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NEW DEVELOPMENTS OF FRACTIONAL INTEGRAL INEQUALITIES VIA DIFFERENT TYPES OF CONVEXITY*

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Dedicated to Professor Constantin Fetecau on his 80th anniversary

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Abstract

Integral inequalities based on generalized convexity are of great importance in both theoretical and applied mathematics. Due to its extensive applications, the study of integral inequalities has become one of the most rapidly advancing fields in mathematics. In this paper, we introduce several new classes of Milne-type inequalities using higher-order strongly exponentially convex functions. Novel relationships between various concepts of higher-order strongly exponentially functions are developed. These findings include several well-known results as well as some newly established results as particular cases. The results obtained in this paper can be viewed as a generalization and extension of previously known results.

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1 Introduction

In recent years, the generalization of convex functions has emerged as a fundamental and influential concept in the theoretical development of mathematical inequalities. Integral inequalities associated with various classes of convex functions play a significant role in numerous areas of mathematics, including mathematical analysis, fractional calculus, and discrete fractional calculus. See the references for further details ([4, 10, 15, 18, 32]).

Karamardian [12] introduced the concept of strongly convex functions and utilized this notion to establish the unique existence of solutions to nonlinear complementarity problems. However, several researchers have noted that the concept of strongly convex functions was originally introduced by Polyak [28]. Lin and Fukushima [14] introduced the notion of higher-order strongly convex functions and established that a continuously differentiable function is higher-order strongly convex if and only if its gradient satisfies the condition of higher-order strong monotonicity. Mishra et al. [17] introduced the concept of higher-order strongly generalized convex functions and subsequently derived related results and properties. In [7], the authors derived some new integral inequalities for higher-order strongly exponentially convex functions. For further fundamental properties and applications of strongly convex functions, the reader is referred to the cited works [5, 16, 20] and the references therein.

It is well recognized that algorithmically convex functions can be utilized to establish sharper and more refined inequalities. The notion of exponentially convex (or exponentially concave) functions is closely associated with that of log-convex functions. The development of exponentially convex functions can be attributed to the pioneering work of Bernstein [6]. Avriel [3]proposed and analyzed the concept of r-convex functions, whereas the notion of (r, p)-convex functions was subsequently explored by Antczak [2]. For a more detailed discussion on the properties of r-convex functions, see [11, 22] and the references provided therein. Exponentially convex functions have substantial applications across diverse domains, such as information theory, big data processing, machine learning, and statistical analysis. Noor and Noor explored the concept of exponentially convex functions and elaborated

on their fundamental properties [23–27]. It is important to emphasize that the class of exponentially convex functions presented in [23–26] differs significantly from the notion of exponentially convex functions introduced by Bernstein [6]. They have established that exponentially convex functions exhibit many of the fundamental and significant properties characteristic of convex functions.

By incorporating the principles of fractional calculus into the framework of generalized convexity, researchers can establish a more extensive class of mathematical inequalities. These inequalities provide a deeper theoretical understanding of fractional integrals and, at the same time, possess considerable practical relevance for the analysis and optimization of complex systems in various fields of science and engineering. The interplay between generalized convexity and fractional integral operators opens up new avenues for research and facilitates the development of more robust mathematical frameworks for the modeling, analysis, and solution of problems in fractional dynamics, see references [1, 34, 37, 38].

Due to the significant impact of fractional integrals across various areas of pure and applied sciences, scholars have extended the classical concept in multiple directions and established new integral inequalities corresponding to these generalized fractional integral operators. Kilbas et al. [13] examined several important properties of various classes of fractional integrals and fractional derivatives and also explored integral transform techniques to obtain explicit solutions of fractional differential equations. Srivastava et al. [35] established a fractional version of the Steffensen–Hayashi inequality and presented several noteworthy applications to different inequalities associated with ν-fractional operators. Srivastava et al. [36] established novel Hermite–Hadamard type inequalities for interval-valued preinvex functions by employing Riemann–Liouville fractional integral operators.

Recently, Mishra et al. [8] formulates an extension of Hermite-Hadamard type inequalities to Katugampola fractional integrals. Budak et al. [9] established on new general inequalities for exponentially s-convex functions via Katugampola fractional integrals. Bisht et al. some new integral inequalities for higher-order strongly exponentially convex functions. Rashid et al. [29] established some trapezoid-type inequalities for generalized fractional integrals and related inequalities via exponentially convex functions. Rashid et al. [30] derived a new integral identity involving Riemann–Liouville fractional integrals and obtained new fractional bounds for the functions having the exponential convexity property.

Inspired by the preceding developments, we aim to obtain new integral inequalities for generalized exponentially convex functions. Several scholars

have developed extensions and generalizations of the Milne–type inequality for generalized convex functions, but higher-order strongly exponentially convex functions have not been considered in their results. Therefore, we derive a Milne–type integral inequality for higher-order strongly exponentially convex functions. Over the past few years, fractional calculus has been widely used to derive several integral inequalities related to generalized convex functions. Despite these developments, fractional inequalities related to higher-order strongly exponentially convex functions remain unexplored. By utilizing Riemann–Liouville fractional integrals, we develop new integral inequalities associated with higher-order strongly exponentially convex functions. In addition, several special cases of the main results are outlined.

The organization of this paper is as follows. In section 2 and 3, we discuss some fundamental definitions and one important generalized lemma as needed for our main result. In section 4 and 5, we state and prove our main results by using the above-mentioned generalized lemma and theorems, and we also derive some new corollaries. Lastly, in section 6, a conclusion is drawn.

2 Preliminaries

In this section, we collect some notations and basic definitions required in the sequel of the paper.

Definition 1. *A function $h : I \to \mathbb{R}$ is said to be a convex function if*

$$h\left(\nu\sigma_1 + (1-\nu)\,\sigma_2\right) \leqslant \nu h\left(\sigma_1\right) + (1-\nu)\,h\left(\sigma_2\right)$$

holds for all $\sigma_1, \sigma_2 \in I$ and $\nu \in [0,1]$.

Definition 2. *A positive real-valued function $h : I \subseteq \mathbb{R} \to (0,\infty)$ is said to be exponentially convex on K if*

$$e^{h(\nu\sigma_1+(1-\nu)\sigma_2)} \leq \nu e^{h(\sigma_1)} + (1-\nu)e^{h(\sigma_2)}.$$

Exponentially convex functions are used in statistical learning, sequential prediction, and stochastic optimization, see [19,31,33,39].

Definition 3. *[21] A positive function h defined on the convex set K is called a higher-order strongly exponentially convex function of order $\sigma > 1$ if there exists a constant $c > 0$ such that*

$$e^{h(\nu\sigma_1+(1-\nu)\sigma_2)} \leq \nu e^{h(\sigma_1)} + (1-\nu)e^{h(\sigma_2)} - c\nu(1-\nu)||\sigma_2 - \sigma_1||^{\sigma},$$

$\forall\ \sigma_1, \sigma_2 \in K$ *and* $\nu \in [0,1]$.

Definition 4. *Let $\Phi \in L^1[\sigma_1, \sigma_2]$. Then the left-sided and right-sided Riemann-Liouville fractional integrals of order $\mu \in \mathbb{R}_+$ are defined by*

$$\mathbb{I}^{\mu}_{\sigma_1^+}\Phi(v_1) = \frac{1}{\Gamma(\mu)}\int_{\sigma_1}^{v_1}(v_1-\eta)^{\mu-1}\Phi(\eta)d\eta, \qquad (0 \leq \sigma_1 < v_1 < \sigma_2)$$

and

$$\mathbb{I}^{\mu}_{\sigma_2^-}\Phi(v_1) = \frac{1}{\Gamma(\mu)}\int_{v_1}^{\sigma_2}(\eta-v_1)^{\mu-1}\Phi(\eta)d\eta, \qquad (0 \leq \sigma_1 < v_1 < \sigma_2)$$

respectively, where $\Gamma(\mu)$ is the Gamma function.

3 Auxiliary results

First, we present the following computed integrals that will be employed in establishing our key results:

$$\begin{aligned}\kappa_1(\mu) &= \int_0^{\frac{1}{2}} |\nu^{\mu} - \frac{2}{3}| d\nu \\ &= \begin{cases} \frac{2\mu}{\mu+1}(\frac{2}{3})^{\frac{1}{\mu}+1} + \frac{1}{2^{\mu+1}(\mu+1)} - \frac{1}{3}, & 0 < \mu \leq \frac{ln\frac{2}{3}}{ln\frac{1}{3}} \\ \frac{1}{3} - \frac{1}{2^{\mu+1}(\mu+1)} & \mu > \frac{ln\frac{2}{3}}{ln\frac{1}{3}}, \end{cases}\end{aligned}$$

$$\begin{aligned}\kappa_2(\mu) &= \int_0^{\frac{1}{2}} \nu|\nu^{\mu} - \frac{2}{3}| d\nu \\ &= \begin{cases} \frac{\mu}{\mu+2}(\frac{2}{3})^{\frac{2}{\mu}+1} + \frac{1}{2^{\mu+2}(\mu+2)} - \frac{1}{12}, & 0 < \mu \leq \frac{ln\frac{2}{3}}{ln\frac{1}{3}} \\ \frac{1}{12} - \frac{1}{2^{\mu+2}(\mu+2)} & \mu > \frac{ln\frac{2}{3}}{ln\frac{1}{3}}, \end{cases}\end{aligned}$$

$$\begin{aligned}\kappa_3(\mu) &= \int_0^{\frac{1}{2}} \nu^2|\nu^{\mu} - \frac{2}{3}| d\nu \\ &= \begin{cases} \frac{2\mu}{\mu+3}(\frac{2}{3})^{\frac{3}{\mu}+1} + \frac{1}{2^{\mu+3}(\mu+3)} - \frac{1}{36}, & 0 < \mu \leq \frac{ln\frac{2}{3}}{ln\frac{1}{3}} \\ \frac{1}{36} - \frac{1}{2^{\mu+3}(\mu+3)} & \mu > \frac{ln\frac{2}{3}}{ln\frac{1}{3}}, \end{cases}\end{aligned}$$

$$\kappa_3^*(\mu) = \int_0^{\frac{1}{2}} \nu^3 |\nu^\mu - \frac{2}{3}| d\nu$$
$$= \begin{cases} \frac{3\mu}{2(\mu+4)}(\frac{2}{3})^{\frac{4}{\mu}+1} + \frac{1}{2^{\mu+4}(\mu+4)} - \frac{1}{96}, 0 < \mu \leq \frac{ln\frac{2}{3}}{ln\frac{1}{3}} \\ \frac{1}{96} - \frac{1}{2^{\mu+4}(\mu+4)} \qquad \mu > \frac{ln\frac{2}{3}}{ln\frac{1}{3}}, \end{cases}$$

$$\kappa_3^{**}(\mu) = \int_0^{\frac{1}{2}} \nu^4 |\nu^\mu - \frac{2}{3}| d\nu$$
$$= \begin{cases} \frac{2\mu}{5(\mu+5)}(\frac{2}{3})^{\frac{5}{\mu}+1} + \frac{1}{2^{\mu+5}(\mu+5)} - \frac{1}{240}, 0 < \mu \leq \frac{ln\frac{2}{3}}{ln\frac{1}{3}} \\ \frac{1}{240} - \frac{1}{2^{\mu+5}(\mu+5)} \qquad \mu > \frac{ln\frac{2}{3}}{ln\frac{1}{3}}, \end{cases}$$

$$\kappa_4(\mu) = \int_{\frac{1}{2}}^1 |\nu^\mu - \frac{1}{3}| d\nu$$
$$= \begin{cases} \frac{1}{\mu+1} - \frac{1}{2^{\mu+1}(\mu+1)} - \frac{1}{6}, \qquad 0 < \mu \leq \frac{ln\frac{1}{3}}{ln\frac{1}{2}} \\ \frac{2\mu}{\mu+1}(\frac{1}{3})^{\frac{1}{\mu}+1} + \frac{1}{2^{\mu+1}(\mu+1)} + \frac{1}{\mu+1} - \frac{1}{2} \ \mu > \frac{ln\frac{1}{3}}{ln\frac{1}{2}}, \end{cases}$$

$$\kappa_5(\mu) = \int_{\frac{1}{2}}^1 \nu |\nu^\mu - \frac{1}{3}| d\nu$$
$$= \begin{cases} \frac{1}{\mu+2} - \frac{1}{2^{\mu+2}(\mu+2)} - \frac{1}{8}, \qquad 0 < \mu \leq \frac{ln\frac{1}{3}}{ln\frac{1}{2}} \\ \frac{\mu}{\mu+2}(\frac{1}{3})^{\frac{2}{\mu}+1} + \frac{2}{2^{\mu+2}(\mu+2)} + \frac{1}{\mu+2} - \frac{5}{24} \ \mu > \frac{ln\frac{1}{3}}{ln\frac{1}{2}}, \end{cases}$$

$$\kappa_6(\mu) = \int_{\frac{1}{2}}^1 \nu^2 |\nu^\mu - \frac{1}{3}| d\nu$$
$$= \begin{cases} \frac{1}{\mu+3} - \frac{1}{2^{\mu+3}(\mu+3)} - \frac{7}{72}, \qquad 0 < \mu \leq \frac{ln\frac{1}{3}}{ln\frac{1}{2}} \\ \frac{2\mu}{\mu+3}(\frac{1}{3})^{\frac{3}{\mu}+1} + \frac{2}{2^{\mu+3}(\mu+3)} + \frac{1}{\mu+3} - \frac{1}{8} \ \mu > \frac{ln\frac{1}{3}}{ln\frac{1}{2}}, \end{cases}$$

$$\kappa_6^*(\mu) = \int_{\frac{1}{2}}^1 \nu^3 |\nu^\mu - \frac{1}{3}| d\nu$$
$$= \begin{cases} \frac{1}{\mu+4} - \frac{1}{2^{\mu+4}(\mu+4)} - \frac{15}{192}, \qquad 0 < \mu \leq \frac{ln\frac{1}{3}}{ln\frac{1}{2}} \\ \frac{3\mu}{2(\mu+4)}(\frac{1}{3})^{\frac{4}{\mu}+1} + \frac{2}{2^{\mu+4}(\mu+4)} + \frac{1}{\mu+4} - \frac{17}{192} \ \mu > \frac{ln\frac{1}{3}}{ln\frac{1}{2}}, \end{cases}$$

$$\kappa_6^{**}(\mu) = \int_{\frac{1}{2}}^{1} \nu^4 |\nu^\mu - \frac{1}{3}| d\nu$$
$$= \begin{cases} \frac{1}{\mu+5} - \frac{1}{2^{\mu+5}(\mu+5)} - \frac{31}{480}, & 0 < \mu \leq \frac{ln\frac{1}{3}}{ln\frac{1}{2}} \\ \frac{6\mu}{5(\mu+5)}(\frac{1}{3})^{\frac{5}{\mu}+1} + \frac{2}{2^{\mu+5}(\mu+5)} + \frac{1}{\mu+5} - \frac{26}{375}, & \mu > \frac{ln\frac{1}{3}}{ln\frac{1}{2}}. \end{cases}$$

Lemma 1. *Let $\eta : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a differentiable mapping on (σ_1, σ_2) such that $h' \in L_1([\sigma_1, \sigma_2])$. Then, for $\mu > 0$, the following equality holds:*

$$\frac{1}{3}\left[2e^{h(\sigma_1)} - e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)} + 2e^{h(\sigma_2)}\right]$$
$$- \frac{\Gamma(\mu+1)}{2(\sigma_2-\sigma_1)^\mu}\left[K^\mu_{\sigma_1^+} e^{h(\sigma_2)} + K^\mu_{\sigma_2^-} e^{h(\sigma_1)}\right]$$
$$= \frac{\sigma_2 - \sigma_1}{2}\Bigg[L_1 + L_2 - L_3 - L_4\Bigg],$$

where,

$$L_1 = \int_0^{\frac{1}{2}} \left(\nu^\mu - \frac{2}{3}\right) e^{h(\nu\sigma_2 + (1-\nu)\sigma_1)} h'(\nu\sigma_2 + (1-\nu)\sigma_1) d\nu,$$
$$L_2 = \int_{\frac{1}{2}}^{1} \left(\nu^\mu - \frac{1}{3}\right) e^{h(\nu\sigma_2 + (1-\nu)\sigma_1)} h'(\nu\sigma_2 + (1-\nu)\sigma_1) d\nu,$$
$$L_3 = \int_0^{\frac{1}{2}} \left(\nu^\mu - \frac{2}{3}\right) e^{h(\nu\sigma_1 + (1-\nu)\sigma_2)} h'(\nu\sigma_1 + (1-\nu)\sigma_2) d\nu,$$
$$L_4 = \int_{\frac{1}{2}}^{1} \left(\nu^\mu - \frac{1}{3}\right) e^{h(\nu\sigma_1 + (1-\nu)\sigma_2)} h'(\nu\sigma_1 + (1-\nu)\sigma_2) d\nu.$$

Proof. Using the integration by parts, we get

$$L_1 = \int_0^{\frac{1}{2}} \left(\nu^\mu - \frac{2}{3}\right) e^{h(\nu\sigma_2 + (1-\nu)\sigma_1)} h'(\nu\sigma_2 + (1-\nu)\sigma_1) d\nu \tag{1}$$
$$= \frac{1}{\sigma_2 - \sigma_1}\left[\left(\nu^\mu - \frac{2}{3}\right) e^{h(\nu\sigma_2 + (1-\nu)\sigma_1)}\right]_0^{\frac{1}{2}}$$
$$- \frac{\mu}{\sigma_2 - \sigma_1} \int_0^{\frac{1}{2}} \nu^{\mu-1} e^{h(\nu\sigma_2 + (1-\nu)\sigma_1)} d\nu$$
$$= \frac{1}{\sigma_2 - \sigma_1}\left(\frac{1}{2^\mu} - \frac{2}{3}\right) e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)} + \frac{2}{3(\sigma_2 - \sigma_1)} e^{h(\sigma_1)}$$
$$- \frac{\mu}{(\sigma_2 - \sigma_1)} \int_0^{\frac{1}{2}} \nu^{\mu-1} e^{h(\nu\sigma_2 + (1-\nu)\sigma_1)} d\nu,$$

$$
\begin{aligned}
L_2 &= \int_{\frac{1}{2}}^{1} \left(\nu^{\mu} - \frac{1}{3}\right) e^{h(\nu\sigma_2+(1-\nu)\sigma_1)} h'(\nu\sigma_2 + (1-\nu)\sigma_1) d\nu \qquad (2)\\
&= \frac{1}{\sigma_2 - \sigma_1}\left[\left(\nu^{\mu} - \frac{1}{3}\right) e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}\right]_{\frac{1}{2}}^{1}\\
&\quad - \frac{\mu}{(\sigma_2 - \sigma_1)} \int_{\frac{1}{2}}^{1} \nu^{\mu-1} e^{h(\nu\sigma_2+(1-\nu)\sigma_1)} d\nu\\
&= \frac{2}{3(\sigma_2 - \sigma_1)} e^{h(\sigma_2)} - \frac{1}{\sigma_2 - \sigma_1}\left(\frac{1}{2^{\mu}} - \frac{1}{3}\right) e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)}\\
&\quad - \frac{\mu}{\sigma_2 - \sigma_1} \int_{\frac{1}{2}}^{1} \nu^{\mu-1} e^{h(\nu\sigma_2+(1-\nu)\sigma_1)} d\nu.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
L_3 &= \int_{0}^{\frac{1}{2}} \left(\nu^{\mu} - \frac{2}{3}\right) e^{h(\nu\sigma_1+(1-\nu)\sigma_2)} h'(\nu\sigma_1 + (1-\nu)\sigma_2) d\nu \qquad (3)\\
&= -\frac{1}{\sigma_2 - \sigma_1}\left(\frac{1}{2^{\mu}} - \frac{2}{3}\right) e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)} - \frac{2}{3(\sigma_2 - \sigma_1)} e^{h(\sigma_2)}\\
&\quad + \frac{\mu}{(\sigma_2 - \sigma_1)} \int_{0}^{\frac{1}{2}} \nu^{\mu-1} e^{h(\nu\sigma_1+(1-\nu)\sigma_2)} d\nu,
\end{aligned}
$$

and

$$
\begin{aligned}
L_4 &= \int_{\frac{1}{2}}^{1} \left(\nu^{\mu} - \frac{1}{3}\right) e^{h(\nu\sigma_1+(1-\nu)\sigma_2)} h'(\nu\sigma_1 + (1-\nu)\sigma_2) d\nu \qquad (4)\\
&= -\frac{2}{3(\sigma_2 - \sigma_1)} e^{h(\sigma_1)} + \frac{1}{\sigma_2 - \sigma_1}\left(\frac{1}{2^{\mu}} - \frac{1}{3}\right) e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)}\\
&\quad + \frac{\mu}{\sigma_2 - \sigma_1} \int_{\frac{1}{2}}^{1} \nu^{\mu-1} e^{h(\nu\sigma_1+(1-\nu)\sigma_2)} d\nu.
\end{aligned}
$$

By the equalities (1) and (2), we get

$$
\begin{aligned}
&(\sigma_2 - \sigma_1)(L_1 + L_2)\\
&= \frac{2}{3}(e^{h(\sigma_1)} + e^{h(\sigma_2)}) - \frac{1}{3} e^{h(\frac{\sigma_1+\sigma_2}{2})} - \mu \int_{0}^{\frac{1}{2}} \nu^{\mu-1} e^{h(\nu\sigma_2+(1-\nu)\sigma_1)} d\nu \qquad (5)\\
&= \frac{2}{3}(e^{h(\sigma_1)} + e^{h(\sigma_2)}) - \frac{1}{3} e^{h(\frac{\sigma_1+\sigma_2}{2})} - \frac{\mu}{(\sigma_2 - \sigma_1)^{\mu}} \int_{\sigma_1}^{\sigma_2} (\ell - \sigma_1)^{\mu-1} e^{(\ell)} d\ell\\
&= \frac{2}{3}(e^{h(\sigma_1)} + e^{h(\sigma_2)}) - \frac{1}{3} e^{h(\frac{\sigma_1+\sigma_2}{2})} - \frac{\Gamma(\mu+1)}{(\sigma_2 - \sigma_1)^{\mu}} K^{\mu}_{\sigma_1^+} e^{h(\sigma_2)}.
\end{aligned}
$$

By the equalities (3) and (4), we get

$$\begin{aligned}&(\sigma_2-\sigma_1)(L_3+L_4)\\&=-\frac{2}{3}(e^{h(\sigma_1)}+e^{h(\sigma_2)})+\frac{1}{3}e^{h(\frac{\sigma_1+\sigma_2}{2})}+\frac{\Gamma(\mu+1)}{(\sigma_2-\sigma_1)^{\mu}}K^{\mu}_{\sigma_2-}e^{h(\sigma_1)}\end{aligned}\tag{6}$$

The equalities (5) and (6) yields the following equality

$$\begin{aligned}\frac{\sigma_2-\sigma_1}{2}\left[L_1+L_2-L_3-L_4\right]&=\frac{1}{3}\left[2e^{h(\sigma_1)}-e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)}+2e^{h(\sigma_2)}\right]\\&-\frac{\Gamma(\mu+1)}{2(\sigma_2-\sigma_1)^{\mu}}\left[K^{\mu}_{\sigma_1+}e^{h(\sigma_2)}+K^{\mu}_{\sigma_2-}e^{h(\sigma_1)}\right].\end{aligned}$$

Thus, Lemma 1 is proved. □

4 Milne type inequalities for exponentially convex functions

This section is devoted to deriving some Milne type inequalities with the help of exponentially convex functions.

Theorem 1. *Let $h:[\sigma_1,\sigma_2]\to\mathbb{R}$ be a differentiable mapping on (σ_1,σ_2). If the function $|h|$ exibits exponentially-convex and $|h'|$ exhibits convexity over the interval $[\sigma_1,\sigma_2]$, then we acquire the following Milne type inequalities for Riemann-Liouville fractional integrals:*

$$\begin{aligned}&\left|\frac{1}{3}\left[2e^{h(\sigma_1)}-e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)}+2e^{h(\sigma_2)}\right]\right.\\&\qquad\left.-\frac{\Gamma(\mu+1)}{2(\sigma_2-\sigma_1)^{\mu}}\left[K^{\mu}_{\sigma_1+}e^{h(\sigma_2)}+K^{\mu}_{\sigma_2-}e^{h(\sigma_1)}\right]\right|\\&\le\frac{\sigma_2-\sigma_1}{2}(\kappa_1(\mu)-2\kappa_2(\mu)+2\kappa_3(\mu)+\kappa_4(\mu)-2\kappa_5(\mu)+2\kappa_6(\mu))\\&\quad\times(|e^{h(\sigma_2)}||h'(\sigma_2)|)+2(\kappa_2(\mu)-\kappa_3(\mu)+\kappa_5(\mu)-\kappa_6(\mu))\\&\quad\times(|e^{h(\sigma_1)}||h'(\sigma_2)|+|e^{h(\sigma_2)}||h'(\sigma_1)|)+(\kappa_1(\mu)-2\kappa_2(\mu)+2\kappa_3\\&\quad+\kappa_4(\mu)-2\kappa_5(\mu)+2\kappa_6(\mu))(|e^{h(\sigma_1)}||h'(\sigma_1)|).\end{aligned}$$

Proof. By taking modulus in Lemma 1, we have

$$\begin{aligned}&\left|\frac{1}{3}\left[2e^{h(\sigma_1)}-e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)}+2e^{h(\sigma_2)}\right]\right.\\&\qquad\left.-\frac{\Gamma(\mu+1)}{2(\sigma_2-\sigma_1)^{\mu}}\left[K^{\mu}_{\sigma_1+}e^{h(\sigma_2)}+K^{\mu}_{\sigma_2-}e^{h(\sigma_1)}\right]\right|\end{aligned}$$

$$\leq \frac{\sigma_2-\sigma_1}{2}(|L_1|+|L_2|+|L_3|+|L_4|). \tag{7}$$

By using exponentially convexity of $|h|$ and convexity of $|h'|$, we have

$$\begin{aligned}
|L_1| &\leq \int_0^{\frac{1}{2}} \left|\left(\nu^\mu-\frac{2}{3}\right)\right| |e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|d\nu \qquad (8)\\
&\leq \int_0^{\frac{1}{2}} \left|\left(\nu^\mu-\frac{2}{3}\right)\right| \left[\left(\nu|e^{h(\sigma_2)}|+(1-\nu)|e^{h(\sigma_1)}|\right)\right.\\
&\qquad \left.\times\left(\nu|h'(\sigma_2)|+(1-\nu)|h'(\sigma_1)|\right)\right]d\nu\\
&\leq \int_0^{\frac{1}{2}} \left|\left(\nu^\mu-\frac{2}{3}\right)\right| \left(\nu^2|e^{h(\sigma_2)}||h'(\sigma_2)|+\nu(1-\nu)(|e^{h(\sigma_1)}||h'(\sigma_2)|\right.\\
&\quad \left.+|e^{h(\sigma_2)}||h'(\sigma_1)|)+(1-\nu)^2|e^{h(\sigma_1)}||h'(\sigma_1)|\right)d\nu\\
&\leq \kappa_3(\mu)|e^{h(\sigma_2)}||h'(\sigma_2)|+(\kappa_2(\mu)-\kappa_3(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2)\\
&\quad +|e^{h(\sigma_2)}||h'(\sigma_1)|)+(\kappa_1(\mu)-2\kappa_2(\mu)+\kappa_3(\mu))|e^{h(\sigma_1)}||h'(\sigma_1)|,
\end{aligned}$$

and, similarly,

$$\begin{aligned}
|L_3| &\leq \int_0^{\frac{1}{2}} \left|\left(\nu^\mu-\frac{2}{3}\right)\right| \left|e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1+(1-\nu)\sigma_2)\right|d\nu \qquad (9)\\
&\leq \kappa_3(\mu)|e^{h(\sigma_1)}||h'(\sigma_1)|+(\kappa_2(\mu)-\kappa_3(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2)\\
&\quad +|e^{h(\sigma_2)}||h'(\sigma_1)|)+(\kappa_1(\mu)-2\kappa_2(\mu)+\kappa_3(\mu))|e^{h(\sigma_2)}||h'(\sigma_2)|.
\end{aligned}$$

Again, by using exponentially convexity of $|h|$ and convexity of $|h'|$, we have

$$\begin{aligned}
|L_2| &\leq \int_{\frac{1}{2}}^1 \left|\left(\nu^\mu-\frac{1}{3}\right)\right| |e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|d\nu \qquad (10)\\
&\leq \kappa_6(\mu)|e^{h(\sigma_2)}||h'(\sigma_2)|+(\kappa_5(\mu)-\kappa_6(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2)\\
&\quad +|e^{h(\sigma_2)}||h'(\sigma_1)|)+(\kappa_4(\mu)-2\kappa_5(\mu)+\kappa_6(\mu))|e^{h(\sigma_1)}||h'(\sigma_1)|,
\end{aligned}$$

and

$$\begin{aligned}
|L_4 &\leq \int_{\frac{1}{2}}^1 \left|\left(\nu^\mu-\frac{1}{3}\right)\right| |e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1+(1-\nu)\sigma_2)|d\nu \qquad (11)\\
&\leq \kappa_6(\mu)|e^{h(\sigma_1)}||h'(\sigma_1)|+(\kappa_5(\mu)-\kappa_6(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2)\\
&\quad +|e^{h(\sigma_2)}||h'(\sigma_1)|)+(\kappa_4(\mu)-2\kappa_5(\mu)+\kappa_6(\mu))|e^{h(\sigma_2)}||h'(\sigma_2)|.
\end{aligned}$$

By substituting the inequalities (8)-(11) in (7), then we obtain the required result. Thus, the required result is proved. □

Corollary 1. *If $\mu = 1$, then Theorem 1 reduces to the following inequality*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2e^{h(\sigma_1)} - e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)} + 2e^{h(\sigma_2)}\right] - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} e^{h(\nu)}d\nu\right| \\
&\quad\leq \frac{\sigma_2-\sigma_1}{288}\Bigg\{43(|e^{h(\sigma_2)}||h'(\sigma_2)|) \\
&\qquad + 17(|e^{h(\sigma_1)}||h'(\sigma_2)| + |e^{h(\sigma_2)}||h'(\sigma_1)|) \\
&\qquad + 43(|e^{h(\sigma_1)}||h'(\sigma_1)|)\Bigg\}.
\end{aligned}$$

Theorem 2. *Let $h : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a differentiable mapping on (σ_1, σ_2). If the function $|h|$ exibits exp-convex and $|h'|^q$, $q > 1$ exhibits convexity over the interval $[\sigma_1, \sigma_2]$, then we acquire the following Milne type inequalities for Riemann-Liouville fractional integrals:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2e^{h(\sigma_1)} - e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)} + 2e^{h(\sigma_2)}\right]\right. \\
&\qquad\left. - \frac{\Gamma(\mu+1)}{2(\sigma_2-\sigma_1)^{\mu}}\left[K^{\mu}_{\sigma_1^+}e^{h(\sigma_2)} + K^{\mu}_{\sigma_2^-}e^{h(\sigma_1)}\right]\right| \\
&\leq \frac{\sigma_2-\sigma_1}{2}\Bigg\{\left(\int_0^{\frac{1}{2}}\left|\nu^{\mu} - \frac{2}{3}\right|^p d\nu\right)^{\frac{1}{p}} \times \left(\left[\frac{1}{24}\left(|e^{h(\sigma_2)}h'(\sigma_2)|^q\right.\right.\right. \\
&\quad + 2(|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + 7|e^{h(\sigma_1)}h'(\sigma_1)|^q\Big)\Big]^{\frac{1}{q}} \\
&\quad + \left[\frac{1}{24}\left(|e^{h(\sigma_1)}h'(\sigma_1)|^q + 2(|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q)\right.\right. \\
&\quad \left.\left.\left. + 7|e^{h(\sigma_2)}h'(\sigma_2)|^q\right)\right]^{\frac{1}{q}}\right) \\
&\quad + \left(\int_{\frac{1}{2}}^1\left|\nu^{\mu} - \frac{1}{3}\right|^p d\nu\right)^{\frac{1}{p}} \times \left(\left[\frac{1}{24}\left(7|e^{h(\sigma_2)}h'(\sigma_2)|^q\right.\right.\right. \\
&\quad + 2(|e^{h(\sigma_1)}h'(\sigma_1)|^q + |e^{h(\sigma_1)}h'(\sigma_2)|^q) + |e^{h(\sigma_1)}h'(\sigma_1)|^q\Big)\Big]^{\frac{1}{q}} \\
&\quad + \left[\frac{1}{24}\left(7|e^{h(\sigma_1)}h'(\sigma_1)|^q + 2(|e^{h(\sigma_1)}h'(\sigma_1)|^q\right.\right.
\end{aligned}$$

$$+|e^{h(\sigma_1)}h'(\sigma_2)|^q)+|e^{h(\sigma_2)}h'(\sigma_2)|^q\Big)\Big]^{\frac{1}{q}}\Big)\Big\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Hölder's inequality and using exponentially convexity of $|h|$ and convexity of $|h'|$, we have

$$\begin{aligned}
|L_1| &\le \int_0^{\frac{1}{2}}\left|\left(\nu^\mu-\frac{2}{3}\right)\right||e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|d\nu \qquad (12)\\
&\le \left(\int_0^{\frac{1}{2}}\left|\nu^\mu-\frac{2}{3}\right|^p d\nu\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}|e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|^q d\nu\right)^{\frac{1}{q}}\\
&\le \left(\int_0^{\frac{1}{2}}\left|\nu^\mu-\frac{2}{3}\right|^p d\nu\right)^{\frac{1}{p}}\left[\int_0^{\frac{1}{2}}\left(\nu|e^{h(\sigma_2)}|^q+(1-\nu)|e^{h(\sigma_1)}|^q\right)\right.\\
&\quad\left.\times\left(\nu|h'(\sigma_2)|^q+(1-\nu)|h'(\sigma_1)|^q\right)d\nu\right]^{\frac{1}{q}}\\
&\le \left(\int_0^{\frac{1}{2}}\left|\nu^\mu-\frac{2}{3}\right|^p d\nu\right)^{\frac{1}{p}}\left[\int_0^{\frac{1}{2}}\left(\nu^2|e^{h(\sigma_2)}h'(\sigma_2)|^q+\nu(\nu-1)\right.\right.\\
&\quad\left.\left.\times(|e^{h(\sigma_1)}h'(\sigma_2)|^q+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+(1-\nu)^2|e^{h(\sigma_1)}h'(\sigma_1)|^q\right)d\nu\right]^{\frac{1}{q}}\\
&\le \left(\int_0^{\frac{1}{2}}\left|\nu^\mu-\frac{2}{3}\right|^p d\nu\right)^{\frac{1}{p}}\left[\frac{1}{24}\left(|e^{h(\sigma_2)}h'(\sigma_2)|^q\right.\right.\\
&\quad\left.\left.+2(|e^{h(\sigma_1)}h'(\sigma_2)|^q+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+7|e^{h(\sigma_1)}h'(\sigma_1)|^q\right)\right]^{\frac{1}{q}}.
\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}
|L_2| &\le \left(\int_{\frac{1}{2}}^1\left|\nu^\mu-\frac{1}{3}\right|^p d\nu\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{\frac{1}{2}}^1|e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|^q d\nu\right)^{\frac{1}{q}} \qquad (13)\\
&\le \left(\int_{\frac{1}{2}}^1\left|\nu^\mu-\frac{1}{3}\right|^p d\nu\right)^{\frac{1}{p}}\left[\frac{1}{24}\left(7|e^{h(\sigma_2)}h'(\sigma_2)|^q\right.\right.
\end{aligned}$$

$$+ 2(|e^{h(\sigma_1)}h'(\sigma_1)|^q + |e^{h(\sigma_1)}h'(\sigma_2)|^q) + |e^{h(\sigma_1)}h'(\sigma_1)|^q\Bigg)\Bigg]^{\frac{1}{q}},$$

$$\begin{aligned}
|L_3| &\leq \left(\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right|^p\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^{\frac{1}{2}} |e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1+(1-\nu)\sigma_2)|^q d\nu\right)^{\frac{1}{q}} \qquad (14)\\
&\leq \left(\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right|^p d\nu\right)^{\frac{1}{p}} \left[\frac{1}{24}\Bigg(|e^{h(\sigma_1)}h'(\sigma_1)|^q \right.\\
&\quad \left. + 2(|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + 7|e^{h(\sigma_2)}h'(\sigma_2)|^q\Bigg)\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
L_4 &\leq \left(\int_{\frac{1}{2}}^1 |\nu^\mu - \frac{1}{3}d\nu|^p\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{\frac{1}{2}}^1 |e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1+(1-\nu)\sigma_2)|^q d\nu\right)^{\frac{1}{q}} \qquad (15)\\
&\leq \left(\int_{\frac{1}{2}}^1 \left|\nu^\mu - \frac{1}{3}\right|^p d\nu\right)^{\frac{1}{p}} \left[\frac{1}{24}\Bigg(7|e^{h(\sigma_1)}h'(\sigma_1)|^q \right.\\
&\quad \left. + 2(|e^{h(\sigma_1)}h'(\sigma_1)|^q + |e^{h(\sigma_1)}h'(\sigma_2)|^q) + |e^{h(\sigma_2)}h'(\sigma_2)|^q\Bigg)\right]^{\frac{1}{q}}.
\end{aligned}$$

Now, if we put the value of (12)-(15) in (7), then we obtain the required above result. Thus, the required result is proved.

□

Corollary 2. *If $\mu = 1$, then Theorem 2 reduces to the following inequality*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2e^{h(\sigma_1)} - e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)} + 2e^{h(\sigma_2)}\right] - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} e^{h(\nu)}d\nu\right| \\
&\quad \leq \frac{\sigma_2-\sigma_1}{2}\left(\frac{4^{p+1}-1}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} \times \Bigg\{\Bigg(\Bigg[\frac{1}{24}\Bigg(|e^{h(\sigma_2)}h'(\sigma_2)|^q \\
&\quad + 2(|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + 7|e^{h(\sigma_1)}h'(\sigma_1)|^q\Bigg)\Bigg]^{\frac{1}{q}} \\
&\quad + \left[\frac{1}{24}\Bigg(|e^{h(\sigma_1)}h'(\sigma_1)|^q + 2(|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q)\right.
\end{aligned}$$

$$+ 7|e^{h(\sigma_2)}h'(\sigma_2)|^q\Big)\Big]^{\frac{1}{q}}\Big) + \Big(\Big[\frac{1}{24}\Big(7|e^{h(\sigma_2)}h'(\sigma_2)|^q$$
$$+ 2(|e^{h(\sigma_1)}h'(\sigma_1)|^q + |e^{h(\sigma_1)}h'(\sigma_2)|^q) + |e^{h(\sigma_1)}h'(\sigma_1)|^q\Big)\Big]^{\frac{1}{q}}$$
$$+ \Big[\frac{1}{24}\Big(7|e^{h(\sigma_1)}h'(\sigma_1)|^q + 2(|e^{h(\sigma_1)}h'(\sigma_1)|^q + |e^{h(\sigma_1)}h'(\sigma_2)|^q)$$
$$+ |e^{h(\sigma_2)}h'(\sigma_2)|^q\Big)\Big]^{\frac{1}{q}}\Big)\Big\}.$$

Theorem 3. *Let $h : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a differentiable mapping on (σ_1, σ_2). If the function $|h|$ exibits exponentially-convex and $|h'|^q$, $q > 1$ exhibits convexity over the interval $[\sigma_1, \sigma_2]$, then we acquire the following Milne type inequalities for Riemann-Liouville fractional integrals:*

$$\Big|\frac{1}{3}\Big[2e^{h(\sigma_1)} - e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)} + 2e^{h(\sigma_2)}\Big]$$
$$- \frac{\Gamma(\mu+1)}{2(\sigma_2-\sigma_1)^\mu}\left[K^\mu_{\sigma_1{}^+}e^{h(\sigma_2)} + K^\mu_{\sigma_2{}^-}e^{h(\sigma_1)}\right]\Big|$$
$$\leq \frac{\sigma_2-\sigma_1}{2}\Big\{(\kappa_1(\mu))^{1-\frac{1}{q}}\Big[\kappa_3(\mu)|e^{h(\sigma_2)}h'(\sigma_2)|^q$$
$$+ (\kappa_2(\mu)-\kappa_3(\mu))(|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q)$$
$$+ (\kappa_1(\mu) - 2\kappa_2(\mu) + \kappa_3(\mu))|e^{h(\sigma_1)}h'(\sigma_1)|^q\Big]^{\frac{1}{q}}$$
$$+ (\kappa_4(\mu))^{1-\frac{1}{q}}\Big[\kappa_6(\mu)|e^{h(\sigma_2)}h'(\sigma_2)|^q + (\kappa_5(\mu)-\kappa_6(\mu))$$
$$\times (|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q)$$
$$+ (\kappa_4(\mu) - 2\kappa_5(\mu) + \kappa_6(\mu))|e^{h(\sigma_1)}h'(\sigma_1)|^q\Big]^{\frac{1}{q}}$$
$$+ (\kappa_1(\mu))^{1-\frac{1}{q}}\Big[\kappa_3(\mu)|e^{h(\sigma_1)}h'(\sigma_1)|^q + (\kappa_2(\mu)-\kappa_3(\mu))$$
$$\times (|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q)$$
$$+ (\kappa_1(\mu) - 2\kappa_2(\mu) + \kappa_3(\mu))|e^{h(\sigma_2)}h'(\sigma_2)|^q\Big]^{\frac{1}{q}}$$
$$+ (\kappa_4(\mu))^{1-\frac{1}{q}}\Big[\kappa_6(\mu)|e^{h(\sigma_1)}h'(\sigma_1)|^q + (\kappa_5(\mu)-\kappa_6(\mu))$$

$$\times (|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + (\kappa_4(\mu) - 2\kappa_5(\mu) + \kappa_6(\mu))|e^{h(\sigma_2)}h'(\sigma_2)|^q\Bigg]^{\frac{1}{q}}\Bigg\}.$$

Proof. By applying the inequality of power-mean using exponentially convexity of $|h|$ and convexity of $|h'|$, we have

$$\begin{aligned}
|L_1| &\leq \int_0^{\frac{1}{2}} \left|\left(\nu^\mu - \frac{2}{3}\right)\right| |e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2 + (1-\nu)\sigma_1)|d\nu \qquad (16)\\
&\leq \left(\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right|^q d\nu\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \left|\left(\nu^\mu - \frac{2}{3}\right)\right| \right.\\
&\quad \left. \times \left|e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2 + (1-\nu)\sigma_1)|^q d\nu\right)^{\frac{1}{q}}\right.\\
&\leq (\kappa_1(\mu))^{1-\frac{1}{q}} \left[\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right| \left\{\left(\nu|e^{h(\sigma_2)}|^q + (1-\nu)|e^{h(\sigma_1)}|^q\right)\right.\right.\\
&\quad \left.\left. \times \left(\nu|h'(\sigma_2)|^q + (1-\nu)|h'(\sigma_1)|^q d\nu\right)\right\}\right]^{\frac{1}{q}}\\
&\leq (\kappa_1(\mu))^{1-\frac{1}{q}}\\
&\quad \times \left[\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right| \left(\nu^2|e^{h(\sigma_2)}h'(\sigma_2)|^q + \nu(\nu-1)(|e^{h(\sigma_1)}h'(\sigma_2)|^q\right.\right.\\
&\quad \left.\left. + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + (1-\nu)^2|e^{h(\sigma_1)}h'(\sigma_1)|^q\right) d\nu\right]^{\frac{1}{q}}\\
&\leq (\kappa_1(\mu))^{1-\frac{1}{q}}\\
&\quad \times \left[\kappa_3(\mu)|e^{h(\sigma_2)}h'(\sigma_2)|^q + (\kappa_2(\mu) - \kappa_3(\mu))(|e^{h(\sigma_1)}h'(\sigma_2)|^q\right.\\
&\quad \left. + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + (\kappa_1(\mu) - 2\kappa_2(\mu) + \kappa_3(\mu))|e^{h(\sigma_1)}h'(\sigma_1)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
|L_2| &\leq \int_{\frac{1}{2}}^1 \left|\left(\nu^\mu - \frac{1}{3}\right)\right| |e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2 + (1-\nu)\sigma_1)|d\nu \qquad (17)\\
&\leq (\kappa_4(\mu))^{1-\frac{1}{q}}\\
&\quad \times \left[\kappa_6(\mu)|e^{h(\sigma_2)}h'(\sigma_2)|^q + (\kappa_5(\mu) - \kappa_6(\mu))(|e^{h(\sigma_1)}h'(\sigma_2)|^q\right.
\end{aligned}$$

$$+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+(\kappa_4(\mu)-2\kappa_5(\mu)+\kappa_6(\mu))|e^{h(\sigma_1)}h'(\sigma_1)|^q\Bigg]^{\frac{1}{q}},$$

$$|L_3|\leq\int_0^{\frac{1}{2}}\left|\left(\nu^\mu-\frac{2}{3}\right)\right||e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1+(1-\nu)\sigma_2)|d\nu \tag{18}$$

$$\leq(\kappa_1(\mu))^{1-\frac{1}{q}}$$

$$\times\Bigg[\kappa_3(\mu)|e^{h(\sigma_1)}h'(\sigma_1)|^q+(\kappa_2(\mu)-\kappa_3(\mu))(|e^{h(\sigma_1)}h'(\sigma_2)|^q$$

$$+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+(\kappa_1(\mu)-2\kappa_2(\mu)+\kappa_3(\mu))|e^{h(\sigma_2)}h'(\sigma_2)|^q\Bigg]^{\frac{1}{q}},$$

$$|L_4|\leq\int_{\frac{1}{2}}^1\left|\left(\nu^\mu-\frac{1}{3}\right)\right||e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1+(1-\nu)\sigma_2)|d\nu \tag{19}$$

$$\leq(\kappa_4(\mu))^{1-\frac{1}{q}}$$

$$\times\Bigg[\kappa_6(\mu)|e^{h(\sigma_1)}h'(\sigma_1)|^q+(\kappa_5(\mu)-\kappa_6(\mu))(|e^{h(\sigma_1)}h'(\sigma_2)|^q$$

$$+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+(\kappa_4(\mu)-2\kappa_5(\mu)+\kappa_6(\mu))|e^{h(\sigma_2)}h'(\sigma_2)|^q\Bigg]^{\frac{1}{q}}.$$

Now, if we put the value of (16)-(19) in (7), then we obtain the required above result. Thus, the required result is proved. □

Corollary 3. *If $\mu=1$, then Theorem 3 reduces to the following inequality*

$$\left|\frac{1}{3}\left[2e^{h(\sigma_1)}-e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)}+2e^{h(\sigma_2)}\right]-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}e^{h(\nu)}d\nu\right|$$

$$\leq\frac{\sigma_2-\sigma_1}{2}\left(\frac{5^{1-\frac{1}{q}}}{24^{1+\frac{1}{q}}}\right)\times\Bigg\{\Bigg[7(\mu)|e^{h(\sigma_2)}h'(\sigma_2)|^q$$

$$+17(|e^{h(\sigma_1)}h'(\sigma_2)|^q+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+79|e^{h(\sigma_1)}h'(\sigma_1)|^q\Bigg]^{\frac{1}{q}}$$

$$+\Bigg[79|e^{h(\sigma_2)}h'(\sigma_2)|^q+17(|e^{h(\sigma_1)}h'(\sigma_2)|^q+|e^{h(\sigma_2)}h'(\sigma_1)|^q)$$

$$+79|e^{h(\sigma_1)}h'(\sigma_1)|^q\Bigg]^{\frac{1}{q}}+\Bigg[7|e^{h(\sigma_1)}h'(\sigma_1)|^q+17(|e^{h(\sigma_1)}h'(\sigma_2)|^q$$

$$+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+79|e^{h(\sigma_2)}h'(\sigma_2)|^q\Bigg]^{\frac{1}{q}}+\Bigg[79|e^{h(\sigma_1)}h'(\sigma_1)|^q$$

$$+ 7(|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + 79|e^{h(\sigma_2)}h'(\sigma_2)|^q\Big]^{\frac{1}{q}}\Bigg\}.$$

5 Milne type inequalities for strong exponentially convex functions

This section is devoted to deriving some Milne type inequalities with the help of strong exponentially convex functions.

Theorem 4. *Let $h : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a differentiable mapping on (σ_1, σ_2). If the function $|h|$ exibits strongly exponentially-convex and $|h'|$ exhibits strongly convexity over the interval $[\sigma_1, \sigma_2]$, then we acquire the following Milne type inequalities for Riemann-Liouville fractional integrals:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2e^{h(\sigma_1)} - e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)} + 2e^{h(\sigma_2)}\right]\right.\\
&\qquad\left. - \frac{\Gamma(\mu+1)}{2(\sigma_2-\sigma_1)^{\mu}}\left[K^{\mu}_{\sigma_1{}^+}e^{h(\sigma_2)} + K^{\mu}_{\sigma_2{}^-}e^{h(\sigma_1)}\right]\right|\\
&\leq \frac{\sigma_2-\sigma_1}{2}\Bigg[\kappa_3(\mu)(|e^{h(\sigma_2)}||h'(\sigma_2)| + |e^{h(\sigma_1)}||h'(\sigma_1)|)\\
&+ 2(\kappa_2(\mu) - \kappa_3(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2) + |e^{h(\sigma_2)}||h'(\sigma_1)|)\\
&+ (\kappa_1(\mu) - 2\kappa_2(\mu) + \kappa_3(\mu))(|e^{h(\sigma_1)}||h'(\sigma_1)|\\
&+ |e^{h(\sigma_2)}||h'(\sigma_2)|) - c_2||\sigma_2 - \sigma_1||^{\ell_2}((\kappa_3(\mu) - \kappa_3^*(\mu))\\
&\times (|e^{h(\sigma_2)}| + |e^{h(\sigma_1)}|) + (\kappa_2(\mu)\\
&- 2\kappa_3(\mu) + \kappa_3^*(\mu))(|e^{h(\sigma_1)}| + |e^{h(\sigma_2)}|))\\
&- c_1||\sigma_2 - \sigma_1||^{\ell_1}((\kappa_3(\mu) - \kappa_3^*(\mu))(|h'(\sigma_2)| + h'(\sigma_1))\\
&+ (\kappa_2(\mu) - 2\kappa_3(\mu) + \kappa_3^*(\mu))(|h'(\sigma_1)| + h'(\sigma_2)))\\
&+ 2c_1c_2||\sigma_2 - \sigma_1||^{\ell_2+\ell_1}(\kappa_3(\mu) - 2\kappa_3^*(\mu) + \kappa_3^{**}(\mu))\\
&+ \kappa_6(\mu)(|e^{h(\sigma_2)}||h'(\sigma_2)| + |e^{h(\sigma_1)}||h'(\sigma_1)|) + 2(\kappa_5(\mu)\\
&- \kappa_6(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2) + |e^{h(\sigma_2)}||h'(\sigma_1)|)\\
&+ (\kappa_4(\mu) - 2\kappa_5(\mu) + \kappa_6(\mu))|e^{h(\sigma_1)}||h'(\sigma_1)|\\
&- c_2||\sigma_2 - \sigma_1||^{\ell_2}((\kappa_6(\mu) - \kappa_6^*(\mu))(|e^{h(\sigma_2)}| + |e^{h(\sigma_1)}|)\\
&+ (\kappa_4(\mu) - 2\kappa_6(\mu) + \kappa_6^*(\mu))(|e^{h(\sigma_1)}| + |e^{h(\sigma_2)}|)\\
&- c_1||\sigma_2 - \sigma_1||^{\ell_1}((\kappa_6(\mu) - \kappa_6^*(\mu))(|h'(\sigma_2)| + |h'(\sigma_1)|)
\end{aligned}$$

$$+ (\kappa_4(\mu) - 2\kappa_6(\mu) + \kappa_6^*(\mu))|h'(\sigma_1)|)$$
$$+ 2c_1c_2||\sigma_2 - \sigma_1||^{\ell_2+\ell_1}(\kappa_6(\mu) - 2\kappa_6^*(\mu) + \kappa_6^{**}(\mu))\Bigg].$$

Proof. We have:

$$\begin{aligned}
|L_1| &\leq \int_0^{\frac{1}{2}} \left|\left(\nu^\mu - \frac{2}{3}\right)\right| |e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2 + (1-\nu)\sigma_1)|d\nu \qquad (20)\\
&\leq \int_0^{\frac{1}{2}} \left|\left(\nu^\mu - \frac{2}{3}\right)\right|\\
&\quad \times \left[\left(\nu|e^{h(\sigma_2)}| + (1-\nu)|e^{h(\sigma_1)}| - c_1'\nu(1-\nu)||\sigma_2 - \sigma_1||^{\ell_1}\right)\right.\\
&\quad \left.\times \left(\nu|h'(\sigma_2)| + (1-\nu)|h'(\sigma_1)| - c_2'\nu(1-\nu)||\sigma_2 - \sigma_1||^{\ell_2}\right)\right]d\nu\\
&\leq \int_0^{\frac{1}{2}} \left|\left(\nu^\mu - \frac{2}{3}\right)\right|\left(\nu^2|e^{h(\sigma_2)}||h'(\sigma_2)|\right.\\
&\quad + \nu(1-\nu)(|e^{h(\sigma_1)}||h'(\sigma_2)| + |e^{h(\sigma_2)}||h'(\sigma_1)|) + (1-\nu)^2\\
&\quad \times |e^{h(\sigma_1)}||h'(\sigma_1)| - c_2||\sigma_2 - \sigma_1||^{\ell_2}((\nu^2 - \nu^3)|e^{h(\sigma_2)}|\\
&\quad + (\nu - 2\nu^2 + \nu^3)|e^{h(\sigma_1)}|) - c_1||\sigma_2 - \sigma_1||^{\ell_1}((\nu^2 - \nu^3)|h'(\sigma_2)|\\
&\quad + (\nu - 2\nu^2 + \nu^3)|h'(\sigma_1)|)\\
&\quad \left. + c_1c_2||\sigma_2 - \sigma_1||^{\ell_2+\ell_1}(\nu^2 - 2v^3 + \nu^4)\right)d\nu\\
&\leq \kappa_3(\mu)|e^{h(\sigma_2)}||h'(\sigma_2)| + (\kappa_2(\mu) - \kappa_3(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2)\\
&\quad + |e^{h(\sigma_2)}||h'(\sigma_1)|) + (\kappa_1(\mu) - 2\kappa_2(\mu) + \kappa_3(\mu))|e^{h(\sigma_1)}||h'(\sigma_1)|\\
&\quad - c_2||\sigma_2 - \sigma_1||^{\ell_2}((\kappa_3(\mu) - \kappa_3^*(\mu))|e^{h(\sigma_2)}| + (\kappa_2(\mu) - 2\kappa_3(\mu)\\
&\quad + \kappa_3^*(\mu))|e^{h(\sigma_1)}|) - c_1||\sigma_2 - \sigma_1||^{\ell_1}((\kappa_3(\mu) - \kappa_3^*(\mu))|h'(\sigma_2)|\\
&\quad + (\kappa_2(\mu) - 2\kappa_3(\mu) + \kappa_3^*(\mu))|h'(\sigma_1)|)\\
&\quad + c_1c_2||\sigma_2 - \sigma_1||^{\ell_2+\ell_1}(\kappa_3(\mu) - 2\kappa_3^*(\mu) + \kappa_3^{**}(\mu)),
\end{aligned}$$

and, similarly,

$$\begin{aligned}
|L_3| &\leq \int_0^{\frac{1}{2}} \left|\left(\nu^\mu - \frac{2}{3}\right)\right|\left|e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1 + (1-\nu)\sigma_2)\right|d\nu \qquad (21)\\
&\leq \kappa_3(\mu)|e^{h(\sigma_1)}||h'(\sigma_1)| + (\kappa_2(\mu) - \kappa_3(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2)\\
&\quad + |e^{h(\sigma_2)}||h'(\sigma_1)|) + (\kappa_1(\mu) - 2\kappa_2(\mu) + \kappa_3(\mu))|e^{h(\sigma_2)}||h'(\sigma_2)|
\end{aligned}$$

$$
\begin{aligned}
&- c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_3(\mu)-\kappa_3^*(\mu))|e^{h(\sigma_1)}|+(\kappa_2(\mu)-2\kappa_3(\mu)\\
&+\kappa_3^*(\mu))|e^{h(\sigma_2)}|)-c_1||\sigma_2-\sigma_1||^{\ell_1}((\kappa_3(\mu)-\kappa_3^*(\mu))|h'(\sigma_1)|\\
&+(\kappa_2(\mu)-2\kappa_3(\mu)+\kappa_3^*(\mu))|h'(\sigma_2)|)\\
&+c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}(\kappa_3(\mu)-2\kappa_3^*(\mu)+\kappa_3^{**}(\mu)).
\end{aligned}
$$

Again, by using strongly exponentially convexity of $|h|$ and strongly convexity of $|h'|$, we have

$$
\begin{aligned}
|L_2| &\leq \int_{\frac{1}{2}}^{1}\left|\left(\nu^\mu-\frac{1}{3}\right)\right||e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|d\nu \qquad (22)\\
&\leq \kappa_6|e^{h(\sigma_2)}||h'(\sigma_2)|+(\kappa_5(\mu)-\kappa_6(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2)\\
&\quad+|e^{h(\sigma_2)}||h'(\sigma_1)|)+(\kappa_4(\mu)-2\kappa_5(\mu)+\kappa_6(\mu))|e^{h(\sigma_1)}||h'(\sigma_1)|\\
&\quad-c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_6(\mu)-\kappa_6^*(\mu))|e^{h(\sigma_2)}|+(\kappa_4(\mu)-2\kappa_6(\mu)\\
&\quad+\kappa_6^*(\mu))|e^{h(\sigma_1)}|)-c_1||\sigma_2-\sigma_1||^{\ell_1}((\kappa_6(\mu)-\kappa_6^*(\mu))|h'(\sigma_2)|\\
&\quad+(\kappa_4(\mu)-2\kappa_6(\mu)+\kappa_6^*(\mu))|h'(\sigma_1)|)\\
&\quad+c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}(\kappa_6(\mu)-2\kappa_6^*(\mu)+\kappa_6^{**}(\mu)),
\end{aligned}
$$

and

$$
\begin{aligned}
|L_4 &\leq \int_{\frac{1}{2}}^{1}\left|\left(\nu^\mu-\frac{1}{3}\right)\right||e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1+(1-\nu)\sigma_2)|d\nu \qquad (23)\\
&\leq \kappa_6|e^{h(\sigma_1)}||h'(\sigma_1)|+(\kappa_5(\mu)-\kappa_6(\mu))(|e^{h(\sigma_1)}||h'(\sigma_2)\\
&\quad+|e^{h(\sigma_2)}||h'(\sigma_1)|)+(\kappa_4(\mu)-2\kappa_5(\mu)+\kappa_6(\mu))|e^{h(\sigma_2)}||h'(\sigma_2)|\\
&\quad-c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_6(\mu)-\kappa_6^*(\mu))|e^{h(\sigma_1)}|+(\kappa_4(\mu)-2\kappa_6(\mu)\\
&\quad+\kappa_6^*(\mu))|e^{h(\sigma_2)}|)-c_1||\sigma_2-\sigma_1||^{\ell_1}((\kappa_6(\mu)-\kappa_6^*(\mu))|h'(\sigma_1)|\\
&\quad+(\kappa_4(\mu)-2\kappa_6(\mu)+\kappa_6^*(\mu))|h'(\sigma_2)|)\\
&\quad+c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}(\kappa_6(\mu)-2\kappa_6^*(\mu)+\kappa_6^{**}(\mu)).
\end{aligned}
$$

Now, if we put the value of (20)-(23) in (7), then we obtain the required above result. Thus, the required result is proved. □

Theorem 5. *Let $h:[\sigma_1,\sigma_2]\to\mathbb{R}$ be a differentiable mapping on (σ_1,σ_2). If the function $|h|$ exibits strongly exponentially-convex and $|h'|^q$, $q>1$ exhibits strongly convexity over the interval $[\sigma_1,\sigma_2]$. Then we acquire the following Milne type inequalities for Riemann-Liouville fractional integrals:*

$$
\left|\frac{1}{3}\left[2e^{h(\sigma_1)}-e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)}+2e^{h(\sigma_2)}\right]\right.
$$

$$
\begin{aligned}
&- \frac{\Gamma(\mu+1)}{2(\sigma_2-\sigma_1)^{\mu}}\left[K^{\mu}_{\sigma_1^+}e^{h(\sigma_2)}+K^{\mu}_{\sigma_2^-}e^{h(\sigma_1)}\right]\Bigg| \\
&\leq \frac{\sigma_2-\sigma_1}{2} \\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left|\nu^{\mu}-\frac{2}{3}\right|^{p}d\nu\right)^{\frac{1}{p}}\left(\left[\frac{1}{24}\left((|e^{h(\sigma_2)}h'(\sigma_2)|^{q}+|e^{h(\sigma_1)}h'(\sigma_1)|^{q}\right)\right.\right.\right. \\
&+4(|e^{h(\sigma_1)}h'(\sigma_2)|^{q}+|e^{h(\sigma_2)}h'(\sigma_1)|^{q})+7(|e^{h(\sigma_1)}h'(\sigma_1)|^{q} \\
&\left.+|e^{h(\sigma_2)}h'(\sigma_2)|^{q})\right)-\frac{16}{960}\left(5c_2||\sigma_2-\sigma_1||^{\ell_2}(25|e^{h(\sigma_2)}| \right. \\
&+55|e^{h(\sigma_1)}|)+5c_1||\sigma_2-\sigma_1||^{\ell_1}(25|h'(\sigma_2)|+55|h'(\sigma_1)|) \\
&\left.\left.\left.-2c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}\right)\right]^{\frac{1}{q}}\right)+\left(\int_{\frac{1}{2}}^{1}\left|\nu^{\mu}-\frac{1}{3}\right|^{p}d\nu\right)^{\frac{1}{p}} \\
&\times\left[\frac{1}{24}\left(7(|e^{h(\sigma_2)}h'(\sigma_2)+e^{h(\sigma_1)}h'(\sigma_1)|^{q})+4(|e^{h(\sigma_1)}h'(\sigma_1)|^{q}\right.\right. \\
&\left.+|e^{h(\sigma_1)}h'(\sigma_2)|^{q})+(|e^{h(\sigma_1)}h'(\sigma_1)|^{q}+e^{h(\sigma_2)}h'(\sigma_2))\right) \\
&-\frac{16}{960}\left(5c_2||\sigma_2-\sigma_1||^{\ell_2}(55|e^{h(\sigma_2)}|+25|e^{h(\sigma_1)}|)\right. \\
&+5c_1||\sigma_2-\sigma_1||^{\ell_1}(55|h'(\sigma_2)|+25|h'(\sigma_1)|) \\
&\left.\left.\left.-2c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}\right)\right]^{\frac{1}{q}}\right\},
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By applying Hölder's inequality and using strongly exponentially convexity of $|h|$ and strongly convexity of $|h'|$, we have

$$
\begin{aligned}
|L_1| &\leq \int_0^{\frac{1}{2}}\left|\left(\nu^{\mu}-\frac{2}{3}\right)\right||e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|d\nu \qquad (24)\\
&\leq \left(\int_0^{\frac{1}{2}}\left|\nu^{\mu}-\frac{2}{3}\right|^{p}d\nu\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^{\frac{1}{2}}|e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|^{q}d\nu\right)^{\frac{1}{q}} \\
&\leq \left(\int_0^{\frac{1}{2}}\left|\nu^{\mu}-\frac{2}{3}\right|^{p}d\nu\right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left[\int_0^{\frac{1}{2}} \left(\nu |e^{h(\sigma_2)}|^q + (1-\nu)|e^{h(\sigma_1)}|^q - c_1' \nu(1-\nu)||\sigma_2 - \sigma_1||^{\ell_1} \right) \right. \\
&\left. \times \left(\nu |h'(\sigma_2)|^q + (1-\nu)|h'(\sigma_1)|^q - c_2' \nu(1-\nu)||\sigma_2 - \sigma_1||^{\ell_2} \right) d\nu \right]^{\frac{1}{q}} \\
&\leq \left(\int_0^{\frac{1}{2}} \left| \nu^\mu - \frac{2}{3} \right|^p d\nu \right)^{\frac{1}{p}} \left[\int_0^{\frac{1}{2}} \left(\nu^2 |e^{h(\sigma_2)} h'(\sigma_2)|^q + \nu(\nu-1) \right. \right. \\
&\times (|e^{h(\sigma_1)} h'(\sigma_2)|^q + |e^{h(\sigma_2)} h'(\sigma_1)|^q) + (1-\nu)^2 |e^{h(\sigma_1)} h'(\sigma_1)|^q \\
&- c_2 ||\sigma_2 - \sigma_1||^{\ell_2} ((\nu^2 - \nu^3)|e^{h(\sigma_2)}| + (\nu - 2\nu^2 + \nu^3)|e^{h(\sigma_1)}|) \\
&- c_1 ||\sigma_2 - \sigma_1||^{\ell_1} ((\nu^2 - \nu^3)|h'(\sigma_2)| + (\nu - 2\nu^2 + \nu^3)|h'(\sigma_1)|) \\
&\left. \left. + c_1 c_2 ||\sigma_2 - \sigma_1||^{\ell_2 + \ell_1} (\nu^2 - 2v^3 + \nu^4) \right) d\nu \right]^{\frac{1}{q}} \\
&\leq \left(\int_0^{\frac{1}{2}} \left| \nu^\mu - \frac{2}{3} \right|^p d\nu \right)^{\frac{1}{p}} \left[\frac{1}{24} \left(|e^{h(\sigma_2)} h'(\sigma_2)|^q \right. \right. \\
&\left. + 2(|e^{h(\sigma_1)} h'(\sigma_2)|^q + |e^{h(\sigma_2)} h'(\sigma_1)|^q) + 7|e^{h(\sigma_1)} h'(\sigma_1)|^q \right) \\
&- \frac{1}{960} \left(c_2 ||\sigma_2 - \sigma_1||^{\ell_2} (25|e^{h(\sigma_2)}| + 55|e^{h(\sigma_1)}|) \right. \\
&+ c_1 ||\sigma_2 - \sigma_1||^{\ell_1} (25|h'(\sigma_2)| + 55|h'(\sigma_1)|) \\
&\left. \left. - 16 c_1 c_2 ||\sigma_2 - \sigma_1||^{\ell_2 + \ell_1} \right) \right]^{\frac{1}{q}}.
\end{aligned}
$$

In a similar way, we obtain

$$
\begin{aligned}
|L_2| &\leq \left(\int_{\frac{1}{2}}^1 \left| \nu^\mu - \frac{1}{3} \right|^p d\nu \right)^{\frac{1}{p}} \\
&\quad \times \left(\int_{\frac{1}{2}}^1 |e^{h(\nu\sigma_2 + (1-\nu)\sigma_1)} h'(\nu\sigma_2 + (1-\nu)\sigma_1)|^q d\nu \right)^{\frac{1}{q}} \qquad (25) \\
&\quad \leq \left(\int_{\frac{1}{2}}^1 \left| \nu^\mu - \frac{1}{3} \right|^p d\nu \right)^{\frac{1}{p}} \left[\frac{1}{24} \left(7|e^{h(\sigma_2)} h'(\sigma_2)|^q \right. \right. \\
&\quad \left. + 2(|e^{h(\sigma_1)} h'(\sigma_1)|^q + |e^{h(\sigma_1)} h'(\sigma_2)|^q) + |e^{h(\sigma_1)} h'(\sigma_1)|^q \right) \\
&\quad - \frac{1}{960} \left(c_2 ||\sigma_2 - \sigma_1||^{\ell_2} (55|e^{h(\sigma_2)}| + 25|e^{h(\sigma_1)}|) \right.
\end{aligned}
$$

$$+ c_1||\sigma_2 - \sigma_1||^{\ell_1}(55|h'(\sigma_2)| + 25|h'(\sigma_1)|)$$
$$- 16c_1c_2||\sigma_2 - \sigma_1||^{\ell_2+\ell_1}\Bigg)\Bigg]^{\frac{1}{q}},$$

$$|L_3| \leq \left(\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right|^p\right)^{\frac{1}{p}}$$
$$\times \left(\int_0^{\frac{1}{2}} |e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1 + (1-\nu)\sigma_2)|^q d\nu\right)^{\frac{1}{q}} \tag{26}$$
$$\leq \left(\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right|^p d\nu\right)^{\frac{1}{p}} \Bigg[\frac{1}{24}\Bigg(|e^{h(\sigma_1)}h'(\sigma_1)|^q$$
$$+ 2(|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + 7|e^{h(\sigma_2)}h'(\sigma_2)|^q\Bigg)$$
$$- \frac{1}{960}\Bigg(c_2||\sigma_2 - \sigma_1||^{\ell_2}(25|e^{h(\sigma_1)}| + 55|e^{h(\sigma_2)}|)$$
$$+ c_1||\sigma_2 - \sigma_1||^{\ell_1}(25|h'(\sigma_1)| + 55|h'(\sigma_2)|)$$
$$- 16c_1c_2||\sigma_2 - \sigma_1||^{\ell_2+\ell_1}\Bigg)\Bigg]^{\frac{1}{q}},$$

$$L_4 \leq \left(\int_{\frac{1}{2}}^1 |\nu^\mu - \frac{1}{3}d\nu|^p\right)^{\frac{1}{p}}$$
$$\times \left(\int_{\frac{1}{2}}^1 |e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1 + (1-\nu)\sigma_2)|^q d\nu\right)^{\frac{1}{q}} \tag{27}$$
$$\leq \left(\int_{\frac{1}{2}}^1 \left|\nu^\mu - \frac{1}{3}\right|^p d\nu\right)^{\frac{1}{p}} \Bigg[\frac{1}{24}\Bigg(7|e^{h(\sigma_1)}h'(\sigma_1)|^q$$
$$+ 2(|e^{h(\sigma_1)}h'(\sigma_1)|^q + |e^{h(\sigma_1)}h'(\sigma_2)|^q) + |e^{h(\sigma_2)}h'(\sigma_2)|^q\Bigg)$$
$$- \frac{1}{960}\Bigg(c_2||\sigma_2 - \sigma_1||^{\ell_2}(55|e^{h(\sigma_1)}| + 25|e^{h(\sigma_2)}|)$$

$$+ c_1||\sigma_2 - \sigma_1||^{\ell_1}(55|h'(\sigma_1)| + 25|h'(\sigma_2)|)$$
$$- 16c_1c_2||\sigma_2 - \sigma_1||^{\ell_2+\ell_1}\Bigg)\Bigg]^{\frac{1}{q}}.$$

Now, if we put the value of (24)-(27) in (7), then we obtain the required above result. Thus, the required result is proved. □

Theorem 6. *Let $h : [\sigma_1, \sigma_2] \to \mathbb{R}$ be a differentiable mapping on (σ_1, σ_2). If the function $|h|$ exibits strongly exponentially-convex and $|h'|^q$, $q > 1$ exhibits strongly convexity over the interval $[\sigma_1, \sigma_2]$, then we acquire the following Milne type inequalities for Riemann-Liouville fractional integrals:*

$$\begin{aligned}
&\left|\frac{1}{3}\left[2e^{h(\sigma_1)} - e^{h\left(\frac{\sigma_1+\sigma_2}{2}\right)} + 2e^{h(\sigma_2)}\right]\right. \\
&\qquad\left. - \frac{\Gamma(\mu+1)}{2(\sigma_2-\sigma_1)^{\mu}}\left[K^{\mu}_{\sigma_1^+}e^{h(\sigma_2)} + K^{\mu}_{\sigma_2^-}e^{h(\sigma_1)}\right]\right| \\
&\leq (\kappa_1(\mu))^{1-\frac{1}{q}} \\
&\quad\times\left[\kappa_3(\mu)|e^{h(\sigma_2)}h'(\sigma_2)|^q + (\kappa_2(\mu)-\kappa_3(\mu))(|e^{h(\sigma_1)}h'(\sigma_2)|^q\right. \\
&\quad + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + (\kappa_1(\mu) - 2\kappa_2(\mu) + \kappa_3(\mu))|e^{h(\sigma_1)}h'(\sigma_1)|^q \\
&\quad - c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_3(\mu)-\kappa_3^*(\mu))|e^{h(\sigma_2)}| + (\kappa_2(\mu) - 2\kappa_3(\mu) \\
&\quad + \kappa_3^*(\mu))|e^{h(\sigma_1)}|) - c_1||\sigma_2-\sigma_1||^{\ell_1}((\kappa_3(\mu)-\kappa_3^*(\mu))|h'(\sigma_2)| \\
&\quad + (\kappa_2(\mu) - 2\kappa_3(\mu) + \kappa_3^*(\mu))|h'(\sigma_1)|) + c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1} \\
&\quad\left.\times(\kappa_3(\mu) - 2\kappa_3^*(\mu) + \kappa_3^{**}(\mu))\right]^{\frac{1}{q}} + (\kappa_4(\mu))^{1-\frac{1}{q}}\left[\kappa_6(\mu)\right. \\
&\quad\times|e^{h(\sigma_2)}h'(\sigma_2)|^q + (\kappa_5(\mu)-\kappa_6(\mu))(|e^{h(\sigma_1)}h'(\sigma_2)|^q \\
&\quad + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + (\kappa_4(\mu) - \kappa_5(\mu) + \kappa_6(\mu))|e^{h(\sigma_1)}h'(\sigma_1)|^q \\
&\quad - c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_6(\mu)-\kappa_6^*(\mu))|e^{h(\sigma_2)}| + (\kappa_4(\mu) - 2\kappa_6(\mu) \\
&\quad + \kappa_6^*(\mu))|e^{h(\sigma_1)}|) - c_1||\sigma_2-\sigma_1||^{\ell_1}((\kappa_6(\mu)-\kappa_6^*(\mu))|h'(\sigma_2)| \\
&\quad + (\kappa_4(\mu) - 2\kappa_6(\mu) + \kappa_6^*(\mu))|h'(\sigma_1)|) + c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1} \\
&\quad\left.\times(\kappa_6(\mu) - 2\kappa_6^*(\mu) + \kappa_6^{**}(\mu))\right]^{\frac{1}{q}} + (\kappa_1(\mu))^{1-\frac{1}{q}} \\
&\quad\times\left[\kappa_3(\mu)|e^{h(\sigma_1)}h'(\sigma_1)|^q + (\kappa_2(\mu)-\kappa_3(\mu))(|e^{h(\sigma_1)}h'(\sigma_2)|^q\right. \\
&\quad\left. + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + (\kappa_1(\mu) - 2\kappa_2(\mu) + \kappa_3(\mu))|e^{h(\sigma_2)}h'(\sigma_2)|^q\right.
\end{aligned}$$

$$
\begin{aligned}
&- c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_6(\mu)-\kappa_6^*(\mu))|e^{h(\sigma_2)}| + (\kappa_4(\mu)-2\kappa_6(\mu)\\
&+ \kappa_6^*(\mu))|e^{h(\sigma_1)}|) - c_1||\sigma_2-\sigma_1||^{\ell_1}((\kappa_6(\mu)-\kappa_6^*(\mu))|h'(\sigma_2)|\\
&+ (\kappa_4(\mu)-2\kappa_6(\mu)+\kappa_6^*(\mu))|h'(\sigma_1)|) + c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}\\
&\times (\kappa_6(\mu)-2\kappa_6^*(\mu)+\kappa_6^{**}(\mu))\Big]^{\frac{1}{q}} + (\kappa_4(\mu))^{1-\frac{1}{q}}\\
&\times \Bigg[\kappa_6(\mu)|e^{h(\sigma_1)}h'(\sigma_1)|^q + (\kappa_5(\mu)-\kappa_6(\mu))(|e^{h(\sigma_1)}h'(\sigma_2)|^q\\
&+ |e^{h(\sigma_2)}h'(\sigma_1)|^q) + (\kappa_4(\mu)-\kappa_5(\mu)+\kappa_6(\mu))|e^{h(\sigma_2)}h'(\sigma_2)|^q\\
&- c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_6(\mu)-\kappa_6^*(\mu))|e^{h(\sigma_1)}| + (\kappa_4(\mu)-2\kappa_6(\mu)\\
&+ \kappa_6^*(\mu))|e^{h(\sigma_2)}|) - c_1||\sigma_2-\sigma_1||^{\ell_1}((\kappa_6(\mu)-\kappa_6^*(\mu))|h'(\sigma_1)|\\
&+ (\kappa_4(\mu)-2\kappa_6(\mu)+\kappa_6^*(\mu))|h'(\sigma_2)|)\\
&+ c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}(\kappa_6(\mu)-2\kappa_6^*(\mu)+\kappa_6^{**}(\mu))\Bigg]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
$$

Proof. By applying power-mean inequality and using strongly exponentially convexity of $|h|$ and strongly convexity of $|h'|$, we have

$$
\begin{aligned}
|L_1| &\le \int_0^{\frac{1}{2}} \left|\left(\nu^\mu - \frac{2}{3}\right)\right| |e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|d\nu \qquad (28)\\
&\le \left(\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right|^q d\nu\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \left|\left(\nu^\mu - \frac{2}{3}\right)\right|\right.\\
&\quad \left.\times |e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|^q d\nu\right)^{\frac{1}{q}}\\
&\le \left(\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right|^q d\nu\right)^{1-\frac{1}{q}} \left(\int_0^{\frac{1}{2}} \left|\left(\nu^\mu - \frac{2}{3}\right)\right| \left|\left(\nu|e^{h(\sigma_2)}|^q\right.\right.\right.\\
&\quad \left.\left.\left. + (1-\nu)|e^{h(\sigma_1)}|^q - c_1\nu(1-\nu)||\sigma_2-\sigma_1||^{\ell_1}\right)\left(\nu|h'(\sigma_2)|^q\right.\right.\right.\\
&\quad \left.\left.\left. + (1-\nu)|h'(\sigma_1)|^q - c_2\nu(1-\nu)||\sigma_2-\sigma_1||^{\ell_2}\right)\right|^q d\nu\right)^{\frac{1}{q}}\\
&\le (\kappa_1(\mu))^{1-\frac{1}{q}}\left[\int_0^{\frac{1}{2}} \left|\nu^\mu - \frac{2}{3}\right|\left(\nu^2|e^{h(\sigma_2)}h'(\sigma_2)|^q\right.\right.\\
&\quad + \nu(\nu-1)(|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q)\\
&\quad \left.\left. + (1-\nu)^2|e^{h(\sigma_1)}h'(\sigma_1)|^q - c_2||\sigma_2-\sigma_1||^{\ell_2}((\nu^2-\nu^3)|e^{h(\sigma_2)}|\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+(\nu-2\nu^2+\nu^3)|e^{h(\sigma_1)}|)-c_1||\sigma_2-\sigma_1||^{\ell_1}((\nu^2-\nu^3)|h'(\sigma_2)|\\
&+(\nu-2\nu^2+\nu^3)|h'(\sigma_1)|)+c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}\\
&\times(\nu^2-2v^3+\nu^4)\Big)d\nu\Big]^{\frac{1}{q}}\\
&\leq(\kappa_1(\mu))^{1-\frac{1}{q}}\Big[\kappa_3(\mu)|e^{h(\sigma_2)}h'(\sigma_2)|^q+(\kappa_2(\mu)-\kappa_3(\mu))\\
&\times(|e^{h(\sigma_1)}h'(\sigma_2)|^q+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+(\kappa_1(\mu)-2\kappa_2(\mu)\\
&+\kappa_3(\mu))|e^{h(\sigma_1)}h'(\sigma_1)|^q-c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_3(\mu)-\kappa_3^*(\mu))|e^{h(\sigma_2)}|\\
&+(\kappa_2(\mu)-2\kappa_3(\mu)+\kappa_3^*(\mu))|e^{h(\sigma_1)}|)-c_1||\sigma_2-\sigma_1||^{\ell_1}\\
&\times((\kappa_3(\mu)-\kappa_3^*(\mu))|h'(\sigma_2)|+(\kappa_2(\mu)-2\kappa_3(\mu)+\kappa_3^*(\mu))|h'(\sigma_1)|)\\
&+c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}(\kappa_3(\mu)-2\kappa_3^*(\mu)+\kappa_3^{**}(\mu))\Big]^{\frac{1}{q}}.
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
|L_2|&\leq\int_{\frac{1}{2}}^{1}\left|\left(\nu^\mu-\frac{1}{3}\right)\right||e^{h(\nu\sigma_2+(1-\nu)\sigma_1)}h'(\nu\sigma_2+(1-\nu)\sigma_1)|d\nu \qquad (29)\\
&\leq(\kappa_4(\mu))^{1-\frac{1}{q}}\Big[\kappa_6(\mu)|e^{h(\sigma_2)}h'(\sigma_2)|^q+(\kappa_5(\mu)-\kappa_6(\mu))\\
&\times(|e^{h(\sigma_1)}h'(\sigma_2)|^q+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+(\kappa_4(\mu)-2\kappa_5(\mu)\\
&+\kappa_6(\mu))|e^{h(\sigma_1)}h'(\sigma_1)|^q-c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_6(\mu)-\kappa_6^*(\mu))\\
&\times|e^{h(\sigma_2)}|+(\kappa_4(\mu)-2\kappa_6(\mu)+\kappa_6^*(\mu))|e^{h(\sigma_1)}|)\\
&-c_1||\sigma_2-\sigma_1||^{\ell_1}((\kappa_6(\mu)-\kappa_6^*(\mu))|h'(\sigma_2)|+(\kappa_4(\mu)-2\kappa_6(\mu)\\
&+\kappa_6^*(\mu))|h'(\sigma_1)|)+c_1c_2||\sigma_2-\sigma_1||^{\ell_2+\ell_1}(\kappa_6(\mu)\\
&-2\kappa_6^*(\mu)+\kappa_6^{**}(\mu))\Big]^{\frac{1}{q}},\\
|L_3|&\leq\int_{0}^{\frac{1}{2}}\left|\left(\nu^\mu-\frac{2}{3}\right)\right||e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1+(1-\nu)\sigma_2)|d\nu \qquad (30)\\
&\leq(\kappa_1(\mu))^{1-\frac{1}{q}}\Big[\kappa_3(\mu)|e^{h(\sigma_1)}h'(\sigma_1)|^q+(\kappa_2(\mu)-\kappa_3(\mu))\\
&\times(|e^{h(\sigma_1)}h'(\sigma_2)|^q+|e^{h(\sigma_2)}h'(\sigma_1)|^q)+(\kappa_1(\mu)-2\kappa_2(\mu)\\
&+\kappa_3(\mu))|e^{h(\sigma_2)}h'(\sigma_2)|^q-c_2||\sigma_2-\sigma_1||^{\ell_2}((\kappa_3(\mu)-\kappa_3^*(\mu))|e^{h(\sigma_1)}|\\
&+(\kappa_2(\mu)-2\kappa_3(\mu)+\kappa_3^*(\mu))|e^{h(\sigma_2)}|)-c_1||\sigma_2-\sigma_1||^{\ell_1}((\kappa_3(\mu)
\end{aligned}
$$

$$
\begin{aligned}
&- \kappa_3^*(\mu))|h'(\sigma_1)| + (\kappa_2(\mu) - 2\kappa_3(\mu) + \kappa_3^*(\mu))|h'(\sigma_2)|) \\
&+ c_1c_2||\sigma_2 - \sigma_1||^{\ell_2+\ell_1}(\kappa_3(\mu) - 2\kappa_3^*(\mu) + \kappa_3^{**}(\mu))\Big]^{\frac{1}{q}}, \\
|L_4| &\leq \int_{\frac{1}{2}}^{1} \left|\left(\nu^\mu - \frac{1}{3}\right)\right| |e^{h(\nu\sigma_1+(1-\nu)\sigma_2)}h'(\nu\sigma_1 + (1-\nu)\sigma_2)|d\nu \qquad (31) \\
&\leq (\kappa_4(\mu))^{1-\frac{1}{q}}\Big[\kappa_6(\mu)|e^{h(\sigma_1)}h'(\sigma_1)|^q + (\kappa_5(\mu) - \kappa_6(\mu)) \\
&\times (|e^{h(\sigma_1)}h'(\sigma_2)|^q + |e^{h(\sigma_2)}h'(\sigma_1)|^q) + (\kappa_4(\mu) - 2\kappa_5(\mu) \\
&+ \kappa_6(\mu))|e^{h(\sigma_2)}h'(\sigma_2)|^q - c_2||\sigma_2 - \sigma_1||^{\ell_2} \\
&\times ((\kappa_6(\mu) - \kappa_6^*(\mu))|e^{h(\sigma_1)}| + (\kappa_4(\mu) - 2\kappa_6(\mu) + \kappa_6^*(\mu))|e^{h(\sigma_2)}|) \\
&- c_1||\sigma_2 - \sigma_1||^{\ell_1}((\kappa_6(\mu) - \kappa_6^*(\mu))|h'(\sigma_1)| \\
&+ (\kappa_4(\mu) - 2\kappa_6(\mu) + \kappa_6^*(\mu))|h'(\sigma_2)|) \\
&+ c_1c_2||\sigma_2 - \sigma_1||^{\ell_2+\ell_1}(\kappa_6(\mu) - 2\kappa_6^*(\mu) + \kappa_6^{**}(\mu))\Big]^{\frac{1}{q}}.
\end{aligned}
$$

Now, if we put the value of (28)-(31) in (7), then we obtain the required above result. Thus, the required result is proved. □

6 Conclusion

This study investigated the class of higher-order strongly exponentially convex functions, representing a generalization of the strongly exponentially convex function concept. We established and proved the Milne type inequality associated with higher-order strongly exponentially convex functions. Moreover, the obtained inequalities are studied by means of fractional integral operators. We developed new connections between various notions of higher-order strongly exponentially convex functions. The obtained results incorporated several well-known results as well as some newly derived ones as particular cases. Hence, the results of this study extended and generalized many existing findings.

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