

Remark 6. (a) We have the inclusion $\tau_{\mathcal{D}} \subset \tilde{\tau}_{(\leq,\mathcal{D})}$ (see Lemma 1 (a)). Moreover, since $\bigcup_{d \in \mathcal{D}} \tau_d \subset \tau_{\mathcal{D}}$, we have $\bigcup_{d \in \mathcal{D}} \tilde{\tau}_{(\leq,d)} \subset \tilde{\tau}_{(\leq,\mathcal{D})}$ (due to Remark 2).

(b) The preorder $\leq$ is self-closed with respect to the topologies $\tau_{\mathcal{D}}$ and $\tilde{\tau}_{(\leq,\mathcal{D})}$. Indeed, for every $x \in P$, taking any $d \in \mathcal{D}$, we know that $P(\geq x)$ is closed for the topologies τ_d and $\tilde{\tau}_{(\leq,d)}$ (see Lemma 4 (a)). In view of the

inclusions pointed out in part (a) above, we conclude that $P(\geq x)$ is also closed for $\tau_{\mathcal{D}}$ and for $\tilde{\tau}_{(\leq,\mathcal{D})}$.

(c) Part (b) of the present remark tells us that if $(x_n) \subset P$ is $\leq$-ascending and $x_n \overset{\mathcal{D}}{\to} x$, then $x_n \leq x$ for all n (we apply Lemma 2 (b) with $\tau = \tau_{\mathcal{D}}$).

Remark 7. If $\leq$ is the trivial preorder on P (i.e., $x \leq y$ for all $x, y \in P$), then a $\leq$-pseudometric $d : P^2_{\leq} = P \times P \to [0, +\infty]$ is an *écart* in the sense of [1, Chapitre IX]. For a family $\mathcal{D}$ of écarts on P, the pair $(P, \mathcal{D})$ is called a uniform space, and $\tau_{\mathcal{D}}$ is the topology on P considered in [1, Chapitre IX]. Thus we retrieve the classical notion of uniform space when we restrict to the case of the trivial preorder on P.

Lemma 5. *Let $(x_n) \subset P$ be a $\leq$-ascending sequence and let $x \in P$. The following conditions are equivalent:*

(i) $x_n \overset{\mathcal{D}}{\to} x$;

(ii) $x_n \to x$ *with respect to the topology* $\tilde{\tau}_{(\leq,\mathcal{D})}$;

(iii) $x_n \overset{d}{\to} x$ *for all* $d \in \mathcal{D}$.

Proof. The equivalence between (i) and (ii) comes from Lemma 1 (b). The implication (i)⇒(iii) is easily implied by the fact that $\tau_d \subset \tau_{\mathcal{D}}$ for all $d \in \mathcal{D}$. It remains to show (iii)⇒(i). So assume that $x_n \to x$ with respect to τ_d for all $d \in \mathcal{D}$. Let $V \subset P$ be a neighborhood of x for the topology $\tau_{\mathcal{D}}$. By definition of that topology, V contains a subset of the form

$$U_1 \cap \ldots \cap U_k$$

where, for every $i \in \{1, \ldots, k\}$, there is $d_i \in \mathcal{D}$ such that U_i is an open neighborhood of x for the topology τ_{d_i}. For every i, since $x_n \overset{d_i}{\to} x$, there is a rank $n_i \in \mathbb{N}$ such that $x_n \in U_i$ whenever $n \geq n_i$. Setting $n_0 = \max\{n_1, \ldots, n_k\}$, we obtain that $x_n \in U_1 \cap \ldots \cap U_k \subset V$ for all $n \geq n_0$. This shows that $x_n \to x$ with respect to $\tau_{\mathcal{D}}$, and (i) is established. □

We need a notion of completeness for the preordered pseudometric spaces $(P, \leq, d)$ and the preordered uniform space $(P, \leq, \mathcal{D})$. Let us present it now.

- A $\leq$-ascending sequence (x_n) is said to be *d-Cauchy* if, for every $\eta \in \mathbb{R}^*_+$, there is a rank n_0 such that, whenever $n_0 \leq m \leq n$, we have $d(x_m, x_n) < \eta$.

- We say that P is $(\leq, d)$-*complete* if every d-Cauchy $\leq$-ascending sequence $(x_n) \subset P$ is convergent with respect to the topology τ_d (or, equivalently, with respect to $\tilde{\tau}_{(\leq,d)}$; see Lemma 1 (b)).
- A $\leq$-ascending sequence (x_n) is said to be $\mathcal{D}$-*Cauchy* if it is d-Cauchy for all $d \in \mathcal{D}$.
- We say that P is $(\leq, \mathcal{D})$-*complete* if every $\mathcal{D}$-Cauchy $\leq$-ascending sequence is convergent with respect to $\tau_{\mathcal{D}}$ (or equivalently to $\tilde{\tau}_{(\leq,\mathcal{D})}$; see Lemma 5).

Remark 8. (a) The fact that P is $(\leq, \mathcal{D})$-complete does not imply that P is $(\leq, d)$-complete for every or even for some $d \in \mathcal{D}$, as shown by the following example. Let $P = (0, +\infty)$ be endowed with the trivial preorder $\leq$ and let $\mathcal{D} = \{d_1, d_2\}$ with

$$d_1(x, y) = |x - y| \quad \text{and} \quad d_2(x, y) = \Big|\frac{1}{x} - \frac{1}{y}\Big| \quad \text{for all } x, y \in P.$$

Thus P is neither $(\leq, d_1)$-complete nor $(\leq, d_2)$-complete. However, we claim that P is $(\leq, \mathcal{D})$-complete. Indeed, if $(x_n) \subset P$ is $\mathcal{D}$-Cauchy, then there must be $n_0 \in \mathbb{N}$ such that $|x_n - x_{n_0}| < 1$ and $|\frac{1}{x_n} - \frac{1}{x_{n_0}}| < 1$ for all $n \geq n_0$, which easily implies that there must be $a, b \in (0, +\infty)$ such that $a \leq x_n \leq b$ for all n. The metrics d_1, d_2 are equivalent on the interval $[a, b]$, and we obtain that there is $x \in [a, b]$ with $x_n \to x$ with respect to both d_1 and d_2, thus also $x_n \overset{\mathcal{D}}{\to} x$ by Lemma 5.

(b) The fact that P is $(\leq, d)$-complete for all $d \in \mathcal{D}$ is also not sufficient for ensuring that P is $(\leq, \mathcal{D})$-complete. Take for instance $P = [0, 1] \cup \{1'\}$ endowed with the partial order $\leq$ whose restriction to $[0, 1]$ is the standard total order and such that $x \leq 1'$ for all $x \in [0, 1)$, and $1, 1'$ are not comparable. Let $d, d' : P^2_{\leq} \to [0, +\infty)$ be the $\leq$-pseudometrics whose restrictions to $[0, 1)$ are given by the standard metric and such that

$$d(x, 1) = d'(x, 1') = 1 - x, \quad d(x, 1') = d'(x, 1) = 2 - x \quad \text{for all } x \in [0, 1).$$

It is easy to see that every $\leq$-ascending sequence $(x_n) \subset P$ converges with respect to both d and d', and this guarantees that P is both $(\leq, d)$-complete and $(\leq, d')$-complete. However the sequence (x_n) given by $x_n = 1 - \frac{1}{n}$ is $\mathcal{D}$-Cauchy (since it is both d-Cauchy and d'-Cauchy) but not convergent with respect to $\tau_{\mathcal{D}}$: this follows from Lemma 5 combined with the observation that 1 (resp., $1'$) is the only limit of (x_n) with respect to d (resp., d') and $1 \neq 1'$.

(c) Here we assume that $\mathcal{D}$ is directed in the sense that, for all $d_1, d_2 \in \mathcal{D}$ there is $d_3 \in \mathcal{D}$ with $d_1 \leq d_3$ and $d_2 \leq d_3$. We assume also that, for all $d \in \mathcal{D}$, every non-stationary $\leq$-ascending sequence (x_n) has at most one limit with respect to d. Then, in this case, the implication

$$(P \text{ is } (\leq, d)\text{-complete for all } d \in \mathcal{D}) \quad \Rightarrow \quad P \text{ is } (\leq, \mathcal{D})\text{-complete} \tag{1}$$

holds. For showing this, let $(x_n) \subset P$ be a $\mathcal{D}$-Cauchy $\leq$-ascending sequence. If (x_n) is stationary then it is immediately convergent with respect to $\mathcal{D}$, so we can assume that (x_n) is not stationary. We have that (x_n) is d-Cauchy for every $d \in \mathcal{D}$. Since P is $(\leq, d)$-complete, there is $x^d \in P$ with $x_n \overset{d}{\to} x^d$: this limit a priori depends on d, but we claim that $x^{d_1} = x^{d_2}$ for all $d_1, d_2 \in \mathcal{D}$. The assumption that $\mathcal{D}$ is directed yields $d_3 \in \mathcal{D}$ such that $d_1 \leq d_3$ and $d_2 \leq d_3$. We have $x_n \overset{d_3}{\to} x^{d_3}$ thus also $x_n \overset{d_1}{\to} x^{d_3}$ and $x_n \overset{d_2}{\to} x^{d_3}$, due to the previous inequalities. Since (x_n) is not stationary, it has at most one limit with respect to d_1, whence $x^{d_1} = x^{d_3}$. Similarly, $x^{d_2} = x^{d_3}$, and finally $x^{d_1} = x^{d_2}$, our claim is established. That claim yields $x \in P$ such that $x_n \overset{d}{\to} x$ for all $d \in \mathcal{D}$ and, therefore, $x_n \overset{\mathcal{D}}{\to} x$ by virtue of Lemma 5. This proves that P is $(\leq, \mathcal{D})$-complete.

Even under the additional assumptions considered in the present part (c), the converse of (1) is not true. Take for instance $(P, \leq)$, d_1, d_2 as in part (a) above, and set in addition $d_3(x, y) = \max\{d_1(x, y), d_2(x, y)\}$ for all $x, y \in P$, and $\mathcal{D} = \{d_1, d_2, d_3\}$. Thus $\mathcal{D}$ is directed and every sequence $(x_n) \subset P$ has at most one limit with respect to d_1, d_2, d_3 (since these are metrics). With the same argument as in part (a), P is $(\leq, \mathcal{D})$-complete. However, as shown in (a), it is neither $(\leq, d_1)$-complete nor $(\leq, d_2)$-complete.

4 Statement of main results

Our first main result concerns a map $f : P \to \mathbb{R} \cup \{+\infty\}$. Let $\mathrm{dom}(f) = f^{-1}(\mathbb{R})$ denote the *domain* of f.

- We say that f is $(\leq, \mathcal{D})$-*submonotone* if, for every $\leq$-ascending sequence $(x_n) \subset P$ and $x \in P$ such that $x_n \overset{\mathcal{D}}{\to} x$ and $(f(x_n))$ is nonincreasing, we have $f(x) \leq f(x_n)$ for all n.

- We say that f is $\geq$-*locally bounded from below* if, for every $x \in \mathrm{dom}(f)$, there is $\mu \in \mathbb{R}$ such that

$$\forall y \in P, \quad x \leq y \quad \Rightarrow \quad f(y) \geq \mu.$$

Theorem 1. *Let $(P, \leq, \mathcal{D})$ be a preordered uniform space which is $(\leq, \mathcal{D})$-complete. Let $f : P \to \mathbb{R} \cup \{+\infty\}$ be $(\leq, \mathcal{D})$-submonotone and $\geq$-locally bounded from below. Let $x \in \mathrm{dom}(f)$. Then there is $y \in \mathrm{dom}(f)$ such that*

(a) *$x \leq y$;*

(b) *$d(x, y) \leq f(x) - f(y)$ for all $d \in \mathcal{D}$;*

(c) *whenever $z \in \mathrm{dom}(f)$ and $z' \in P$ are such that $y \leq z \leq z'$, we have the implication*

$$\exists d \in \mathcal{D},\ d(z, z') > 0$$
$$\Rightarrow \quad \exists d \in \mathcal{D},\ (d(y, z) > f(y) - f(z) \quad or \quad d(z, z') > f(z) - f(z')).$$

For our second main result, instead of a single map f, we consider a family $\mathcal{F} = \{f_d\}_{d \in \mathcal{D}}$ of maps $f_d : P \to \mathbb{R} \cup \{+\infty\}$.

- We say that $\mathcal{F}$ is $(\leq, \mathcal{D})$-*submonotone* if, for every $\leq$-ascending sequence $(x_n) \subset P$ and $x \in P$ such that $x_n \overset{\mathcal{D}}{\to} x$ and $(f_d(x_n))$ is non-increasing for all $d \in \mathcal{D}$, we have $f_d(x) \leq f_d(x_n)$ for all n and all $d \in \mathcal{D}$.

- We say that $\mathcal{F}$ is $\geq$-*locally bounded from below* if, for every $x \in \mathrm{dom}(\mathcal{F}) := \bigcap_{d \in \mathcal{D}} \mathrm{dom}(f_d)$, there is a family $\{\mu_d\}_{d \in \mathcal{D}} \subset \mathbb{R}$ such that

$$\forall y \in P, \quad x \leq y \quad \Rightarrow \quad (\forall d \in \mathcal{D},\ f_d(y) \geq \mu_d).$$

Theorem 2. *Let $(P, \leq, \mathcal{D})$ be a preordered uniform space which is $(\leq, \mathcal{D})$-complete. Assume that $\mathcal{D}$ is (at most) countable. Let $\mathcal{F} = \{f_d\}_{d \in \mathcal{D}}$ be a family of maps $f_d : P \to \mathbb{R} \cup \{+\infty\}$ which is $(\leq, \mathcal{D})$-submonotone and $\geq$-locally bounded from below. Let $x \in \mathrm{dom}(\mathcal{F})$. Then there is $y \in \mathrm{dom}(\mathcal{F})$ such that*

(a) *$x \leq y$;*

(b) *$d(x, y) \leq f_d(x) - f_d(y)$ for all $d \in \mathcal{D}$;*

(c) *whenever $z \in \mathrm{dom}(\mathcal{F})$ and $z' \in P$ are such that $y \leq z \leq z'$, we have the implication*

$$\exists d \in \mathcal{D},\ d(z, z') > 0$$
$$\Rightarrow \quad \exists d \in \mathcal{D},\ (d(y, z) > f_d(y) - f_d(z) \quad or \quad d(z, z') > f_d(z) - f_d(z')).$$

The proofs of Theorems 1 and 2 are given in Section 5.

5 Proofs

5.1 Key tool: Ekeland variational principle over monoids

The key tool for proving our main results is a version of the Ekeland variational principle over monoids, established in [3]. We give the full statement here for making this paper as self-contained as possible (also, the result presented below is slightly rephrased).

- Let $(\mathbb{K}, +)$ be a commutative monoid, with neutral element 0 and we denote by $\mathbb{K}'$ its subset of invertible elements. The set $\mathbb{K}$ is also equipped with a preorder $\leq$ and a countable collection of subsets $\mathcal{V}$ which is stable under finite intersections and such that

$$\bigcap_{V \in \mathcal{V}} V = \{0\}.$$

  We will write

$$\mathbb{K}_+ := \{\alpha \in \mathbb{K} : 0 \leq \alpha\}, \quad \mathbb{K}'_+ := \mathbb{K}' \cap \mathbb{K}_+.$$

  For $(\alpha_n)_{n\in\mathbb{N}} \subset \mathbb{K}$, we write $\alpha_n \to 0$ if every $V \in \mathcal{V}$ contains α_n for all but finitely many n. We assume the following:

  (A) $\forall \alpha \in \mathbb{K}', \quad \{\beta \in \mathbb{K} : \beta \leq \alpha\} \subset \mathbb{K}'$.

  (B) $\forall \alpha, \beta, \gamma \in \mathbb{K}, \quad \alpha \leq \beta \quad \Rightarrow \quad \alpha + \gamma \leq \beta + \gamma$.

  (C) $\forall V \in \mathcal{V}, \quad \forall \alpha \in \mathbb{K}' \cap V, \quad \{\beta \in \mathbb{K} : 0 \leq \beta \leq \alpha\} \subset V$.

  (D) $\forall V \in \mathcal{V}, \ \forall \alpha \in \mathbb{K}', \ \exists N \geq 1, \ \forall \beta_1, \ldots, \beta_N \in \mathbb{K}'_+ \setminus V, \ \beta_1 + \ldots + \beta_N \not\leq \alpha$.

  (E) $\forall \alpha \in \mathbb{K}, \ \forall (\beta_n)_{n\in\mathbb{N}} \subset \mathbb{K}_+, \ (\alpha \leq \beta_n \ \forall n \text{ and } \beta_n \to 0) \ \Rightarrow \ \alpha \leq 0$.

- Let P be a nonempty set endowed with a preorder $\leq$ and a map

$$\delta : P^2_{\leq} = \{(x, y) \in P \times P : x \leq y\} \to \mathbb{K}_+$$

  such that $\delta(x, x) = 0$ for all x and $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$ whenever $x \leq y \leq z$.

- In this context, a $\leq$-ascending sequence $(x_n) \subset P$ is said to be

  - *δ-convergent to x* if $x_n \leq x$ for all n and $\delta(x_n, x) \to 0$;
  - *δ-Cauchy* if for every $V \in \mathcal{V}$ there is n_0 such that $\delta(x_m, x_n) \in V$ whenever $n_0 \leq m \leq n$.

Finally P is called $(\leq, \delta)$-*complete* if every δ-Cauchy $\leq$-ascending sequence $(x_n) \subset P$ is δ-convergent to some x.

- Given a map $f : P \to \mathbb{K}$,
  - we write $\mathrm{dom}(f) = f^{-1}(\mathbb{K}')$;
  - we say that f is $(\leq, \delta)$-*submonotone* if, for every $\leq$-ascending sequence $(x_n) \subset \mathrm{dom}(f)$ which is δ-convergent to x,

$$(f(x_n) \leq f(x_m)\ \forall m, n \ \text{ with } \ m \leq n) \quad \Rightarrow \quad f(x) \leq f(x_m)\ \forall m;$$

  - we say that f is *locally bounded from below* if, for every $x \in \mathrm{dom}(f)$, there is $\mu \in \mathbb{K}'$ such that

$$\forall y \in P, \quad (x \leq y \quad \text{and} \quad f(y) \leq f(x)) \quad \Rightarrow \quad f(y) \geq \mu.$$

  (The latter notion is a special case of the notion of "locally finitely bounded from below" map considered in [3].)

In this setting, we have (see [3, Theorem 1]):

Theorem 3. *Let $\mathbb{K}$, P be as above, assume that P is $(\leq, \delta)$-complete and let $f : P \to \mathbb{K}$ be $(\leq, \delta)$-submonotone and locally bounded from below. Then, for every $x \in \mathrm{dom}(f)$, there is $y \in \mathrm{dom}(f)$ such that*

(a) $x \leq y$;

(b) $\delta(x, y) \leq f(x) - f(y)$;

(c) *whenever $z, z' \in \mathrm{dom}(f)$ satisfy $y \leq z \leq z'$ and $\delta(z, z') \neq 0$, we have*

$$\delta(y, z) \not\leq f(y) - f(z) \quad \text{or} \quad \delta(z, z') \not\leq f(z) - f(z').$$

Example 1. (a) The typical example of monoid to which the result applies is $\mathbb{K} = \mathbb{R} \cup \{+\infty\}$ endowed with the standard addition $+$ and total order $\leq$, and with the countable collection of subsets $\mathcal{V} := \{(-r, r)\}_{r \in \mathbb{Q}_+^*}$. Then conditions (A)–(E) are easily fulfilled. Note also that $\mathbb{K}' = \mathbb{R}$, $\mathbb{K}_+ = [0, +\infty]$, and $\mathbb{K}'_+ = [0, +\infty)$.

(b) As another example, consider the set of sequences

$$\mathbb{K} = (\mathbb{R} \cup \{+\infty\})^{\mathbb{N}} = \{(\alpha_n)_{n \in \mathbb{N}} : \alpha_n \in \mathbb{R} \cup \{+\infty\}\}$$

which is a commutative monoid for the standard componentwise addition, the neutral element is the zero constant sequence. We endow $\mathbb{K}$ with the partial order given by

$$(\alpha_n) \leq (\beta_n) \quad \text{if} \quad \alpha_n \leq \beta_n \ \forall n \in \mathbb{N}.$$

We consider the countable collection of subsets $\mathcal{V} = \{V_{I,r}\}$ parametrized by pairs (I, r) formed by a nonempty finite subset $I \subset \mathbb{N}$ and a map $r : I \to \mathbb{Q}_+^*$, $i \mapsto r_i$, and with

$$V_{I,r} := \{(\alpha_n) \in \mathbb{K} : \alpha_i \in (-r_i, r_i) \ \forall i \in I\}.$$

Here we have $\mathbb{K}' = \{(\alpha_n) : \alpha_n < +\infty \ \forall n\} = \mathbb{R}^{\mathbb{N}}$, $\mathbb{K}_+ = [0, +\infty]^{\mathbb{N}}$, and $\mathbb{K}'_+ = [0, +\infty)^{\mathbb{N}}$. Again it is straightforward to check that (A)–(C) hold true. For showing (D), let $V = V_{I,r} \in \mathcal{V}$, with $I \subset \mathbb{N}$ nonempty finite and $r : I \to \mathbb{Q}_+^*$. Let $\alpha = (\alpha_n) \in \mathbb{K}' = \mathbb{R}^{\mathbb{N}}$. For every $i \in I$, since $r_i \in \mathbb{Q}_+^*$ and $\alpha_i \in \mathbb{R}$, we can find $N_i \in \mathbb{N}^*$ such that $N_i > \frac{\alpha_i}{r_i}$. We set $N_0 = \max\{N_i : i \in I\}$ and $N = N_0 \operatorname{card}(I)$. Now let $\beta_1, \ldots, \beta_N \in \mathbb{K}'_+ \setminus V$. For $m \in \{1, \ldots, N\}$, we write $\beta_m = (\beta_{m,n})_{n\in\mathbb{N}}$. For every $m \in \{1, \ldots, N\}$, the assumption that $\beta_m \in \mathbb{K}'_+ \setminus V$ yields

$$I_m := \{i \in I : \beta_{m,i} \geq r_i\} \neq \emptyset.$$

Fix any $i_m \in I_m$, so that we obtain a map

$$F : \{1, \ldots, N\} \to I, \ m \mapsto i_m.$$

Since $N = N_0 \operatorname{card}(I)$, there is $i \in I$ with $\operatorname{card}(F^{-1}(i)) \geq N_0$. Since $\beta_{m,i} \geq 0$ for all $m \in \{1, \ldots, N\}$ (because $\beta_m \in \mathbb{K}'_+$), we obtain

$$\sum_{m=1}^{N} \beta_{m,i} \geq \sum_{m\in F^{-1}(i)} \beta_{m,i} \geq N_0 r_i > \alpha_i.$$

Whence $\beta_1 + \ldots + \beta_N \not\leq \alpha$. This shows (D).

For showing (E), let $\alpha \in \mathbb{K}$ and $(\beta_n)_{n\in\mathbb{N}} \subset \mathbb{K}_+$ be such that $\alpha \leq \beta_n$ for all n and $\beta_n \to 0$. We write $\beta_n = (\beta_{n,i})_{i\in\mathbb{N}}$. For every $i \in \mathbb{N}$, we aim to show that $\alpha_i \leq 0$. To this end, let $\varepsilon \in \mathbb{Q}_+^*$, set $I = \{i\}$ and $r : I \to \mathbb{Q}_+^*$, $i \mapsto r_i = \varepsilon$. Since $V := V_{I,r} \in \mathcal{V}$ and $\beta_n \to 0$, there is $n_0 \in \mathbb{N}$ such that $\beta_n \in V$ for all $n \geq n_0$. Using in addition that $\beta_n \in \mathbb{K}_+$, this yields $0 \leq \beta_{n,i} < \varepsilon$ for any $n \geq n_0$ and, therefore, $\alpha_i \leq \beta_{n,i} < \varepsilon$. Since this holds for any $\varepsilon \in \mathbb{Q}_+^*$, we conclude that $\alpha_i \leq 0$ and the argument is complete.

5.2 Proof of Theorem 1

We aim to apply Theorem 3 to the monoid $\mathbb{K} = \mathbb{R} \cup \{+\infty\}$ equipped with the standard addition $+$ and total order $\leq$, as in Example 1 (a) and with $\mathcal{V} = \{(-\varepsilon, \varepsilon)\}_{\varepsilon \in \mathbb{Q}_+^*}$. As explained in Example 1 (a), conditions (A)–(E) are fulfilled. For applying Theorem 3, we also consider our preordered set $(P, \leq)$ endowed with the map

$$\delta : P_{\leq}^2 \to [0, +\infty], \quad (x, y) \mapsto \sup_{d \in \mathcal{D}} d(x, y).$$

We clearly have $\delta(x, x) = 0$ for all $x \in P$, moreover it is straightforward to check that $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$ whenever $x \leq y \leq z$.

For checking the applicability of Theorem 3 to the considered setting, we need to show that

- P is $(\leq, \delta)$-complete;
- f is $(\leq, \delta)$-submonotone.

For checking the first assertion, let $(x_n) \subset P$ be a $\leq$-ascending sequence which is δ-Cauchy. For every $d \in \mathcal{D}$, the fact that $d(x, y) \leq \delta(x, y)$ for all $(x, y) \in P_{\leq}^2$ easily implies that (x_n) is d-Cauchy. Hence (x_n) is $\mathcal{D}$-Cauchy. Knowing that P is $(\leq, \mathcal{D})$-complete, we find $x \in P$ such that $x_n \overset{\mathcal{D}}{\to} x$, in particular $x_n \leq x$ for all n (see Remark 6 (c)). We claim that (x_n) is δ-convergent to x. To see this, fix $V \in \mathcal{V}$, $V = (-\varepsilon, \varepsilon)$ with $\varepsilon \in \mathbb{Q}_+^*$. Using that (x_n) is δ-Cauchy, there is $n_0 \in \mathbb{N}$ such that $\delta(x_m, x_n) < \frac{\varepsilon}{2}$ whenever $n_0 \leq m \leq n$. We claim that

$$\delta(x_m, x) < \varepsilon \quad \text{whenever } m \geq n_0. \tag{2}$$

Indeed, let $m \geq n_0$ and let $d \in \mathcal{D}$. Since $x_n \overset{\mathcal{D}}{\to} x$, by virtue of Lemma 5 we have also $x_n \overset{d}{\to} x$, so there is $n_0' \in \mathbb{N}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ whenever $n \geq n_0'$. Now we set $n = \max\{m, n_0'\}$ and we obtain

$$d(x_m, x) \leq d(x_m, x_n) + d(x_n, x) \leq \delta(x_m, x_n) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since this holds for all $d \in \mathcal{D}$, we get (2). The fact that (x_n) is δ-convergent to x is now justified, and we have established that P is $(\leq, \delta)$-complete.

For showing the second assertion, let $(x_n) \subset \mathrm{dom}(f)$ be a $\leq$-ascending sequence, which is δ-convergent to $x \in P$ (in particular, $x_n \leq x$ for all n) and such that $f(x_n) \leq f(x_m)$ for all m, n with $m \leq n$. For every $d \in \mathcal{D}$,

since $d(x,y) \leq \delta(x,y)$ for all $(x,y) \in P^2_{\leq}$, we obtain that $x_n \overset{d}{\to} x$. Whence $x_n \overset{\mathcal{D}}{\to} x$ (see Lemma 5). Using that f is $(\leq, \mathcal{D})$-submonotone, we conclude that $f(x) \leq f(x_m)$ for all m, and the fact that f is $(\leq, \delta)$-submonotone is established.

We are in position to apply Theorem 3, which, given $x \in \mathrm{dom}(f)$, provides us with an element $y \in \mathrm{dom}(f)$ satisfying conditions (a)–(c) therein. Let us deduce that the same element y satisfies conditions (a)–(c) of Theorem 1.

Condition (a) of Theorem 1 coincides with condition (a) of Theorem 3.

Condition (b) of Theorem 3 tells us that $\delta(x,y) \leq f(x) - f(y)$. By definition of δ, this means that $d(x,y) \leq f(x) - f(y)$ for all $d \in \mathcal{D}$, which is exactly condition (b) of Theorem 1.

Finally, let $z \in \mathrm{dom}(f)$ and $z' \in P$ be such that $y \leq z \leq z'$ and assume that $d(z,z') > 0$ for some $d \in \mathcal{D}$. By definition of δ, this yields $\delta(z,z') \neq 0$. If $f(z') = +\infty$ then the inequality $d'(z,z') > f(z) - f(z')$ is immediate for all $d' \in \mathcal{D}$, so we can assume that $z' \in \mathrm{dom}(f)$. Condition (c) of Theorem 3 now implies that $\delta(y,z) > f(y) - f(z)$ or $\delta(z,z') > f(z) - f(z')$ (recall that in our situation the monoid $\mathbb{K} = \mathbb{R} \cup \{+\infty\}$ is endowed with its *total* order $\leq$). Invoking again the definition of δ, these formulas mean that there is $d \in \mathcal{D}$ such that $d(y,z) > f(y) - f(z)$ or there is $d \in \mathcal{D}$ such that $d(z,z') > f(z) - f(z')$, which matches with the conclusion of part (c) of Theorem 1. The proof of Theorem 1 is complete.

5.3 Proof of Theorem 2

We write $\mathcal{D} = \{d_n\}_{n\in\mathbb{N}}$ (that is, we fix a surjective map $\mathbb{N} \to \mathcal{D}$, which is possible since $\mathcal{D}$ is at most countable). Also we write for simplicity $f_n = f_{d_n}$ for all $n \in \mathbb{N}$, so that $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$. We consider the monoid $\mathbb{K} = (\mathbb{R} \cup \{+\infty\})^{\mathbb{N}}$ equipped with the addition $+$, the partial order $\leq$, and the set $\mathcal{V}$ as in Example 1 (b). Furthermore, let $\delta : P^2_{\leq} \to \mathbb{K}_+ = [0,+\infty]^{\mathbb{N}}$, $(x,y) \mapsto (d_n(x,y))_{n\in\mathbb{N}}$. Clearly, $\delta(x,x) = 0$ for all $x \in P$, where 0 denotes the neutral element in $\mathbb{K}$ (i.e., the zero constant sequence). Moreover, whenever $x \leq y \leq z$, we have $d_n(x,z) \leq d_n(x,y) + d_n(y,z)$ for all $n \in \mathbb{N}$, whence $\delta(x,z) \leq \delta(x,y) + \delta(y,z)$.

We aim to apply Theorem 3 with $\mathbb{K}$ and δ as above and to the map

$$f : P \to \mathbb{K},\ x \mapsto (f_n(x))_{n\in\mathbb{N}}.$$

We need to check that

- P is $(\leq, \delta)$-complete;

- f is $(\leq, \delta)$-submonotone;
- f is locally bounded from below.

Let us prove the first assertion. Let $(x_n) \subset P$ be a $\leq$-ascending sequence and assume that (x_n) is δ-Cauchy. Let us show that (x_n) is $\mathcal{D}$-Cauchy. To do this, for every $d = d_i \in \mathcal{D}$ with $i \in \mathbb{N}$, we have to show that (x_n) is d-Cauchy. So let $\varepsilon \in \mathbb{Q}_+^*$. Set $I = \{i\}$ and $r : I \to \mathbb{Q}_+^*$, $i \mapsto r_i = \varepsilon$. Considering $V = V_{I,r}$ (see Example 1 (b)), we find $n_0 \in \mathbb{N}$ such that whenever $n_0 \leq m \leq n$, we have $\delta(x_m, x_n) \in V$. The latter inclusion means that $d(x_m, x_n) < \varepsilon$. We obtain that (x_n) is d-Cauchy for all $d \in \mathcal{D}$, and so $\mathcal{D}$-Cauchy as claimed. Knowing that P is $(\leq, \mathcal{D})$-complete, it follows that there is $x \in P$ such that $x_n \overset{\mathcal{D}}{\to} x$, in particular $x_n \leq x$ for all n (see Remark 6 (c)).

We verify that (x_n) is δ-convergent to x and this will conclude the argument for showing that P is $(\leq, \delta)$-complete. Let $V = V_{I,r} \in \mathcal{V}$ with $I \subset \mathbb{N}$ finite and $r : I \to \mathbb{Q}_+^*$. For every $i \in I$, we know that $x_n \overset{d_i}{\to} x$ (see Lemma 5). Hence there is $n_i \in \mathbb{N}$ such that $d_i(x_n, x) < r_i$ whenever $n \geq n_i$. Set $n_0 = \max\{n_i : i \in I\}$. For every $n \geq n_0$, we then obtain that $d_i(x_n, x) < r_i$ for all $i \in I$, whence $\delta(x_n, x) \in V$. Therefore, (x_n) is δ-convergent to x.

Next we show that f is $(\leq, \delta)$-submonotone. To do this, let $(x_n) \subset \mathrm{dom}(f)$ be a $\leq$-ascending sequence which is δ-convergent to $x \in P$ (thus $x_n \leq x$ for all n) and such that $f(x_n) \leq f(x_m)$ for all $m, n \in \mathbb{N}$ with $m \leq n$. By definition of f, the latter inequality tells us that $f_i(x_n) \leq f_i(x_m)$ for all $m, n \in \mathbb{N}$ with $m \leq n$ and all $i \in \mathbb{N}$, i.e., $(f_i(x_n))_{n \in \mathbb{N}}$ is nonincreasing for all $i \in \mathbb{N}$.

Moreover, note that $x_n \overset{\mathcal{D}}{\to} x$. Indeed, by Lemma 5, it suffices to see that $x_n \overset{d_i}{\to} x$ for every $i \in \mathbb{N}$. For seeing this, let $\varepsilon \in \mathbb{Q}_+^*$. Set $I = \{i\}$ and $r : I \to \mathbb{Q}_+^*$, $i \mapsto \varepsilon$. Since (x_n) is δ-convergent to x, there is $n_0 \in \mathbb{N}$ such that $\delta(x_n, x) \in V_{I,r}$ whenever $n \geq n_0$; whence $d_i(x_n, x) < \varepsilon$ whenever $n \geq n_0$. Therefore, $x_n \overset{\mathcal{D}}{\to} x$.

Now we can invoke the assumption that $\mathcal{F}$ is $(\leq, \mathcal{D})$-submonotone, from which we obtain that $f_i(x) \leq f_i(x_m)$ for all $i \in \mathbb{N}$ and all $m \in \mathbb{N}$, hence $f(x) \leq f(x_m)$ for all $m \in \mathbb{N}$. The argument for showing that f is $(\leq, \delta)$-submonotone is complete.

Let us now establish the third assertion saying that f is locally bounded from below. Let $x \in \mathrm{dom}(f)$. This means that $f_i(x) < +\infty$ for all $i \in \mathbb{N}$, i.e., $x \in \bigcap_{i \in \mathbb{N}} \mathrm{dom}(f_i) = \mathrm{dom}(\mathcal{F})$. Using that $\mathcal{F}$ is $\geq$-locally bounded from below, there is a family $\{\mu_i\}_{i \in \mathbb{N}} \subset \mathbb{R}$ such that $\mu_i \leq f_i(y)$ whenever $y \in P$ satisfies $x \leq y$. Set $\mu = (\mu_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} = \mathbb{K}'$. We obtain that for every $y \in P$ such that $x \leq y$, we have $\mu \leq f(y)$. The argument is complete.

The above assertions allow us to apply Theorem 3. Let $x \in \mathrm{dom}(\mathcal{F}) = \mathrm{dom}(f)$. There exists $y \in \mathrm{dom}(f) = \mathrm{dom}(\mathcal{F})$ fulfilling (a)–(c) of Theorem 3. We claim that the same element y also fulfils (a)–(c) of Theorem 2. Condition (a) is exactly the same for both theorems so we only have to consider conditions (b) and (c).

Theorem 3 (b) tells us that the relation $\delta(x,y) \leq f(x) - f(y)$ holds in the monoid $\mathbb{K}$. This means that $d_i(x,y) \leq f_i(x) - f_i(y)$ for all $i \in \mathbb{N}$, in other words $d(x,y) \leq f_d(x) - f_d(y)$ for all $d \in \mathcal{D}$. Condition (b) of Theorem 2 is established.

Finally, let $z \in \mathrm{dom}(\mathcal{F}) = \mathrm{dom}(f)$ and $z' \in P$ be such that $y \leq z \leq z'$ and $d(z,z') > 0$ for some $d = d_i \in \mathcal{D}$. Since $\delta(z,z') = (d_j(z,z'))_{j\in\mathbb{N}}$, we obtain that $\delta(z,z')$ is different from the zero constant sequence (the neutral element of $\mathbb{K}$). If $z' \notin \mathrm{dom}(f)$, then there is $d' = d_j \in \mathcal{D}$ such that $f_{d'}(z') = f_j(z') = +\infty$, thus $d'(z,z') > f_{d'}(z) - f_{d'}(z')$. Now assume that $z' \in \mathrm{dom}(f)$. By Theorem 3 (c), we then have $\delta(y,z) \not\leq f(y) - f(z)$ or $\delta(z,z') \not\leq f(z) - f(z')$. Due to the definition of the partial order $\leq$ of $\mathbb{K}$, we conclude that there is $d' = d_j \in \mathcal{D}$ such that $d'(y,z) > f_{d'}(y) - f_{d'}(z)$ or there is $d' \in \mathcal{D}$ such that $d'(z,z') > f_{d'}(z) - f_{d'}(z')$. Thus condition (c) of Theorem 2 holds, and the proof of the theorem is complete.

References

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