

NONUNIFORM POLYNOMIAL BEHAVIOR FOR LINEAR RANDOM ONE-SIDED DISCRETE-TIME SYSTEMS*

Oana Albescu[†] Larisa Elena Biriş[‡] Traian Ceaşu[§]
Nicolae Marian Seimeanu[¶]

Dedicated to the memory of Professor Mihail Megan

DOI 10.56082/annalsarscimath.2026.3.105

Abstract

This paper investigates the concepts of nonuniform polynomial stability and instability for linear random one-sided discrete-time systems. We establish necessary and sufficient conditions for these properties and provide Lyapunov-type characterizations in terms of Lyapunov functions and Lyapunov norms.

Keywords: polynomial stability, polynomial instability, Lyapunov functions, Lyapunov norms.

MSC: 34D05, 93C55, 93D05.

*Accepted for publication on March 17, 2026

[†]oana.albescu@unitbv.ro, Faculty of Mathematics and Computer Science, Transilvania University of Brasov, Romania

[‡]larisa.biris@e-uvt.ro, West University of Timisoara, Department of Mathematics, Timisoara, Romania

[§]traian.ceaşu@e-uvt.ro, West University of Timisoara, Department of Mathematics, Timisoara, Romania

[¶]nicusei@yahoo.com, West University of Timisoara, Department of Mathematics, Timisoara, Romania

1 Introduction

Time-varying linear systems and non-autonomous difference equations play a fundamental role in the theory of discrete dynamical systems; see, for instance, [1], [2], [15], [19], [21]. In the deterministic framework, qualitative properties such as exponential stability, splitting, and dichotomy have been extensively investigated. The skew-product approach has proved particularly effective in this context, providing a natural setting for the analysis of non-invertible systems and non-autonomous dynamics (see [4], [5], [17], [22]).

Polynomial dichotomies and related asymptotic behaviors for discrete systems on the half-line were further studied in [6], [7], [12] extending the classical exponential theory. The stochastic extension of hyperbolic theory is grounded in the multiplicative ergodic theorem of Oseledets, which characterizes asymptotic behavior in terms of Lyapunov exponents and lays the foundation for nonuniform exponential phenomena. Building on this framework, Arnold [3] developed the theory of random dynamical systems using the skew-product formulation, which has become central in the study of linear random cocycles. Subsequent contributions have addressed Lyapunov exponents, topological dynamics, and qualitative properties of random systems (see [8], [10]).

Exponential stability and tempered exponential dichotomies for random difference equations and cocycles were investigated in [11], [20], [25].

While the nonuniform exponential theory is by now well developed, considerably less attention has been devoted to polynomial rates of growth and decay in the random discrete setting. In many cases, exponential dichotomies fail to occur, whereas polynomial behavior naturally describes the asymptotic dynamics.

In the deterministic case, nonuniform polynomial instability and Lyapunov-type techniques have been analyzed in [7], [18]. However, a systematic treatment of nonuniform polynomial stability and instability for discrete random semi-dynamical systems, formulated within the skew-product framework, is still lacking.

The present paper is devoted to the study of the aforementioned problem within this framework. We investigate nonuniform polynomial stability and nonuniform polynomial instability for linear random one-sided discrete-time systems. We establish necessary and sufficient conditions for these properties and provide Zabczyk-type as well as Lyapunov-type characterizations in terms of Lyapunov functions and Lyapunov norms. We note that, for polynomial instability, only necessary conditions are obtained. Moreover, we analyze the corresponding properties for the adjoint and inverse systems

associated with the original one. The results obtained here broaden the scope of the exponential theory and contribute to a deeper understanding of nonuniform behavior in random dynamical systems.

2 Preliminaries

Let $\mathbb{Z}_+$ be the set of all positive integers and $(X, \|\cdot\|)$ a Banach space. By $\mathcal{B}(X)$ we denote the Banach algebra of all bounded linear operators acting from X into X . Also, by $(\Omega, \mathfrak{F}, \mathbb{P})$ we denote a probability space. We consider a measurable application $\theta^n : \Omega \rightarrow \Omega$, for all $n \in \mathbb{Z}_+$. This application preserves the probability measure, i.e. $\mathbb{P} \circ \theta^n = \mathbb{P}$, thus $\mathbb{P}(\theta^n B) = \mathbb{P}(B)$, for all $B \in \mathfrak{F}$. Moreover, we have that $\theta^0 = I_\Omega$ and $\theta^n \circ \theta^m = \theta^{n+m} = \theta^{n+m} = \theta^m \circ \theta^n$, for all $n, m \in \mathbb{Z}_+$. Finally, we point out that the application $\mathbb{Z}_+ \times \Omega \ni (n, \omega) \rightarrow \theta^n \omega \in \Omega$ is measurable for all $n \in \mathbb{Z}_+$.

For simplicity, the family of applications $(\theta^n)_{n \in \mathbb{Z}_+}$ will be denoted by θ . In this case, for the measurable family $\theta : \Omega \rightarrow \Omega$ we consider the metric semidynamical system which is denoted by $(\Omega, \mathfrak{F}, \mathbb{Z}_+, \theta)$. The application $\varphi : \Omega \rightarrow (0, +\infty)$ is called θ -invariant if verifies the property $\varphi(\theta^t \omega) = \varphi(\omega)$, for all $n \in \mathbb{Z}_+$ and $\omega \in \Omega$.

We consider the measurable application $\phi : \mathbb{Z}_+ \times \Omega \rightarrow \mathcal{B}(X)$ which satisfies the properties:

- (a) $\phi(0, \omega) = I_X$, for all $\omega \in \Omega$;
- (b) $\phi(n + m, \omega) = \phi(n, \theta^m \omega) \phi(m, \omega)$, for all $n, m \in \mathbb{Z}_+$ and $\omega \in \Omega$.

Obviously, the relation (b) implies

$$\phi(n, \theta^m \omega) \phi(m, \omega) = \phi(n + m, \omega) = \phi(m + n, \omega) = \phi(m, \theta^n \omega) \phi(n, \omega),$$

for all $n, m \in \mathbb{Z}_+$ and all $\omega \in \Omega$. Throughout this work, the pair (θ, ϕ) is used to denote a linear random one-sided discrete-time system.

3 Nonuniform polynomial stability

Definition 1. We say that a linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially stable if there exist a θ -invariant random variable $\alpha : \Omega \rightarrow (0, +\infty)$ and a function $N : \Omega \rightarrow [1, +\infty)$ such that

$$\|\phi(n, \omega)x\| \leq N(\omega)(n + 1)^{-\alpha(\omega)} \|x\|, \quad (1)$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proposition 1. *The following statements are equivalent:*

- (a) *the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially stable;*
- (b) *there exist a function $N : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\alpha : \Omega \rightarrow (0, +\infty)$ such that*

$$\|\phi(n, \omega)\| \leq N(\omega)(n+1)^{-\alpha(\omega)}, \quad (2)$$

for all $(n, \omega) \in \mathbb{Z}_+ \times \Omega$;

- (c) *there exist a function $N : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\alpha : \Omega \rightarrow (0, +\infty)$ such that*

$$\|\phi(n+m, \omega)x\| \leq N(\theta^m \omega)(n+1)^{-\alpha(\omega)} \|\phi(m, \omega)x\|, \quad (3)$$

for all $(m, n, \omega, x) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times \Omega \times X$;

Proof. Let $(n, \omega) \in \mathbb{Z}_+ \times \Omega$. First, one can easily see that the implication (a) $\Rightarrow$ (b) can be verified for $\|x\| = 1$. We have

$$\begin{aligned} \|\phi(n, \omega)\| &= \sup_{\|x\|=1} \|\phi(n, \omega)x\| \\ &\leq \sup_{\|x\|=1} N(\omega)(n+1)^{-\alpha(\omega)} \|x\| \\ &= N(\omega)(n+1)^{-\alpha(\omega)}. \end{aligned}$$

For the second implication (b) $\Rightarrow$ (a), from (2), we obtain

$$\|\phi(n, \omega)x\| \leq \|\phi(n, \omega)\| \cdot \|x\| \leq N(\omega)(n+1)^{-\alpha(\omega)} \|x\|, \quad \text{for all } x \in X.$$

To prove (c) $\Rightarrow$ (a), let $m = 0$ in (3). It follows that

$$\begin{aligned} \|\phi(n, \omega)x\| &= \|\phi(n+0, \omega)x\| \\ &\leq N(\theta^0 \omega)(n+1)^{-\alpha(\omega)} \|\phi(0, \omega)x\| \\ &= N(\omega)(n+1)^{-\alpha(\omega)} \|x\|. \end{aligned}$$

For the implication (a) $\Rightarrow$ (c). From (a), we obtain

$$\|\phi(n, \theta^m \omega)x\| \leq N(\theta^m \omega)(n+1)^{-\alpha(\theta^m \omega)} \|x\|.$$

Since the variable α is θ -invariant, i.e. $\alpha(\theta^m \omega) = \alpha(\omega)$, it follows that:

$$\|\phi(n, \theta^m \omega)x\| \leq N(\theta^m \omega)(n+1)^{-\alpha(\omega)} \|x\|.$$

From the previous inequality, we have

$$\|\phi(n, \theta^m \omega) \phi(m, \omega) x\| \leq N(\theta^m \omega) (n+1)^{-\alpha(\omega)} \|\phi(m, \omega) x\|.$$

Using the hypothesis, it results

$$\begin{aligned} \|\phi(n+m, \omega) x\| &= \|\phi(n, \theta^m \omega) \phi(m, \omega) x\| \\ &\leq N(\theta^m \omega) (n+1)^{-\alpha(\omega)} \|\phi(m, \omega) x\|, \end{aligned}$$

for all $(m, n, \omega, x) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \times \Omega \times X$. $\square$

3.1 Zabczyk type theorems

Theorem 1. *Let (θ, ϕ) be a linear random one-sided discrete-time system. Then (θ, ϕ) is nonuniformly polynomially stable if and only if there exist a θ -invariant random variable $\beta : \Omega \rightarrow (0, +\infty)$ and two functions $M, N : \Omega \rightarrow [1, +\infty)$ such that*

$$\sum_{n=m}^{+\infty} (n-m+1)^{\beta(\omega)-1} N(\theta^m \omega)^{-1} \|\phi(n, \omega) x\| dt \leq M(\omega) \|\phi(m, \omega) x\|, \quad (4)$$

for all $(m, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proof. Necessity: Let N and α as in Definition 1. Let $\beta : \Omega \rightarrow (0, +\infty)$ a θ -invariant random variable such that $0 < \beta < \alpha$. For every $n \geq m$ we have

$$\begin{aligned} \|\phi(n, \omega) x\| &= \|\phi(n-m+m, \omega) x\| \\ &= \|\phi(n-m, \theta^m \omega) \phi(m, \omega) x\| \\ &\leq N(\theta^m \omega) (n-m+1)^{-\alpha(\omega)} \|\phi(m, \omega) x\|. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{n=m}^{+\infty} (n-m+1)^{\beta(\omega)-1} N(\theta^m \omega)^{-1} \|\phi(n, \omega) x\| \\ &\leq \sum_{n=m}^{+\infty} (n-m+1)^{\beta(\omega)-1-\alpha(\omega)} \|\phi(m, \omega) x\| \\ &= \|\phi(m, \omega) x\| \sum_{n=m}^{+\infty} \frac{1}{(n-m+1)^{1+\alpha(\omega)-\beta(\omega)}} \\ &= M(\omega) \|\phi(m, \omega) x\|. \end{aligned}$$

Sufficiency: We assume that there exist the functions $M, N : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\beta : \Omega \rightarrow (0, +\infty)$ such that the inequality given in the statement holds. For $m = 0$ we have

$$\sum_{n=0}^{+\infty} (n+1)^{\beta(\omega)-1} \|\phi(n, \omega)x\| \leq M(\omega) \|x\|.$$

We aim to establish the existence of a function $D : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\alpha : \Omega \rightarrow (0, +\infty)$ such that

$$\|\phi(n, \omega)x\| \leq D(\omega)(n+1)^{-\alpha(\omega)} \|x\|$$

holds, for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. The argument is proved by contradiction. For any function $D : \Omega \rightarrow [1, +\infty)$ and any θ -invariant random variable $\alpha : \Omega \rightarrow (0, +\infty)$ we have

$$\|\phi(n, \omega)x\| > D(\omega)(n+1)^{-\alpha(\omega)} \|x\|,$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. In particular, taking $D(\omega) = 1$ and $\alpha(\omega) = \beta(\omega)$ we obtain

$$\begin{aligned} M(\omega) \|x\| &\geq \sum_{n=0}^{+\infty} (n+1)^{\beta(\omega)-1} \|\phi(n, \omega)x\| > \\ &> D(\omega) \|x\| \sum_{n=0}^{+\infty} \frac{1}{n+1} = +\infty. \end{aligned}$$

This leads to a contradiction, and therefore we conclude that there exist a function $D : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\alpha : \Omega \rightarrow (0, +\infty)$ such that the inequality (1) holds. $\square$

Theorem 2. *Let (θ, ϕ) be a linear random one-sided discrete-time system. Then (θ, ϕ) is nonuniformly polynomially stable if and only if there exist a θ -invariant random variable $\beta : \Omega \rightarrow (0, +\infty)$ and a function $M : \Omega \rightarrow [1, +\infty)$ such that*

$$\sum_{n=0}^{+\infty} (n+1)^{\beta(\omega)-1} \|\phi(n, \omega)x\| \leq M(\omega) \|x\|, \quad (5)$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proof. Necessity: We consider $\beta : \Omega \rightarrow (0, +\infty)$ such that $0 < \beta < \alpha$. Then

$$\begin{aligned} & \sum_{n=0}^{+\infty} (n+1)^{\beta(\omega)-1} \|\phi(n, \omega)x\| \\ & \leq \sum_{n=0}^{+\infty} (n+1)^{\beta(\omega)-1} N(\omega) (n+1)^{-\alpha(\omega)} \|x\| \\ & = N(\omega) \|x\| \sum_{n=0}^{+\infty} \frac{1}{(n+1)^{1+\alpha(\omega)-\beta(\omega)}} \\ & = M(\omega) \|x\|. \end{aligned}$$

Sufficiency: The proof is analogous to that of the sufficiency part of the previous theorem and is therefore omitted. $\square$

3.2 Lyapunov functions

Definition 2. We say that $L : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}_+$ is a Lyapunov function for the linear random one-sided discrete-time system (θ, ϕ) if there exist a function $D : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\alpha : \Omega \rightarrow (0, +\infty)$ such that

$$L(n, \omega, x) \leq D(\omega) (1+n)^{-\alpha(\omega)} \|x\|, \quad (6)$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Theorem 3. The following statements are equivalent:

- (a) the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially stable;
- (b) there exist a Lyapunov function $L : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}_+$ and a θ -invariant random variable $\eta : \Omega \rightarrow (0, +\infty)$ such that

$$L(0, \omega, x) - L(n, \omega, x) (1+n)^{\eta(\omega)} \geq \sum_{k=0}^{n-1} (1+k)^{\eta(\omega)-1} \|\phi(k, \omega)x\|, \quad (7)$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$;

- (c) there exist a Lyapunov function $L_1 : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}_+$ and a θ -invariant random variable, $\eta : \Omega \rightarrow (0, +\infty)$ such that

$$L_1(n, \omega, x) + \sum_{k=0}^{n-1} (1+k)^{\eta(\omega)-1} \|\phi(k, \omega)x\| \leq L_1(0, \omega, x), \quad (8)$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proof. To prove the implication (a) $\Rightarrow$ (b), we consider the θ -invariant random variable $\beta : \Omega \rightarrow (0, +\infty)$ with $0 < \beta < \alpha$. We define $L : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}_+$ by

$$L(m, \omega, x) = (1 + m)^{-\beta(\omega)} \sum_{k=m}^{+\infty} (1 + k)^{\beta(\omega)-1} \|\phi(k, \omega)x\|,$$

for all $(m, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. Then

$$\begin{aligned} L(n, \omega, x) &= (1 + n)^{-\beta(\omega)} \sum_{k=n}^{+\infty} (1 + k)^{\beta(\omega)-1} \|\phi(k, \omega)x\| \\ &\leq (1 + n)^{-\beta(\omega)} \sum_{k=n}^{+\infty} (1 + k)^{\beta(\omega)-1} N(\omega) (1 + k)^{-\alpha(\omega)} \|x\| \\ &= (1 + n)^{-\beta(\omega)} N(\omega) \|x\| \sum_{k=n}^{+\infty} \frac{1}{(1 + k)^{1+\alpha(\omega)-\beta(\omega)}} \\ &\leq (1 + n)^{-\beta(\omega)} N(\omega) \|x\| \sum_{k=0}^{+\infty} \frac{1}{(1 + k)^{1+\alpha(\omega)-\beta(\omega)}} \\ &= D(\omega) (1 + n)^{-\beta(\omega)} \|x\|. \end{aligned}$$

We obtain that

$$L(n, \omega, x) (1 + n)^{\beta(\omega)} = \sum_{k=n}^{+\infty} (1 + k)^{\beta(\omega)-1} \|\phi(k, \omega)x\|,$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. Hence

$$\begin{aligned} L(n, \omega, x) (1 + n)^{\beta(\omega)} &+ \sum_{k=0}^{n-1} (1 + k)^{\beta(\omega)-1} \|\phi(k, \omega)x\| = \\ &= \sum_{k=0}^{+\infty} (1 + k)^{\beta(\omega)-1} \|\phi(k, \omega)x\| = L(0, \omega, x). \end{aligned}$$

To prove the converse implication (b) $\Rightarrow$ (a), we deduce from (7) that

$$\sum_{k=0}^{n-1} (1 + k)^{\eta(\omega)-1} \|\phi(k, \omega)x\| \leq L(0, \omega, x).$$

Taking the limit as $n \rightarrow +\infty$ we conclude that

$$\sum_{k=0}^{+\infty} (1+k)^{\eta(\omega)-1} \|\phi(k, \omega)x\| \leq L(0, \omega, x).$$

Using (6), this yields to

$$\sum_{k=0}^{+\infty} (1+k)^{\eta(\omega)-1} \|\phi(k, \omega)x\| \leq L(0, \omega, x) \leq D(\omega)\|x\|.$$

According to Theorem 2 it follows that linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially stable.

To prove the implication (a) $\Rightarrow$ (c), we consider $\beta : \Omega \rightarrow (0, +\infty)$ such that $0 < 2\beta < \alpha$. We define $L_1 : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}_+$ by

$$L_1(m, \omega, x) = \sum_{k=m}^{+\infty} (1+k)^{\beta(\omega)-1} \|\phi(k, \omega)x\|,$$

for all $(m, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. Then $L_1 : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}_+$ is a Lyapunov function since

$$\begin{aligned} L_1(n, \omega, x) &= (1+n)^{-\beta(\omega)} \sum_{k=n}^{+\infty} (1+n)^{\beta(\omega)} (1+k)^{\beta(\omega)-1} \|\phi(k, \omega)x\| \\ &\leq (1+n)^{-\beta(\omega)} \sum_{k=n}^{+\infty} (1+n)^{\beta(\omega)} (1+k)^{\beta(\omega)-1} N(\omega) (1+k)^{\alpha(\omega)} \|x\| \\ &= N(\omega) \|x\| (1+n)^{-\beta(\omega)} \sum_{k=n}^{+\infty} (1+k)^{2\beta(\omega)} \frac{1}{(1+k)^{1+\alpha(\omega)}} \\ &\leq N(\omega) \|x\| (1+n)^{-\beta(\omega)} \sum_{k=0}^{+\infty} \frac{1}{(1+k)^{1+\alpha(\omega)-2\beta(\omega)}} \\ &\leq D(\omega) \|x\| (1+n)^{-\beta(\omega)}. \end{aligned}$$

The function $L_1 : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}$ satisfies (8) since

$$\begin{aligned} L_1(n, \omega, x) + \sum_{k=0}^{+\infty} (1+k)^{\beta(\omega)-1} \|\phi(k, \omega)x\| &= \\ &= \sum_{k=n}^{+\infty} (1+k)^{\beta(\omega)-1} \|\phi(k, \omega)x\| + \sum_{k=0}^{n-1} (1+k)^{\beta(\omega)-1} \|\phi(k, \omega)x\| \end{aligned}$$

$$= \sum_{k=0}^{+\infty} (1+k)^{\beta(\omega)-1} \|\phi(k, \omega)x\| = L_1(0, \omega, x).$$

Finally, we prove (c) $\Rightarrow$ (a). From (8) we have that

$$\sum_{k=0}^{n-1} (1+k)^{\eta(\omega)-1} \|\phi(k, \omega)x\| \leq L_1(0, \omega, x).$$

Taking the limit as $n \rightarrow +\infty$ we conclude that

$$\sum_{k=0}^{+\infty} (1+k)^{\eta(\omega)-1} \|\phi(k, \omega)x\| \leq L_1(0, \omega, x).$$

Using (6), this yields to

$$\sum_{k=0}^{+\infty} (1+k)^{\eta(\omega)-1} \|\phi(k, \omega)x\| \leq L_1(0, \omega, x) \leq D(\omega) \|x\|.$$

According to Theorem 2 it follows that the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially stable. $\square$

3.3 Lyapunov norms

Definition 3. We say that the linear random one-sided discrete-time system (θ, ϕ) has polynomial growth if there exist a θ -invariant random variable $\beta : \Omega \rightarrow (0, +\infty)$ and a function $M : \Omega \rightarrow [1, +\infty)$ such that

$$\|\phi(n, \omega)x\| \leq M(\omega)(n+1)^{\beta(\omega)} \|x\|, \quad (9)$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proposition 2. The following statements are equivalent:

- (a) the linear random one-sided discrete-time system (θ, ϕ) has polynomial growth;
- (b) there exist a function $M : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\beta : \Omega \rightarrow (0, +\infty)$ such that

$$\|\phi(n, \omega)\| \leq M(\omega)(n+1)^{\beta(\omega)}, \quad (10)$$

for all $(n, \omega) \in \mathbb{Z}_+ \times \Omega$;

(c) there exist a function $M : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\beta : \Omega \rightarrow (0, +\infty)$ such that

$$\|\phi(n+m, \omega)x\| \leq M(\theta^m \omega)(n+1)^{\beta(\omega)} \|\phi(m, \omega)x\|, \quad (11)$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proof. Let $(n, \omega) \in \mathbb{Z}_+ \times \Omega$. Now we prove the implication (a) $\Rightarrow$ (b).

If

$$\|\phi(n, \omega)x\| \leq M(\omega)(n+1)^{\beta(\omega)} \|x\|,$$

then by taking the supremum over $\|x\| = 1$ we obtain

$$\begin{aligned} \|\phi(n, \omega)\| &= \sup_{\|x\|=1} \|\phi(n, \omega)x\| \\ &\leq \sup_{\|x\|=1} M(\omega)(n+1)^{\beta(\omega)} \|x\| \\ &= M(\omega)(n+1)^{\beta(\omega)}. \end{aligned}$$

For (b) $\Rightarrow$ (a), if (10) holds, then

$$\|\phi(n, \omega)x\| \leq \|\phi(n, \omega)\| \cdot \|x\| \leq M(\omega)(n+1)^{\beta(\omega)} \|x\|,$$

for all $x \in X$.

We establish now (c) $\Rightarrow$ (a). Let $m = 0$ in (11). It follows that

$$\begin{aligned} \|\phi(n+0, \omega)x\| &= \|\phi(n, \omega)x\| \\ &\leq M(\theta^0 \omega)(n+1)^{\beta(\omega)} \|\phi(0, \omega)x\| \\ &= M(\omega)(n+1)^{\beta(\omega)} \|x\|. \end{aligned}$$

For the implication (a) $\Rightarrow$ (c), we note that

$$\|\phi(n, \theta^m \omega)x\| \leq M(\theta^m \omega)(n+1)^{\beta(\theta^m \omega)} \|x\|.$$

Since the random variable β is θ -invariant, we obtain that

$$\|\phi(n, \theta^m \omega)x\| \leq M(\theta^m \omega)(n+1)^{\beta(\omega)} \|x\|.$$

From the previous inequality, we have

$$\|\phi(n, \theta^m \omega)\phi(m, \omega)x\| \leq M(\theta^m \omega)(n+1)^{\beta(\omega)} \|\phi(m, \omega)x\|.$$

Finally, using

$$\phi(n+m, \omega)x = \phi(n, \theta^m \omega)\phi(m, \omega)x,$$

it follows that

$$\begin{aligned} \|\phi(n+m, \omega)x\| &= \|\phi(n, \theta^m \omega)\phi(m, \omega)x\| \\ &\leq M(\theta^m \omega)(n+1)^{\beta(\omega)} \|\phi(m, \omega)x\|, \end{aligned}$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. $\square$

Definition 4. Let (θ, ϕ) be a linear random one-sided discrete-time system with polynomial growth. For every $\omega \in \Omega$, the mapping $\|\cdot\| : X \rightarrow \mathbb{R}_+$ defined for all $x \in X$ by

$$\|x\|_\omega = \sup_{n \in \mathbb{N}} (1+n)^{-\beta(\omega)} \|\phi(n, \omega)x\|$$

is called the Lyapunov norm generated by the linear random one-sided discrete-time system (θ, ϕ) .

Remark 1. The following statements are true:

(a) if $\|\phi(0, \omega)x\| \geq \|x\|$ then

$$\|x\| \leq \|x\|_\omega \leq M(\omega)\|x\|.$$

(b) if the linear random one-sided discrete-time system is nonuniformly polynomially stable, with $M = N$ and $\beta = \alpha$, then

$$\|x\| \leq \|x\|_\omega \leq N(\omega)\|x\|.$$

(c) the topology generated by the family of norms $\{\|\cdot\|_\omega : \omega \in \Omega\}$ does not depend on the θ -invariant random variable $\beta : \Omega \rightarrow (0, +\infty)$.

Indeed, let $x \in X$ and

$$\|x\|_{\omega, \beta} = \sup_{n \in \mathbb{N}} (1+n)^{-\beta(\omega)} \|\phi(n, \omega)x\|.$$

Now, let $\beta_1, \beta_2 : \Omega \rightarrow (0, +\infty)$ be two θ -invariant random variables such that $\beta_1 \leq \beta_2$. From

$$(1+n)^{-\beta_2(\omega)} \leq (1+n)^{-\beta_1(\omega)}$$

it follows that

$$\|x\|_{\omega, \beta_2} \leq \|x\|_{\omega, \beta_1}.$$

Banach's theorem for equivalent norms implies that $\|\cdot\|_{\omega, \beta_1}$ and $\|\cdot\|_{\omega, \beta_2}$ are indeed equivalent. We now assume that the θ -invariant random variables $\beta_1 : \Omega \rightarrow (0, +\infty)$ and $\beta_2 : \Omega \rightarrow (0, +\infty)$ are arbitrary. We consider θ -invariant random variable $\beta_3 = \max\{\beta_1, \beta_2\} : \Omega \rightarrow (0, +\infty)$. Then, from $\beta_1 \leq \beta_3$ we obtain the equivalence of the norms $\|\cdot\|_{\omega, \beta_1}$ and $\|\cdot\|_{\omega, \beta_3}$. Moreover, from $\beta_2 \leq \beta_3$ we have the equivalence of the norms $\|\cdot\|_{\omega, \beta_2}$ and $\|\cdot\|_{\omega, \beta_3}$. Thus, the norms $\|\cdot\|_{\omega, \beta_1}$ and $\|\cdot\|_{\omega, \beta_2}$ are equivalent. Consequently, the norms $\|\cdot\|_{\omega, \beta}$ are equivalent and this equivalence is independent of the θ -invariant random variable $\beta : \Omega \rightarrow (0, +\infty)$.

Theorem 4. *If the linear random one-sided discrete-time system (θ, ϕ) has polynomial growth, then*

$$\|\phi(m_0, \omega)x\|_{\theta^{m_0}\omega} \leq (1 + m_0)^{\beta(\omega)} \|x\|_{\omega},$$

for all $(m_0, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proof. A direct computation yields

$$\begin{aligned} \|\phi(m_0, \omega)x\|_{\theta^{m_0}\omega} &= \sup_{k \in \mathbb{N}} (1 + k)^{-\beta(\theta^{m_0}\omega)} \|\phi(k, \theta^{m_0}\omega)\phi(m_0, \omega)x\| \\ &= \sup_{k \in \mathbb{N}} (1 + k)^{-\beta(\omega)} \|\phi(k + m_0, \omega)x\| \\ &= \sup_{n \geq m_0} (1 + n - m_0)^{-\beta(\omega)} \|\phi(n, \omega)x\| \\ &= \sup_{n \geq m_0} \left(\frac{1 + n}{1 + n - m_0} \right)^{\beta(\omega)} (1 + n)^{-\beta(\omega)} \|\phi(n, \omega)x\| \\ &\leq \sup_{n \geq m_0} (1 + m_0)^{\beta(\omega)} (1 + n)^{-\beta(\omega)} \|\phi(n, \omega)x\| \\ &\leq (1 + m_0)^{\beta(\omega)} \sup_{n \in \mathbb{N}} (1 + n)^{-\beta(\omega)} \|\phi(n, \omega)x\| \\ &= (1 + m_0)^{\beta(\omega)} \|x\|_{\omega}, \end{aligned}$$

since

$$\frac{1 + n}{1 + n - m_0} \leq 1 + m_0,$$

for all $n \geq m_0$. □

Remark 2. *The next result is a theorem of Zabczyk-type for Lyapunov norms.*

Theorem 5. *Let (θ, ϕ) be a linear random one-sided discrete-time system with polynomial growth. Then the system (θ, ϕ) is nonuniformly polynomially stable if and only if there exist a θ -invariant random variable $\eta : \Omega \rightarrow (0, +\infty)$ and a function $D : \Omega \rightarrow [1, +\infty)$ such that*

$$\sum_{n=0}^{+\infty} (1+n)^{\eta(\omega)-1} \|\phi(n, \omega)x\|_{\theta^n \omega} \leq D(\omega) \|x\|_{\omega},$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proof. Necessity. There exist a θ -invariant random variable $\alpha : \Omega \rightarrow (0, +\infty)$ and a function $N : \Omega \rightarrow [1, +\infty)$ satisfying

$$\|\phi(n, \omega)x\| \leq N(\omega)(1+n)^{-\alpha(\omega)} \|x\|,$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$. We obtain

$$\begin{aligned} \|\phi(n, \omega)x\|_{\theta^n \omega} &= \sup_{k \in \mathbb{N}} (1+k)^{-\beta(\omega)} \|\phi(k, \theta^n \omega)\phi(n, \omega)x\| \\ &= \sup_{k \in \mathbb{N}} (1+k)^{-\beta(\omega)} \|\phi(k+n, \omega)x\| \\ &\leq \sup_{k \in \mathbb{N}} (1+k)^{-\beta(\omega)} N(\omega) \|x\| (k+n+1)^{-\alpha(\omega)} \\ &= N(\omega) \|x\| \sup_{m \geq n} (1+m-n)^{-\beta(\omega)} (m+1)^{-\alpha(\omega)} \\ &\leq N(\omega) \|x\| (n+1)^{-\alpha(\omega)}. \end{aligned}$$

Therefore, for any θ -invariant random variable $\eta : \Omega \rightarrow (0, +\infty)$ with $0 < \eta < \alpha$ we have that

$$\begin{aligned} &\sum_{n=0}^{+\infty} (1+n)^{\eta(\omega)-1} \|\phi(n, \omega)x\|_{\theta^n \omega} \leq \\ &\leq N(\omega) \|x\| \cdot \sum_{n=0}^{+\infty} \frac{1}{(1+n)^{1+\alpha(\omega)-\eta(\omega)}} = \\ &= D(\omega) \|x\| \leq D(\omega) \|x\|_{\omega}, \end{aligned}$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Sufficiency. By direct computation we have that

$$\sum_{n=0}^{+\infty} (1+n)^{\eta(\omega)-1} \|\phi(n, \omega)x\| \leq \sum_{n=0}^{+\infty} (1+n)^{\eta(\omega)-1} \|\phi(n, \omega)x\|_{\theta^n \omega} \leq$$

$$\leq D(\omega)||x||_{\omega} \leq D(\omega)M(\omega)||x|| = K(\omega)||x||.$$

Finally, using Theorem 2, it follows that the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially stable. $\square$

4 Nonuniform polynomial instability

Definition 5. We say that the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially unstable if there exist a θ -invariant random variable $\sigma : \Omega \rightarrow (0, +\infty)$ and a function $T : \Omega \rightarrow [1, +\infty)$ such that

$$T(\omega)||\phi(n, \omega)x|| \geq (1+n)^{\sigma(\omega)}||x|| \quad (12)$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Remark 3. From the previous inequality, by taking the supremum over $||x|| = 1$ we obtain the inequality

$$T(\omega)||\phi(n, \omega)|| \geq (1+n)^{\sigma(\omega)}.$$

Note that this inequality does not imply the inequality given in the previous definition.

Proposition 3. The following statements are equivalent:

- (a) the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially unstable;
- (b) there exist a function $T : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\sigma : \Omega \rightarrow (0, +\infty)$ such that

$$T(\theta^m \omega)||\phi(m+n, \omega)x|| \geq (1+n)^{\sigma(\omega)}||\phi(m, \omega)x||,$$

for all $m, n \in \mathbb{Z}_+$, $\omega \in \Omega$ and $x \in X$;

- (c) there exist a function $T : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\sigma : \Omega \rightarrow (0, +\infty)$ such that

$$T(\theta^n \omega)||\phi(n+m, \omega)x|| \geq (1+m)^{\sigma(\omega)}||\phi(n, \omega)x||,$$

for all $n, m \in \mathbb{Z}_+$, $\omega \in \Omega$ and $x \in X$;

(d) there exist a function $T : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\sigma : \Omega \rightarrow (0, +\infty)$ such that

$$T(\omega) \|\phi(n+m, \omega)x\| \geq (1+n+m)^{\sigma(\omega)} \|x\|,$$

for all $n, m \in \mathbb{Z}_+$, $\omega \in \Omega$ and $x \in X$.

Proof. Let T and σ be given as in Definition 5. Then

$$\begin{aligned} T(\theta^m \omega) \|\phi(m+n, \omega)x\| &= T(\theta^m \omega) \|\phi(n, \theta^m \omega) \phi(m, \omega)x\| \geq \\ &\geq (1+n)^{\sigma(\theta^m \omega)} \|\phi(m, \omega)x\| = (1+n)^{\sigma(\omega)} \|\phi(m, \omega)x\|. \end{aligned}$$

The implication (a) $\Rightarrow$ (c) follows by an analogous argument. The implication (b) $\Rightarrow$ (a) holds for $m = 0$. For $n = 0$ we obtain the implication (c) $\Rightarrow$ (a). The implication (a) $\Rightarrow$ (d) is immediate. The implication (d) $\Rightarrow$ (a) follows for $m = 0$. $\square$

Theorem 6. *If the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially unstable then exist a θ -invariant random variable $\delta : \Omega \rightarrow (0, +\infty)$ and two functions $P, Q : \Omega \rightarrow [1, +\infty)$ such that*

$$\sum_{n=0}^m (m-n+1)^{\delta(\omega)-1} P(\theta^n \omega)^{-1} \|\phi(n, \omega)x\| \leq Q(\omega) \|\phi(m, \omega)x\|, \quad (13)$$

for all $(m, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proof. Let T and σ be given as in Definition 5. Then

$$\begin{aligned} \|\phi(m, \omega)x\| &= \|\phi(m-n+n, \omega)x\| = \\ &= \|\phi(m-n, \theta^n \omega) \phi(n, \omega)x\| \\ &\geq T(\theta^n \omega)^{-1} (m-n+1)^{\sigma(\theta^n \omega)} \|\phi(n, \omega)x\| \\ &= T(\theta^n \omega)^{-1} (m-n+1)^{\sigma(\omega)} \|\phi(n, \omega)x\|, \end{aligned}$$

for all $(m, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

We consider $P = T : \Omega \rightarrow [1, +\infty)$ a θ -invariant random variable $\delta : \Omega \rightarrow$

$(0, +\infty)$ such that $0 < \delta < \sigma$. Then

$$\begin{aligned}
& \sum_{n=0}^m (m-n+1)^{\delta(\omega)-1} P(\theta^n \omega)^{-1} \|\phi(n, \omega)x\| \leq \\
& \leq \sum_{n=0}^m (m-n+1)^{\delta(\omega)-1-\sigma(\omega)} \|\phi(m, \omega)x\| \\
& = \|\phi(m, \omega)x\| \cdot \sum_{k=1}^{m+1} \frac{1}{k^{1+\sigma(\omega)-\delta(\omega)}} \\
& \leq \|\phi(m, \omega)x\| \cdot \sum_{k=1}^{+\infty} \frac{1}{k^{1+\sigma(\omega)-\delta(\omega)}} \\
& \leq Q(\omega) \|\phi(m, \omega)x\|,
\end{aligned}$$

were

$$Q(\omega) = 1 + \sum_{k=1}^{+\infty} \frac{1}{k^{1+\sigma(\omega)-\delta(\omega)}}.$$

□

Definition 6. We say that $L_2 : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}_+$ is a Lyapunov function for the linear random one-sided discrete-time system if there exist a θ -invariant random variable $\xi : \Omega \rightarrow (0, +\infty)$ and a function $N : \Omega \rightarrow [1, +\infty)$ such that

$$L_2(0, \omega, x) + \sum_{n=0}^m (m-n+1)^{\xi(\omega)-1} N(\theta^n \omega)^{-1} \|\phi(n, \omega)x\| \leq L_2(m, \omega, x), \quad (14)$$

for all $(m, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Theorem 7. If the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially unstable, then there exist a Lyapunov function $L_2 : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}_+$ and a function $K : \Omega \rightarrow [1, +\infty)$ such that

$$L_2(n, \omega, x) \leq K(\omega) \|\phi(n, \omega)x\|,$$

for all $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$.

Proof. If the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially unstable then from Theorem 6 there exist a θ -invariant

random variable $\delta : \Omega \rightarrow (0, +\infty)$ and two functions $P, Q : \Omega \rightarrow [1, +\infty)$ such that

$$\sum_{n=0}^m (m - n + 1)^{\delta(\omega)-1} P(\theta^n \omega)^{-1} \|\phi(n, \omega)x\| \leq Q(\omega) \|\phi(m, \omega)x\|.$$

Let $N = P : \Omega \rightarrow [1, +\infty)$ and $\xi = \delta : \Omega \rightarrow (0, +\infty)$. We define

$$L_2(m, \omega, x) = \begin{cases} 0 & , m = 0 \\ \sum_{n=0}^m (m - n + 1)^{\xi(\omega)-1} N(\theta^n \omega)^{-1} \|\phi(n, \omega)x\| & , m > 0 \end{cases}$$

Then $L_2 : \mathbb{Z}_+ \times \Omega \times X \rightarrow \mathbb{R}_+$ is a Lyapunov function that satisfies the conditions of the statement with $K = Q : \Omega \rightarrow [1, +\infty)$. $\square$

Remark 4. *For nonuniform polynomial instability, both Zabczyk-type and Lyapunov-type criteria provide only necessary conditions. Sufficiency is still an open problem.*

5 Applications to the adjoint and inverse system

Given a linear random one-sided discrete-time system (θ, ϕ) , two associated linear random one-sided discrete-time systems can be defined: the inverse system (θ, ϕ^{-1}) and the adjoint system (θ, ϕ^*) .

Let $(n, \omega) \in \mathbb{Z}_+ \times \Omega$. Then $\phi(n, \omega)^{-1} : X \rightarrow X$ satisfies

$$P(n, \omega) \phi(n, \omega)^{-1} = \phi(n, \omega)^{-1} \phi(n, \omega)x = x,$$

for all $x \in X$. If X^* is the topological dual of the space X , then

$$\phi(n, \omega)^* : X^* \rightarrow X^*$$

satisfies

$$\phi(n, \omega)^* y^*(x) = y^*(\phi(n, \omega)x),$$

for all $y^* \in X^*$.

Proposition 4. *The linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially stable if and only if the adjoint system (θ, ϕ^*) is nonuniformly polynomially stable.*

Proof. Necessity. Let $(n, \omega, x) \in \mathbb{Z}_+ \times \Omega \times X$ and $y^* \in X^*$. Let N and α as in Definition 1. We deduce that

$$\begin{aligned} \|\phi(n, \omega)^* y^*(x)\| &= \|y^*(\phi(n, \omega)x)\| \leq \\ &\leq \|y^*\| \cdot \|\phi(n, \omega)x\| \leq N(\omega)(n+1)^{-\alpha(\omega)} \|y^*\| \cdot \|x\|. \end{aligned}$$

Taking the supremum over $\|x\| = 1$ we obtain that

$$\|\phi(n, \omega)^* y^*\| \leq N(\omega)(n+1)^{-\alpha(\omega)} \|y^*\|.$$

This proves that (θ, ϕ^*) is nonuniformly polynomially stable.

Sufficiency. We assume that (θ, ϕ^*) is nonuniformly polynomially stable. Let $N : \Omega \rightarrow [1, +\infty)$ and $\alpha : \Omega \rightarrow (0, +\infty)$ a θ -invariant random variable such that

$$\|\phi(n, \omega)^* y^*\| \leq N(\omega)(n+1)^{-\alpha(\omega)} \|y^*\|.$$

Taking the supremum over $\|y^*\| = 1$ we obtain that

$$\|\phi(n, \omega)\| = \|\phi(n, \omega)^*\| \leq N(\omega)(n+1)^{-\alpha(\omega)}.$$

Now from Proposition 1 we deduce that the system (θ, ϕ) is nonuniformly polynomially stable. $\square$

Theorem 8. *The following statements are equivalent:*

- (a) *the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially stable if and only if the inverse system (θ, ϕ^{-1}) is nonuniformly polynomially unstable;*
- (b) *the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially stable if and only if the inverse system (θ, ϕ^{-1}) is nonuniformly polynomially stable.*

Proof. (a) Let $(n, \omega) \in \mathbb{Z}_+ \times \Omega$ and $y \in X$. There exists a unique $x \in X$ such that

$$\phi(n, \omega)x = y \Leftrightarrow x = \phi(n, \omega)^{-1}y.$$

If N and α are as in Definition 1, then

$$\begin{aligned} \|y\| &= \|\phi(n, \omega)x\| \leq N(\omega)(1+n)^{-\alpha(\omega)} \|x\| \\ &= N(\omega)(1+n)^{-\alpha(\omega)} \|\phi(n, \omega)^{-1}y\|. \end{aligned}$$

Therefore,

$$N(\omega) \|\phi(n, \omega)^{-1}y\| \geq (1+n)^{\alpha(\omega)} \|y\|.$$

Conversely, let $(n, \omega) \in \mathbb{Z}_+ \times \Omega$ and $x \in X$. There exists a unique $y \in X$ such that

$$\phi(n, \omega)^{-1}y = x \Leftrightarrow \phi(n, \omega)x = y.$$

We assume that the function $S : \Omega \rightarrow [1, +\infty)$ and the θ -invariant random variable $\xi : \Omega \rightarrow (0, +\infty)$ satisfy

$$S(\omega) \|\phi(n, \omega)^{-1}y\| \geq (1+n)^{\xi(\omega)} \|y\|.$$

Hence

$$S(\omega) \|x\| \geq (1+n)^{\xi(\omega)} \|\phi(n, \omega)x\|,$$

which is equivalent to

$$\|\phi(n, \omega)x\| \leq S(\omega)(1+n)^{-\xi(\omega)} \|x\|.$$

(b) Let $(n, \omega) \in \mathbb{Z}_+ \times \Omega$ and let $y \in X$. There exists a unique $x \in X$ such that

$$\phi(n, \omega)x = y \Leftrightarrow x = \phi(n, \omega)^{-1}y.$$

Let $T : \Omega \rightarrow [1, +\infty)$ and $\sigma : \Omega \rightarrow (0, +\infty)$ be as in Definition 5. It follows that

$$\begin{aligned} T(\omega) \|y\| &= T(\omega) \|\phi(n, \omega)x\| \geq (1+n)^{\sigma(\omega)} \|x\| = \\ &= (1+n)^{\sigma(\omega)} \|\phi(n, \omega)^{-1}y\|. \end{aligned}$$

Hence

$$\|\phi(n, \omega)^{-1}y\| \leq T(\omega)(1+n)^{-\sigma(\omega)} \|y\|.$$

Conversely, let $(n, \omega) \in \mathbb{Z}_+ \times \Omega$ and $x \in X$. There exists a unique $y \in X$ such that

$$\phi(n, \omega)^{-1}y = x \Leftrightarrow \phi(n, \omega)x = y.$$

Consider the function $S : \Omega \rightarrow [1, +\infty)$ and a θ -invariant random variable $\xi : \Omega \rightarrow (0, +\infty)$ such that

$$\|\phi(n, \omega)^{-1}y\| \leq S(\omega)(1+n)^{-\xi(\omega)} \|y\|.$$

Hence

$$\|x\| \leq S(\omega)(1+n)^{-\xi(\omega)} \|\phi(n, \omega)x\|,$$

which is equivalent to

$$S(\omega) \|\phi(n, \omega)x\| \geq (1+n)^{\xi(\omega)} \|x\|.$$

□

Theorem 9. *The linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially unstable if and only if the adjoint system (θ, ϕ^*) is nonuniformly polynomially unstable.*

Proof. We have that the linear random one-sided discrete-time system (θ, ϕ) is nonuniformly polynomially unstable if and only if the system (θ, ϕ^{-1}) is nonuniformly polynomially stable. Next, the system (θ, ϕ^{-1}) is nonuniformly polynomially stable if and only if the system $(\theta, (\phi^{-1})^*) = (\theta, (\phi^*)^{-1})$ is nonuniformly polynomially stable. Finally, the system $(\theta, (\phi^*)^{-1})$ is nonuniformly polynomially stable if and only if the system $(\theta, ((\phi^*)^{-1})^{-1}) = (\theta, \phi^*)$ is nonuniformly polynomially unstable, which concludes the proof. $\square$

References

- [1] R.P. Agarwal, *Difference Equations and Inequalities*, In: Theory, Methods and Applications, Marcel Dekker, New York, 1992.
- [2] V.C. Aitken and H.M. Schwartz, On the exponential stability of discrete-time systems with applications in observer design, *IEEE Trans. Autom. Control* 39 (1994), 1959-1962.
- [3] L. Arnold, *Random Dynamical Systems*, Springer, New York, 1998.
- [4] B. Aulbach and J. Kalkbrenner, Exponential forward splitting for non-invertible difference equations, *Comput. Math. Appl.* 42 (2001), 743-754.
- [5] B. Aulbach and S. Siegmund, The dichotomy spectrum for noninvertible systems of difference equations, *J. Differ. Equations Appl.* 7 (2001), 895-913.
- [6] L.E. Biriş, T. Ceaşu, I.-L. Popa and N. Seimeanu, On uniform polynomial splitting of variational nonautonomous difference equations in Banach spaces, *An. Univ. Vest Timiş. Ser. Mat.-Inform.* 58 (2022), 22-37.
- [7] R. Boruga and M. Megan, Nonuniform polynomial dichotomy with Lyapunov type norms, *Ann. Acad. Rom. Sci. Ser. Math. Appl.* 12 (2020), 329-343.

- [8] N.D. Cong, *Topological Dynamics of Random Dynamical Systems*, Clarendon, Oxford, 1997.
- [9] R. Datko, Uniform asymptotic stability of evolutionary processes in a Banach space, *SIAM J. Math. Anal.* 3 (1972), 428-445.
- [10] T.S. Doan, Lyapunov exponents for random dynamical systems, PhD Thesis, Technischen Universität Dresden, 2009.
- [11] D. Dragičević, On the exponential stability and hyperbolicity of linear cocycles, *Linear Multilinear Algebra* 69 (2021), 259-277.
- [12] D. Dragičević, A.L. Sasu and B. Sasu, On polynomial dichotomies of discrete nonautonomous systems on the half-line, *Carpathian J. Math.* 38 (2022), 663-680.
- [13] D. Dragičević, A.L. Sasu, B. Sasu and A. Şirianu, Nonuniform criteria for exponential stability of discrete and continuous variational dynamics, *J. Math. Anal. Appl.* 554 (2026), 129964.
- [14] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics, Vol. 840, Springer, Berlin, 1981.
- [15] N. Lupa and L.H. Popescu, A complete characterization of exponential stability for discrete dynamics, *J. Differ. Equations Appl.* 23 (2017), 2072-2092.
- [16] M. Megan, A.L. Sasu and B. Sasu, On uniform exponential stability of linear skew-product semiflows in Banach spaces, *Bull. Belg. Math. Soc. - Simon Stevin* 9 (2002), 143-154.
- [17] M. Megan and I.-L. Popa, Exponential splitting for nonautonomous linear discrete-time systems in Banach spaces. *J. Comput. Appl. Math.* 312 (2017), 181-191.
- [18] I.-L. Popa, T. Ceaşu and M. Megan, Nonuniform power instability and Lyapunov sequences, *Appl. Math. Comput.* 247 (2014), 969-975.
- [19] I.-L. Popa, T. Ceaşu and M. Megan, On exponential stability for linear discrete-time systems in Banach spaces, *Comput. Math. Appl.* 63 (2012), 1497-1503.
- [20] I.-L. Popa, Lyapunov functions for random semidynamical systems in terms of tempered exponential splitting, *Math. Meth. Appl. Sci.* 44 (2021), 11923-11932.

- [21] C. Pötzsche, *Geometric Theory of Discrete Nonautonomous Dynamical Systems*, Lecture Notes in Mathematics, Springer 2010.
- [22] R.L. Sacker and G.R. Sell, Lifting Properties in skew-product flows with applications to differential equations, *Mem. Am. Math. Soc.* 11 (1977), 190.
- [23] A.L. Sasu, M. Megan and B. Sasu, On Rolewicz-Zabczyk techniques in the stability theory of dynamical systems, *Fixed Point Theory* 13 (2012), 205-236.
- [24] Z. Zabczyk, Remarks on the control of discrete-time distributed parameter systems, *SIAM J. Control Optim.* 12 (1974), 721-735.
- [25] L. Zhou, K. Lu and W. Zhang, Roughness of tempered exponential dichotomies for infinite-dimensional random difference equations, *J. Differ. Equations* 254 (2013), 4024-4046.

