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NEW ASPECTS OF PREINVEX EQUILIBRIUM EQUATIONS*

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Abstract

Some classes of general preinvex equilibrium equations are introduced and investigated. Using fixed point, auxiliary principle technique and dynamical systems coupled with finite difference approach, we suggest and analyze a number of several multistep iterative methods for solving the general preinvex equilibrium equations. Convergence analysis of these methods is investigated under suitable conditions. Sensitivity analysis of the preinvex equilibrium equations is also considered. Various special cases are discussed as applications of the main results. Several open problems are suggested for future research.

Keywords: equilibrium equations, convex functions, fixed point iterative methods, auxiliary principle technique, dynamical system, convergence, sensitivity analysis.

MSC: 26D15, 26D10, 49J40, 65N35, 49J40, 90C26, 90C30.

1 Introduction

Equilibrium problems theory introduced by Blum et al. [5], Noor [30–34] and Noor et al. [51] provides us with a unified, natural, innovative and general

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framework to study a wide class of problems arising in economics, finance, transportation, network, cooperative game, Nash equilibrium, data sciences, structural analysis, elasticity and optimization. Equilibrium problems theory has witnessed an explosive growth in theoretic advances, algorithmic aspects and applications across all discipline of pure and applied sciences. This theory provides a novel and unified treatment of problems arising in The ideas and techniques of this theory are being used in a variety of diverse areas and proved to be productive and innovative. Equilibrium problems have been extended and generalized in many directions using novel and innovative techniques, see [5, 11, 13, 15, 16, 20, 22, 29–34, 41, 48, 49, 51–53, 59, 60, 67] and the references therein.

Several complex problems, which arise in different branches of mathematical and engineering sciences such as physical, biomedical, regional, optimization, ecology, economics and engineering sciences, can be studied in the general and unified framework of nonlinear equations. Different techniques including including Taylor series, quadrature formulas, Trapezoidal rule, variation iterations, homotopy perturbation, decomposition methods and Lax-Milgram lemma are being suggested and analyzed in recent years to find the approximate solutions of the nonlinear equation. For more detail, see [6,7,14,15,21,23,35–41,43–47,49,50,53,62,63] and the references therein. It has been proved that the general nonlinear equations are equivalent to the fixed point problems. This equivalent formulation has been exploited to suggest and analyze several multi-step explicit and implicit iterative methods for solving nonlinear nonlinear equations. In particular, these multi step methods can be viewed as novel modifications and generalizations of Noor (three step) iterations, the origin can be traced back to Noor [27, 31]. These Noor (three-step) iterations include Ishikawa (two-step) iterations, Mann (one-step) iteration and Picard method as special cases. Three step Noor iterations are a class of iterative methods used to find fixed points of nonexpansive mappings in Banach spaces. These methods are particularly useful in various fields such as finance, economics, network analysis, elasticity, optimization, water resources, medical images, and structural analysis. The iterations converge to a fixed point of the nonexpansive mapping, which is a solution of a variational inequality, under certain mild conditions [15]. The convergence criteria for these new iterations are supported by recent results conducted by various authors in the field of functional analysis and optimization. Noor iterations, a powerful three-step method generalizing Mann/Ishikawa, offer novel aspects like generating complex fractal structures (Julia/Mandelbrot sets) with unique boundaries resembling natural forms and improved convergence for solving nonlinear problems, finding applications in signal recovery,

image inpainting, polynomiography, solar energy optimization and controlling chaotic systems (cardiac arrhythmia) [3]. These orbits reveal richer dynamics than simpler methods, linking fixed-point theory to engineering and climate modeling, with recent work focusing on faster convergence and applications in quantum/computational fields. For the applications, generalizations, modifications, extensions and other aspects of Noor (three-step) iterations, see [2–4, 7, 15, 17, 18, 27, 31, 33, 43–46, 48, 53–60, 64, 66] and the references therein.

Motivated and inspired by the research activities in this fast developing field, we introduce and study some new system of general preinvex equilibrium equations involving some arbitrary nonlinear operators. We establish the equivalence between the general preinvex equilibrium equations and the fixed point problem. This alternate equivalence is used to prove the existence of the solution as well as to suggest multi step methods. Convergence criteria is also analyzed under some weaker conditions. To discuss the asymptotic stability of the solution and to suggest multi-step iterative methods, we apply the dynamical system technique, which is mainly due to Dupuis et al. [10] as developed in [10,16,28,31,42,43,46–48,52,53,64,65]. This is entirely new way of considering the multi-step methods for solving general preinvex equilibrium equations. We introduce first order, second order and third order initial value and boundary value problems for computing the approximate solution of the preinvex equilibrium equations, which is main domain of differential equations. This technique is used to consider the the qualitative results such as the existence, uniqueness continuous dependence and the global stability of the solution of preinvex equilibrium equations. Making use of finite forward backward interpolation and discretizing the dynamical systems, several multi-step iterative methods for solving the preinvex equilibrium equations analyzed. In section 5, we explore the auxiliary principle technique to suggest some iterative schemes for solving the preinvex equilibrium equations along with convergence criteria under weaker conditions. The main idea involving this technique is to first consider an auxiliary problem and then to show that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach. Noor [25] has modified the auxiliary principle technique involving an operator and discussed its applications in information theory and machine learning [57]. It has been shown that these new iterative methods include a wide class of known and new iterative methods as special cases. Dafermos [9] considered the sensitivity analysis considered the sensitivity of the variational inequalities using essentially the projection method. The sensitivity analysis helps us to estimate the problem data, measuring errors and predict the future

changes of the equilibrium as a result of changes in the governing system. From mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied and can stimulate new ideas and techniques for problem solving the problems. We also study the sensitivity analysis of the preinvex equilibrium equations in section 6. We have only investigated the theoretical aspect of the results regarding applications of various techniques for solving the preinvex equilibrium equations. Developments of the numerically applicable efficient methods and comparison with methods and their applications need further research efforts and can be considered important open problems. It is expected the techniques and ideas of this paper may be starting point for further research.

2 Formulations and basic facts

Let $\mathcal{H}$ be a real Hilbert space and $K \subseteq \mathcal{H}$ be a closed convex set. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm, respectively. We also need the some basic concepts from the convex analysis [6, 19] and Noor et al. [53].

For a given nonlinear operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$ and continuous bifunctions $E(.,.), \eta(.,.) : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$, we consider the problem of finding $\mu \in \mathcal{H}$ such that

$$E(\mathcal{T}(\mu), \eta(\nu, \mu)) \quad = 0, \qquad \forall \nu \in \mathcal{H}, \tag{1}$$

which is called the preinvex equilibrium equation.

For an arbitrary operator $g : \mathcal{H} \longrightarrow \mathcal{H}$, the problem (1) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) = g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu)), \quad \forall \nu \in \mathcal{H}, \tag{2}$$

where ρ is a parameter, is called the general preinvex equilibrium equation.

Special cases

We now point out some very important and interesting problems, which can be obtained as special cases of the problem (2).

(I). If $g = I$, then the problem (2) reduces to finding $\mu \in \mathcal{H}$, such that

$$\mu = \mu - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu)), \quad \forall \nu \in \mathcal{H}, \tag{3}$$

which is the system of preinvex equilibrium equations.

(II). If $E(\mathcal{T}(\mu), \eta(\nu, \mu)) = \mathcal{T}(\mu)$, then the problem (2) reduces to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) = g(\mu) - \rho\mathcal{T}(\mu), \quad \forall \nu \in \mathcal{H}, \tag{4}$$

which are called the general nonlinear equations.

(III). For $E(\mathcal{T}(\mu), \eta(\nu, \mu)) = \langle \mathcal{T}(\mu) - f, \nu - \mu \rangle$, the problem (3) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\langle \mathcal{T}(\mu), \nu - \mu \rangle = \langle f, \nu - \mu \rangle, \quad \forall \nu \in \mathcal{H} \tag{5}$$

is called the generalized Lax-Milgram Lemma. It is shown [20, 24, 31, 52, 53] that the symmetric and positive boundary value problems can only be studied in the general frame work of the Lax-Milgram Lemma.

We need the following definitions and results in obtaining our results.

Definition 1. *An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:*

1. *Strongly monotone, if there exists a constant $\alpha > 0$, such that*

$$\langle \mathcal{T}(\mu) - \mathcal{T}(\nu), \mu - \nu \rangle \geq \alpha \|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}.$$

2. *Lipschitz continuous, if there exist a constants $\beta > 0$, such that*

$$\|\mathcal{T}(\mu) - \mathcal{T}(\nu)\| \leq \beta \|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$

3. *Monotone, if*

$$\langle \mathcal{T}(\mu) - \mathcal{T}(\nu), \mu - \nu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

Definition 2. *The bifunction $E(\mathcal{T}(.), .) : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be:*

1. *Strongly monotone, if there exists a constant $\alpha > 0$, such that*

$$\langle E(\mathcal{T}(\mu), \eta(\nu, \mu)) - E(\mathcal{T}(\zeta), \eta(\nu, \zeta), \mu - \zeta \rangle \geq \alpha \|\mu - \zeta\|^2, \quad \forall \zeta, \mu, \nu \in \mathcal{H}.$$

2. *Lipschitz continuous, if there exists a constant $\beta > 0$, such that*

$$\|E(\mathcal{T}(\mu), \eta(\nu, \mu)) - E(\mathcal{T}(\zeta), \eta(\nu, \zeta))\| \leq \beta \|\mu - \zeta\|, \quad \forall \zeta, \mu, \nu \in \mathcal{H}.$$

3. monotone, if

$$\langle E(\mathcal{T}(\mu),\eta(\nu,\mu)) - E(\mathcal{T}(\zeta),\eta(\nu,\zeta)),\mu-\zeta)\rangle \geq 0, \quad \forall \zeta,\mu,\nu \in \mathcal{H}.$$

Remark 1. *Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.*

3 Iterative methods and convergence analysis

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the general preinvex equilibrium nonlinear equations.

Let $\mu \in \mathcal{H}$ be the solution of (2). Then

$$g(\mu) = g(\mu) - \rho E(\mathcal{T}(\mu),\eta(\nu,\mu)), \qquad \forall \nu \in \mathcal{H},$$

which can be written as

$$\mu = \mu - g(\mu) + \Big[g(\mu) - \rho E(\mathcal{T}(\mu),\eta(\nu,\mu))\Big], \qquad \forall \nu \in \mathcal{H}. \tag{6}$$

This implies that the general preinvex equilibrium equation (2) is equivalent to the fixed point problem. (6). We define the function Φ associated with (6) as

$$\Phi(\mu) = \mu - g(\mu) + \Big[g(\mu) - \rho E(\mathcal{T}(\mu),\eta(\nu,\mu))\Big], \qquad \forall \nu \in \mathcal{H}. \tag{7}$$

To prove the unique existence of the solution of the problem (2), it is enough to show that the map F defined by (7) has a fixed point.

Theorem 1. *Let the operator g be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$, respectively. Let the bifunction $E(\mathcal{T}(.),.)$ be Lipschitz continuous with constant $\beta > 0$. If there exists a parameter $\rho > 0$, such that*

$$\rho < \frac{1-k}{\beta}, \quad k < 1, \quad \delta < 1, \tag{8}$$

where

$$\theta = \rho(\beta) + k \tag{9}$$

$$k = \sqrt{1 - 2\sigma + \delta^2} + \delta, \tag{10}$$

then there exists a unique solution of the problem (2).

Proof. **Uniqueness.**
Let $\mu_1 \neq \mu_2$ be the two solution of the problem (6), Then

$$\mu_1 = \mu_1 - g(\mu_1) + \rho E(\mathcal{T}(\mu_1), \eta(\nu, \mu_1)), \qquad \forall \nu \in \mathcal{H}, \tag{11}$$

and

$$\mu_2 = \mu_2 - g(\mu_2) + \rho E(\mathcal{T}(\mu_2), \eta(\nu, \mu_2)), \qquad \forall \nu \in \mathcal{H}. \tag{12}$$

Combining (11) and (12), we have

$$\begin{aligned} \|\mu_1 - \mu_2\| &= \|\mu_1 - \mu_2 - (g(\mu_1) - g(\mu_2)) \\ &\quad + \rho(E(\mathcal{T}(\mu_1), \eta(\nu, \mu_1)) - E(\mathcal{T}(\mu_2), \eta(\nu, \mu_2)))\| \\ &\leq \|\mu_1 - \mu_2\| + \|g(\mu_1) - g(\mu_2)\| \\ &\quad + \rho\|E(\mathcal{T}(\mu_1), \eta(\nu, \mu_1)) - E(\mathcal{T}(\mu_2), \eta(\nu, \mu_2))\| \\ &\leq (1 + \delta + \rho\beta)\|\|\mu_1 - \mu_2\| = \theta_1\|\mu_1 - \mu_2\|, \end{aligned} \tag{13}$$

where $\theta_1 = 1 + \delta + \rho\beta$.
For $\theta_1 \leq 1$, we have $1 + \delta + \rho\beta \leq 1$. Thus

$$\rho\beta \leq -\delta,$$

which is impossible. Consequently, we have the uniqueness of the solution.
Existence.

Since the general preinvex equilibrium equations (6) and (2) are equivalent, so it is enough to show that the map $\Phi(\mu)$, defined by (7 has a fixed point. For $\forall \zeta \neq \mu \in \mathcal{H}$, we have

$$\begin{aligned} \Big\|\Phi(\mu) - \Phi(\zeta)\Big\| &\leq \Big\|\mu - \zeta - (g(\mu) - g(\zeta))\Big\| \\ &\quad + \Big\|\Big[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu))\Big] - \Big[g(\zeta) - \rho E(\mathcal{T}(\zeta), \eta(\nu, \zeta))\Big]\Big\| \\ &\leq \Big\|\mu - \zeta - ((g(\mu) - g(\zeta))\Big\| + \Big\|g(\zeta) - g(\mu) \\ &\quad - \rho\Big(E(\mathcal{T}(\mu), \eta(\nu, \mu)) - E(\mathcal{T}(\zeta), \eta(\nu, \zeta))\Big)\Big\| \\ &\leq \Big\|\mu - \zeta - (g(\mu) - (\zeta))\Big\| + \Big\|g(\zeta) - g(\mu)\Big\| \\ &\quad + \rho\Big\|E(\mathcal{T}(\mu), \eta(\nu, \mu)) - E(\mathcal{T}(\zeta), \eta(\nu, \zeta))\Big\| \\ &\leq \Big\|\mu - \zeta - (g(\mu) - g(\zeta))\Big\| + \rho\beta\Big\|\zeta - \mu\Big\| + \delta\Big\|\zeta - \mu\Big\|, \end{aligned} \tag{14}$$

where $\beta > 0, \delta > 0$ are Lipschitz continuity constants of the bifunction $E(.,.)$ and the operator g, respectively.

Since the operator g is strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$, it follows that

$$\begin{aligned}\left\|\mu-\zeta-(g(\mu)-g(\zeta))\right\|^2 &\leq \left\|\mu-\zeta\right\|^2-2\langle g(\mu)-g(\zeta),\mu-\zeta\rangle \\ &\quad +\left\|g(\mu)-g(\zeta)\right\|^2 \\ &\leq (1-2\sigma+\delta^2)\left\|\mu-\zeta\right\|^2. \end{aligned} \tag{15}$$

From (14) and (15), we have

$$\left\|\Phi(\mu)-\Phi(\nu)\right\| \leq \left\{\sqrt{(1-2\sigma+\delta^2)}+\rho\beta\right\}\left\|\mu-\eta\right\| = \theta\left\|\mu-\eta\right\|,$$

where

$$\begin{aligned}\theta &= \rho\beta+k, \\ k &= \sqrt{1-2\sigma+\delta^2}+\delta.\end{aligned}$$

From (8), it follows that $\theta < 1$, which implies that the map $\Phi(\mu)$ defined by (7) has a fixed point, which is the unique solution of (2). □

We now suggested and analyzed the three step iterative methods for solving the general preinvex equilibrium equations (2).

Algorithm 1. *For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned}y_n &= (1-\gamma_n)\mu_n \\ &\quad +\gamma_n\left\{\mu_n-g(\mu_n)+\left[g(\mu_n)-\rho E(\mathcal{T}(\mu_n),\eta(\nu.\mu_n))\right]\right\} \qquad (16) \\ z_n &= (1-\beta_n)\mu_n \\ &\quad +\beta_n\left\{y_n-g(y_n)+\left[g(y_n)-\rho E(\mathcal{T}(y_n),\eta(\nu,y_n))\right]\right\} \qquad (17) \\ \mu_{n+1} &= (1-\alpha_n)\mu_n \\ &\quad +\alpha_n\left\{z_n-g(z_n)+\left[g(z_n)-\rho E(\mathcal{T}(z_n),\eta(\nu,z_n))\right]\right\}, \qquad (18)\end{aligned}$$

which are known as the modified Noor iterations.
For $\gamma_n = 0$, Algorithm 1 reduces to:

Algorithm 2. *For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned} z_n &= (1-\beta_n)\mu_n + \beta_n\Big\{\mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho E(\mathcal{T}(\mu_n), \eta(\nu, \mu_n))\Big]\Big\} \\ \mu_{n+1} &= (1-\alpha_n)\mu_n + \alpha_n\Big\{z_n - g(z_n) + \Big[g(z_n) - \rho E(\mathcal{T}(z_n), \eta(\nu, z_n))\Big]\Big\}, \end{aligned}$$

which is known as the Ishikawa iterative scheme for the problem (2).
Note that, for $\gamma_n = 0 \quad and \quad \beta_n = 0$, Algorithm 1 is called the Mann iterative method, that is.

Algorithm 3. *For a given μ_0, compute $\{\mu_{n+1}\}$ by the iterative schemes*

$$\mu_{n+1} = (1-\beta_n)\mu_n + \beta_n\Big\{\mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho E(\mathcal{T}(\mu_n), \eta(\nu, \mu_n))\Big]\Big\}.$$

We now study the convergence analysis of Algorithm 1, which is the main motivation of our next result.

Theorem 2. *Let the $\mathcal{T}(.),.)$, bifunction $E(.,.)$ and the operator g satisfy all the assumptions of Theorem 1. If the condition (8) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 1 converges to the exact solution $\mu \in \mathcal{H}$ of the preinvex equilibrium equations (2) strongly in $\mathcal{H}$.*

Proof. From Theorem 1, we see that there exists a unique solution $\mu \in \mathcal{H}$ of the general preinvex equilibrium equations (2). Let $\mu \in \mathcal{H}$ be the unique solution of (2). Then, using the fixed point problem (6), we have

$$\mu = (1-\alpha_n)\mu + \alpha_n\Big\{\mu - g(\mu) + \Big[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu))\Big]\Big\} \quad (19)$$

$$= (1-\beta_n)\mu + \beta_n\Big\{\mu - g(\mu) + \Big[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu))\Big]\Big\} \quad (20)$$

$$= (1-\gamma_n)\mu + \gamma_n\Big\{\mu - g(\mu) + \Big[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu))\Big]\Big\}. \quad (21)$$

From (18) and (19), we have

$$\begin{aligned}
\Big\|\mu_{n+1}-\mu\Big\| &= \Big\|(1-\alpha_n)(\mu_n-\mu)+\alpha_n(z_n-\mu-(g(z_n)-h(\mu)))\\
&\quad+\alpha_n\Big\{\Big[g(\mu_n)-\rho E(\mu_n,\eta(\nu,\mu_n))\Big]\\
&\quad-\Big[g(\mu)-\rho E(\mathcal{T}(\mu),\eta(\nu,\mu))\Big]\Big\}\Big\|\\
&\le (1-\alpha_n)\Big\|\mu_n-\mu\Big\|+\alpha_n\Big\|\mu_n-\mu-(g(z_n)-g(\mu))\Big\|\\
&\quad+\alpha_n\Big\{\Big\|g(\mu_n)-g(\mu)-\rho\big(E(\mathcal{T}(\mu_n),\eta(\nu,\mu_n))\\
&\quad-E(\mathcal{T}(\mu),\eta(\nu,\mu))\big)\Big\|\Big\}\\
&\le (1-\alpha_n)\Big\|\mu_n-\mu\Big\|+\alpha_n(k+\rho\beta)\Big\|\mu_n-\mu\Big\|\\
&= (1-\alpha_n)\Big\|u_n-\mu\Big\|+\alpha_n\theta\Big\|\mu_n-\mu\Big\|, \qquad (22)
\end{aligned}$$

where θ is defined by (9).
In a similar way, from (16) and (20), we have

$$\begin{aligned}
\Big\|z_n-\mu\Big\| &\le (1-\beta_n)\Big\|\mu_n-\mu\Big\|+2\beta_n\theta\Big\|y_n-\mu-(g(y_n)-g(\mu))\Big\|\\
&\quad+\beta_n\Big\|g(y_n)-g(\mu)-\rho\big(E(\mathcal{T}(y_n),\eta(\nu,y_n))-E(\mathcal{T}(\mu),\eta(\nu,\mu))\big)\Big\|\\
&\quad+\delta\Big\|y_n-\mu\Big\|\\
&\le (1-\beta_n)\Big\|\mu_n-\mu\Big\|+\beta_n(k+\rho\beta)\Big\|y_n-\mu\Big\|,\\
&\le (1-\beta_n)\Big\|\mu_n-\mu\Big\|+\beta_n\theta\Big\|y_n-\mu\Big\|, \qquad (23)
\end{aligned}$$

where θ is defined by (9).
From (16) and (21), we obtain

$$\begin{aligned}
\Big\|y_n-\mu\Big\| &\le (1-\gamma_n)\Big\|\mu_n-\mu\Big\|+\gamma_n\theta\Big\|\mu_n-\mu\Big\|\\
&\le (1-(1-\theta)\gamma_n)\Big\|\mu_n-\mu\Big\| \le ||\mu_n-\mu||. \qquad (24)
\end{aligned}$$

From (23) and (24), we obtain

$$\begin{aligned}\left\|z_n-\mu\right\| &\leq (1-\beta_n)\left\|\mu_n-\mu\right\|+\beta_n\theta\left\|\mu_n-\mu\right\| \\ &= (1-(1-\theta)\beta_n)\left\|\mu_n-\mu\right\| \leq ||\mu_n-\mu||. \end{aligned} \tag{25}$$

Form the above equations, we have

$$\begin{aligned}\left\|\mu_{n+1}-\mu\right\| &\leq (1-\alpha_n)\left\|\mu_n-\mu\right\|+\alpha_n\theta\left\|\mu_n-\mu\right\| \\ &= [1-(1-\theta)\alpha_n]\left\|\mu_n-\mu\right\| \\ &\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\left\|\mu_0-\mu\right\|.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\lim_{i\to 0}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{u_n\}$ convergence strongly to μ. From (24), and (25), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

We suggest new perturbed iterative schemes for solving the general preinvex equilibrium equations (2).

Algorithm 4. *For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the iterative schemes*

$$\begin{aligned} y_n &= (1-\gamma_n)\mu_n \\ &\quad +\gamma_n\Big\{\mu_n-g(\mu_n)+\Big[g(\mu_n)-\rho E(\mathcal{T}(\mu_n),\eta(\nu,\mu))\Big]\Big\}+\gamma_n h_n \\ z_n &= (1-\beta_n)\mu_n \\ &\quad +\beta_n\Big\{y_n-g(y_n)+\Big[g(y_n)-\rho E(\mathcal{T}(y_n),\eta(\nu,y_n))\Big]\Big\}+\beta_n f_n \\ \mu_{n+1} &= (1-\alpha_n)\mu_n \\ &\quad +\alpha_n\Big\{z_n-g(z_n)+\Big[g(z_n)-\rho E(\mathcal{T}(z_n),\eta(\nu,z_n))\Big]\Big\}+\alpha_n e_n, \end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0\leq\alpha_n,\beta_n,\gamma_n\leq 1;\quad \forall n\geq 0,\quad \sum_{n=0}^{\infty}\alpha_n=\infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving the general preinvex equilibrium equations (2).

Also, we can suggest the following iterative methods for solving the general preinvex equilibrium equations.

Algorithm 5. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho E(\mathcal{T}(\mu_{n+1}), \eta(\nu, \mu_{n+1}))\Big],$$

which is equivalent to the following two-step method.

Algorithm 6. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho E(\mathcal{T}(\mu_n), \eta(\nu, \mu_n))\Big] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho E(\mathcal{T}(z_n), \eta(\nu, z_n))\Big]. \end{aligned}$$

We can rewrite the equation (6) as:

$$\mu = \mu - g(\mu) + \Big[g\big(\frac{\mu+\mu}{2}\big) - \rho E\big(\mathcal{T}\big(\big(\frac{\mu+\mu}{2}\big), \eta(\nu, \frac{\mu+\mu}{2}\big)\big)\Big].$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 7. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \mu_{n+1} &= \mu_n - g(\mu_n) + \Big[g\big(\frac{\mu_n + \mu_{n+1}}{2}\big) \\ &\quad -\rho E\big(\mathcal{T}\big(\frac{\mu_n + \mu_{n+1}}{2}\big), \eta(\nu, \frac{\mu_n + \mu_{n+1}}{2}\big)\big)\Big]. \end{aligned}$$

Applying the predictor-corrector technique, we suggest the following inertial iterative method for solving the problem (2).

Algorithm 8. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} z_n &= \mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho E(\mathcal{T}(\mu_n), \eta(\nu, \mu_n))\Big] \\ \mu_{n+1} &= \mu_n - n(\mu_n) + \Big[g\big(\frac{z_n + \mu_n}{2}\big) - \rho E\big(\mathcal{T}\big(\frac{z_n + \mu_n}{2}\big), \eta(\nu, \frac{z_n + \mu_n}{2}\big)\big)\Big]. \end{aligned}$$

We now suggest multi-step inertial methods for solving the general preinvex equilibrium equations (2).

Algorithm 9. *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$
\begin{aligned}
&z_n = \mu_n - \theta_n(\mu_n - \mu_{n-1}), \quad n = 1, 2, \ldots\ldots \\
&y_n = (1-\gamma_n)z_n \\
&+\gamma_n\Big\{z_n - g(z_n) + \Big[g\big(\frac{z_n+\mu_n}{2}\big) - \rho E\big(\mathcal{T}\big(\big(\frac{z_n+\mu_n}{2}\big), \eta\big(\nu, \frac{z_n+\mu_n}{2}\big)\big)\Big]\Big\}, \\
&t_n = (1-\beta_n)y_n + \beta_n\Big\{y_n - g(y_n) \\
&+\Big[g\big(\frac{y_n+z_n+\mu_n}{3}\big) - \rho E\big(\mathcal{T}\big(\big(\frac{y_n+z_n+\mu_n}{3}\big), \eta\big(\nu, \frac{y_n+z_n+\mu_n}{3}\big)\big)\Big]\Big\}, \\
&\mu_{n+1} = (1-\alpha_n)z_n + \alpha_n\big(t_n - g(t_n)\big) \\
&+\alpha_n\Big[g\big(\frac{z_n+y_n+t_n+\mu_n}{4}\big) \\
&-\rho E\big(\mathcal{T}\big(\big(\frac{y_n+z_n+t_n+\mu_n}{4}\big), \eta\big(\nu, \frac{y_n+z_n+t_n+\mu_n}{4}\big)\big)\Big],
\end{aligned}
$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1$.

We can rewrite the equation (6) as:

$$
\begin{aligned}
\mu = \mu - g((1-\alpha)\mu + \alpha\mu) + \big[&g((1-\alpha)\mu + \alpha\mu) \\
&-\rho E\big(\mathcal{T}((1-\alpha)\mu + \alpha\mu)), \eta(\nu, (1-\alpha)\mu + \alpha\mu)\big)\big], \quad \forall \nu \in \mathcal{H},
\end{aligned}
$$

where α is a constant. This form is used to suggest the following iterative method.

Algorithm 10. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$
\begin{aligned}
\mu_{n+1} = \mu_n - ((1-\alpha)\mu_{n-1} + \alpha\mu_n) + \big[&g((1-\alpha)\mu_{n-1} + \alpha\mu_n) \\
&-\rho E\big(\mathcal{T}((1-\alpha)\mu_{n-1} + \alpha\mu_n)), \eta(\nu, (1-\alpha)\mu_{n-1} + \alpha\mu_n)\big)\big], \quad \forall \nu \in \mathcal{H},
\end{aligned}
$$

which is equivalent to the following inertial iterative method.

Algorithm 11. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$
\begin{aligned}
y_n &= (1-\alpha)\mu_{n-1} + \alpha\mu_n) \quad n = 1, 2, \ldots\ldots \\
z_n &= \mu_n - g(y_n) + \big[g(y_n) - \rho E\big(\mathcal{T}(y_n), \eta(\nu, y_n)\big)\big], \\
t_n &= \mu_n - g(y_n + z_n) + \big[g(y_n + z_n) - \rho E\big(\mathcal{T}(y_n + z_n), \eta(\nu, y_n + z_n)\big)\big] \\
\mu_{n+1} &= \mu_n - g(y_n + z_n + t_n) + \big[g(y_n + z_n + t_n) \\
&\qquad -\rho E\big(\mathcal{T}(y_n + z_n + t_n), \eta(\nu, y_n + z_n + t_n)\big)\big], \quad \forall \nu \in \mathcal{H}.
\end{aligned}
$$

Using the above technique, one can investigate the convergence analysis of these inertial iterative methods. We would like to mention that Algorithm 9, Algorithm 8 and Algorithm 11 can be viewed as the new generalizations of Noor (three-step) iterations [28, 31] for solving the fixed point problem. Similar multi-step hybrid iterative methods can be proposed and analyzed for solving the general preinvex equilibrium equations (2), which is an interesting problem.

4 Dynamical systems technique

In this section, we consider the dynamical systems technique for solving the general nonlinear equilibrium equations. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [10]. It is worth mentioning that the dynamical system is a first order initial value problem. Consequently, variational inequalities and nonlinear problems arising in various branches in pure and applied sciences can now be studied via the differential equations. It has been shown [10, 16, 28, 31, 42, 43, 46–48, 52, 53, 64, 65] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. We introduce and consider some first order, second order and third order dynamical systems associated with the general preinvex equilibrium equations, which are used to study the global stability of the existence of the solution of the general preinvex equilibrium equations and to suggest several iterative methods for solving these preinvex equilibrium equations.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \left\{ \left[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu)) \right] - g(\mu) \right\}, \quad \forall \nu \in \mathcal{H}. \tag{26}$$

One can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem (2), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{27}$$

We now consider a dynamical system associated with the general preinvex equilibrium equations. Using the equivalent formulation (3.1), we suggest a class of dynamical systems as

$$\frac{d\mu}{dt} = \lambda \left\{ \left[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu)) \right] - g(\mu) \right\}, \quad \mu(t_0) = \alpha, \tag{28}$$

where λ and α are constant. The system of type (28) is called the system of general dynamical system associated with the problem (2). From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (2) can be studied. The equilibrium point of the dynamical system (28) is defined as follows.

The equilibrium point of the dynamical system (28) is defined as follows.

Definition 3. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (28), if,*

$$\frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the general preinvex equilibrium equations (2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point. This implies that $\mu \in \mathcal{H}$ is a solution of the preinvex equilibrium equations (2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Definition 4. *[10] The dynamical system is said to converge to the solution set S^* of (28), if , irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t\to\infty} dist(\mu(t), S^*) = 0, \tag{29}$$

where

$$dist(\mu, S^*) = inf_{\nu\in S^*}\|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (29) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 5. *The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|\mu(t) - \mu^*\| \leq \mu_1\|\mu(t_0) - \mu^*\|exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where μ_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 1. *(Gronwall Lemma) [10,17]. Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t-t_0|)$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t)exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now establish that the trajectory of the solution of the dynamical system (28) converges to the unique solution of the general preinvex equilibrium equation (2). The analysis is in the spirit of Noor [31,32], Noor et al. [42,43,52,53] and Xia and Wang [65].

Theorem 3. *Let the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ and the bifunction $E(\mathcal{T}(.),.))$ be Lipschitz continuous with constants $\delta > 0, \beta > 0$, respectively. If $\lambda(2\delta + \rho\beta) < 1$, then, for each $\mu \in \mathcal{H}$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (28) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \lambda\Big\{\big[g(\mu) - \rho\big(E(\mathcal{T}(\mu), \eta(\nu, \mu))\big] - g(\mu)\Big\}, \quad \forall \mu, \nu \in H,$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$.

For $\forall \mu \neq \eta \in H$, we have

$$\begin{aligned}
\Big\|G(\mu) - G(\zeta)\Big\| &\leq \lambda\Big\{\Big\|\Big[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu))\Big] - \Big[g(\nu) \\
&\quad -\rho(E(\mathcal{T}(\nu), \eta(\nu, \mu)))\Big]\Big\| + \Big\|g(\mu) - g(\zeta)\Big\|\Big\} \\
&\leq \lambda\Big\{\Big\|g(\mu) - g(\zeta)\Big\| \\
&\quad + \Big\|\Big[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu))\Big] - \Big[g(\zeta) - \rho E(\mathcal{T}(\zeta), \nu)\Big]\Big\|\Big\} \\
&\leq \lambda\Big\{\Big\|g(\mu) - g(\zeta)\Big\| + \Big\|g(\mu) - g(\nu) - \rho(E(\mathcal{T}(\mu), \nu) - (E(\mathcal{T}(\eta), \nu)\Big\|\Big\} \\
&\leq \lambda\Big\{\Big\|g(\mu) - g(\zeta)\Big\| + \Big\|g(\mu) - g(\zeta)\Big\| + \rho\Big\|E(\mathcal{T}(\mu), \nu) - E(\mathcal{T}(\zeta), \nu)\Big\|\Big\} \\
&\leq \lambda\{(2\delta + \rho\beta)\}\Big\|\mu - \eta\Big\|.
\end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant $\lambda\{(2\delta + \rho\beta)\} < 1$ and for each $\mu \in \mathcal{H}$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (28), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider, for any $\mu \in \mathcal{H}$,

$$\Big\|G(\mu)\Big\| = \Big\|\frac{d\mu}{dt}\Big\| \leq \lambda\Big\|\big[g(\mu) - \rho\big(E(\mathcal{T}(\mu), \eta(\nu, \mu))\big] - g(\mu)\Big\| \leq \lambda\rho\beta.$$

Then

$$\big\|\mu(t)\big\| \leq \big\|\mu_0\big\| + \int_{t_0}^{t} \big\|\mu(s)\big\|ds \leq (\big\|\mu_0\big\| + k_2 \int_{t_0}^{t} \big\|\mu(s)\big\|ds,$$

where $k_2 = \lambda\rho\beta$.
Hence by the Gronwall Lemma 1, we have

$$\|\mu(t)\| \leq \|\mu_0\|e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 4. *Let the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ be strongly monotone with constant $\sigma > 0$ and $\delta > 0$. If the bifunction $E(.,.)$ is Lipschitz continuous with constants $\beta > 0$, then the dynamical system (28) converges globally exponentially to the unique solution of the general preinvex equilibrium equations (2).*

Proof. Since the the bifunction $E(.,.)$ and the operator g are Lipschitz continuous, it follows from Theorem 3 that the dynamical system (28) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in \mathcal{H}$. Let $\mu(t)$ be a solution of the initial value problem (28). For a given $\mu^* \in H$ satisfying (2), consider the Lyapunov function

$$L(\mu) = \lambda \|\mu(t) - \mu^*\|^2, \quad u(t) \in \mathcal{H}. \tag{30}$$

From (28) and (30), we have

$$\begin{aligned}
\frac{dL}{dt} &= 2\lambda \left\langle \mu(t) - \mu^*, \frac{d\mu}{dt} \right\rangle \\
&= 2\lambda \left\langle \mu(t) - \mu^*, \Big[g(\mu(t)) - \rho E(\mathcal{T}(\mu(t)), \eta(\nu, \mu(t))) \Big] - g(\mu(t)) \right\rangle \\
&= 2\lambda \Big\langle \mu(t) - \mu^*, \Big[g(\mu(t)) - \rho E(\mathcal{T}(\mu(t)), \eta(\nu, \mu(t)) \Big] \\
&\quad - g(\mu^*) + g(\mu^*(t)) - g(\mu(t)) \Big\rangle \\
&= -2\lambda \Big\langle \mu(t) - \mu^*(t), g(\mu(t)) - g(\mu^*(t)) \Big\rangle \\
&\quad + 2\lambda \Big\langle \mu(t) - \mu^*(t), \Big[g(\mu(t)) - \rho E(\mathcal{T}(\mu(t)), \eta(\nu, \mu(t))) \Big] - g(\mu^*(t)) \Big\rangle \\
&\leq -2\lambda \Big\langle \rho(E\big(\mathcal{T}(\mu(t)), \eta(\nu, \mu(t)) \\
&\qquad - E(\mathcal{T}(\mu^*(t)), \eta(\nu, \mu^*(t)), g(\mu(t)) - g(\mu^*) \Big\rangle \\
&\qquad + 2\lambda \Big\langle \mu(t) - \mu^*(t), \Big[g(\mu(t)) - \rho E\big(\mathcal{T}(\mu(t)), \eta(\nu, \mu(t))\big) \Big] - \Big[g(\mu^*(t)) \\
&\qquad - \rho E(\mathcal{T}(\mu^*(t)), \eta(\nu, \mu^*(t))\big) \Big] \Big\rangle, \\
&\leq -2\lambda\sigma \Big\| \mu(t) - \mu^* \Big\|^2 + \lambda \Big\| g(\mu(t)) - g(\mu^*) \Big\|^2 \\
&\qquad + \lambda \Big\| \Big[g(\mu(t)) - \rho E\big(\mathcal{T}(\mu(t)), \eta(\nu, \mu(t))\big) \Big] \\
&\qquad\quad - \Big[g(\mu^*(t)) - \rho E(\mathcal{T}(\mu^*(t)), \eta(\nu, \mu^*(t))\big) \Big] \Big\|^2.
\end{aligned} \tag{31}$$

Using the Lipschitz continuity of the bifunction $E(\mathcal{T}(.,.)$ and the operator

g, we have

$$\begin{aligned}
&\Big\|\Big[g(\mu(t)) - \rho E\big(\mathcal{T}(\mu(t)), \eta(\nu, \mu(t))\big)\Big] \\
&\qquad -\Big[g(\mu^*(t)) - \rho E\big(\mathcal{T}(\mu^*(t)), \eta(\nu, \mu^*(t))\big)\Big]\Big\| \\
&\leq \Big\|g(\mu(t)) - g(\mu^*(t)) - \rho(E\big(\mathcal{T}(\mu(t)), \eta(\nu, \mu(t))\big) \\
&\qquad -E\big(\mathcal{T}(\mu^*(t)), \eta(\nu, \mu^*(t))\big)\Big\| \\
&\leq \Big\|g(\mu(t)) - g(\mu^*(t))\Big\| + \rho\Big\|E\big(\mathcal{T}(\mu(t)), \eta(\nu, \mu(t))\big) \\
&\qquad -E\big(\mathcal{T}(\mu^*(t)), \eta(\nu, \mu^*(t))\big)\Big\| \\
&\leq (\delta + \rho\beta)\Big\|\mu(t) - \mu^*(t)\Big\|. \qquad (32)
\end{aligned}$$

From (31) and (32), we have

$$\frac{d}{dt}\Big\|\mu(t) - \mu^*\Big\| \leq 2\xi\lambda\Big\|\mu(t) - \mu^*\Big\|,$$

where

$$\xi = (\delta + \rho\beta - 2\sigma).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\Big\|\mu(t) - \mu^*\Big\| \leq \Big\|\mu(t_0) - \mu^*\Big\|e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (28) converges globally exponentially to the unique solution of the general preinvex equilibrium equations (2). □

We use the dynamical system (28) to suggest some iterative for solving the general preinvex euilibrium equations (2). These methods can be viewed in the sense of Noor [27,31] involving the double projection.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (28) becomes

$$\frac{d\mu}{dt} + g(\mu) = \Big[g(\mu) - \rho E\big(\mathcal{T}(\mu), \eta(\nu, \mu)\big)\Big], \quad \mu(t_0) = \alpha. \qquad (33)$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (33), we have

$$\frac{\mu_{n+1}-\mu_n}{h_1}+g(\mu_n)=\Big[g(\mu_n)-\rho E(\mathcal{T}(\mu_{n+1}),\eta(\nu,\mu_{n+1}))\Big], \tag{34}$$

where h_1 is the step size.

Now, we can suggest the following implicit iterative method for general preinvex equilibrium equations (2).

Algorithm 12. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n-g(\mu_n)+\left[g(\mu_n)-\rho E(\mathcal{T}(\mu_{n+1}),\eta(\nu,\mu_{n+1}))-\frac{\mu_{n+1}-\mu_n}{h_1}\right].$$

This is an implicit method and is equivalent to the following two-step method.

Algorithm 13. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n-g(\mu_n)+\Big[g(\mu_n)-\rho E\big(\mathcal{T}(\mu_n),\eta(\nu,\mu_n)\big)\Big]\\ \mu_{n+1} &= \mu_n-g(\mu_n)+\left[g(\mu_n)-\rho E\big(\mathcal{T}(y_n),\eta(\nu,y_n)\big)-\frac{y_n-\mu_n}{h_1}\right]. \end{aligned}$$

Discretizing (28), we now suggest an other implicit iterative method for solving the general preinvex equilibrium equations (2).

$$\frac{\mu_{n+1}-\mu_n}{h_1}+g(\mu_n)=\Big[g(\mu_{n+1})-\rho E\big(\mathcal{T}(\mu_{n+1}),\eta(\nu,\mu_{n+1})\big)\Big], \tag{35}$$

where h_1 is the step size.

This formulation enables us to suggest the two-step iterative method.

Algorithm 14. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n-g(\mu_n)+\Big[g(\mu_n)-\rho E\big(\mathcal{T}(\mu_n),\eta(\nu,\mu_n)\big)\Big]\\ \mu_{n+1} &= \mu_n-g(\mu_n)+\left[g(y_n)-\rho E\big(\mathcal{T}(y_n),\eta(\nu,y_n)\big)-\frac{y_n-\mu_n}{h_1}\right]. \end{aligned}$$

From (33), we have

$$\begin{aligned} \frac{d\mu}{dt}+g(\mu)=\Big[&g((1-\alpha)\mu+\alpha\mu)\\ &-\rho E\big(\mathcal{T}((1-\alpha)\mu+\alpha\mu)),\eta(\nu,(1-\alpha)\mu+\alpha\mu)\big)\Big], \end{aligned} \tag{36}$$

where $\alpha \in [0,1]$ is a constant.

Discretizing (36) and taking $h_1 = 1$, we have

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Big[g\big((1-\alpha)\mu_n + \alpha\mu_{n-1}\big) - \rho E\big(\mathcal{T}((1-\alpha)\mu_n) + \alpha\mu_{n-1}), \eta(\nu, (1-\alpha)\mu_n + \alpha\mu_{n-1})\big)\Big],$$

which is an inertial type iterative method for solving the general preinvex equilibrium equations (2). Using the predictor-corrector techniques, we have

Algorithm 15. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1}, \quad n = 1, 2, \ldots \\ z_n &= \mu_n - g(\mu_n) + \Big[g(y_n) - \rho E\big(\mathcal{T}(y_n), \eta(\nu, y_n)\big)\Big], \\ \mu_{n+1} &= \mu_n - g(z_n) + \Big[g(y_n) - \rho E(\mathcal{T}(z_n), \eta(\nu, z_n))\Big], \end{aligned}$$

which is known as the inertial three-step iterative method.

We now introduce the second order dynamical system associated with the general preinvex equilibrium equations (2). To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$ such that

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = \lambda\Big\{\Big[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu))\Big] - g(\mu)\Big\}, \qquad (37)$$
$$\mu(a) = \alpha, \mu(b) = \beta,$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (37) is indeed a second order boundary vale problem. In a similar way, we can define the second order initial value problem associated with the dynamical system.

The equilibrium point of the dynamical system (37) is defined as follows.

Definition 6. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (37), if,*

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the system of general preinvex equilibrium equations (2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

From (37), we have

$$g(\mu) = [g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu))].$$

Thus, we can rewrite (37) as follows:

$$g(\mu) = \left[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu)) + \gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\right]. \tag{38}$$

For $\lambda = 1$, the problem (37) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$\begin{aligned} \gamma \frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} + g(\mu) &= \left[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu, \mu))\right], \\ \mu(a) = \alpha, \quad \mu(b) &= \beta. \end{aligned} \tag{39}$$

The problem (39) is called the second order dynamical system, which is in fact a second order boundary value problem. This interlink among various fields of mathematical and engineering sciences is fruitful in developing implementable numerical methods for finding the approximate solutions of the general preinvex equilibrium equations. Consequently, one can explore the ideas and techniques of the differential equations to suggest and propose hybrid proximal point methods for solving the general preinvex equilibrium equations and related optimization problems.

We discretize the second-order dynamical systems (39) using central finite difference and backward difference schemes to have

$$\gamma \frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h_1^2} + \frac{\mu_n - \mu_{n-1}}{h_1} + g(\mu_n) = \left[\mu_n - \rho E\big(\mathcal{T}(\mu_{n+1}), \eta(\nu, \mu_{n+1})\big)\right], \tag{40}$$

where h is the step size.

If $\gamma = 1, h_1 = 1$, then, from equation(40) we have:

Algorithm 16. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \left[g(\mu_n) - \rho E(\mathcal{T}(\mu_{n+1}), \eta(\nu, \mu_{n+1}))\right],$$

which is the iterative method for solving the general preinvex equilibrium equations (1). Algorithm 16 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 17. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n\mu_{n-1}, \quad n = 1, 2, \ldots \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Big[g(\mu_n) - \rho E\big(\mathcal{T}(y_n), \eta(\nu, y_n)\big)\Big], \end{aligned}$$

is called the two-step inertial iterative method, where $\theta_n \in [0,1]$ is a constant.

We discretize the second-order dynamical systems (28) using central finite difference and backward difference schemes to have

$$\begin{aligned} &\gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h_1^2} + \frac{\mu_n - \mu_{n-1}}{h_1} + g(\mu_{n+1}) \\ &\qquad = \Big[g(\mu_n) - \rho E\big(\mathcal{T}(\mu_{n+1}), \eta(\nu, \mu_{n+1})\big)\Big], \end{aligned}$$

where h is the step size.

Using this discrete form, we can suggest the following an iterative method for solving the general preinvex equilibrium equations (2).

Algorithm 18. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \mu_{n+1} &= \mu_n - g(\mu_{n+1}) \\ &\quad + \Big[g(\mu_{n+1}) - \rho E\big(\mathcal{T}(\mu_{n+1}), \eta(\nu, \mu_{n+1})\big) \\ &\quad - \gamma\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h_1^2} + \frac{\mu_n - \mu_{n-1}}{h_1}\Big]. \end{aligned}$$

Algorithm 18 is called the hybrid inertial proximal method for solving the general preinvex equilibrium equations and related optimization problems. This is a new proposed method.

Note that, for $\gamma = 0, h_1 = 1$, Algorithm 18 reduces to the following iterative method.

Algorithm 19. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - (\mu_{n+1}) + \Big[g(\mu_{n+1}) - \rho E\big(\mathcal{T}(\mu_{n+1}), \eta(\nu, \mu_{n+1})\big) + (\mu_n - \mu_{n-1})\Big],$$

which is called the inertial iterative method.

We now consider the third order dynamical systems associated with the

general preinvex equilibrium equations of the type (2).
To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\begin{aligned} \gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} + g(\mu) \\ = \Big[g(\mu) - \rho E\big(\mathcal{T}(\mu), \eta(\nu,\mu)\big)\Big], \\ u(a) = \alpha,\ \dot{\mu}(a) = \beta,\ \dot{\mu}(b) = 0, \end{aligned} \tag{41}$$

where $\gamma > 0, \zeta, \xi$ and $\rho > 0$ are constants. Problem (41) is called third order dynamical system associated with general preinvex equilibrium equations (2).
The equilibrium point of the dynamical system (41) is defined as follows.

Definition 7. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (37), if,*

$$\gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} = 0.$$

Thus, it is clear that $\mu \in \mathcal{H}$ is a solution of the general preinvex equilibrium equations (2), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.

Consequently, the problem (41) can be written as

$$\begin{aligned} g(\mu) = \Big[g(\mu) - \rho E(\mathcal{T}(\mu), \eta(\nu,\mu)) + \gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt}\Big]. \\ u(a) = \alpha,\ \dot{\mu}(a) = \beta,\ \dot{\mu}(b) = 0. \end{aligned} \tag{42}$$

We discretize the third-order dynamical systems (41) using central finite difference and backward difference schemes to have

$$\begin{aligned} \mu_{n+1} = \mu_n - g(\mu_n) \\ + \Big[g(\mu_n) - \rho E(\mathcal{T}(\mu_{n+1}), \nu) + \gamma\frac{\mu_{n+2} - 2\mu_{n+1} + 2\mu_{n-1} - \mu_{n-2}}{2h_1^3} \\ + \zeta\frac{\mu_{n+1} - 2\mu_n + \mu_{n-1}}{h_1^2} + \xi\frac{3\mu_n - 4\mu_{n-1} + \mu_{n-2}}{2h_1} + \Big], \end{aligned} \tag{43}$$

where h_1 is the step size.
If $\gamma = 1, h_1 = 1, \zeta = 1, \xi = 1$, then, from equation(43) after adjustment, we have:

Algorithm 20. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - h(\mu_n) + \Big[g(\mu_n) - \rho E\big(\mathcal{T}(\mu_{n+1}), \eta(\nu, \mu_{n+1})\big) + \frac{\mu_{n-1} - 3\mu_n}{2}\Big],$$

which is an inertial type hybrid implicit iterative method for solving the general preinvex equilibrium equations (2).

Using the predictor-corrector technique, we now suggest multi step inertial iterative method for solving the general preinvex equilibrium equations (2).

Algorithm 21. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}
z_n &= \mu_n - \theta_n(\mu_n - \mu_{n-1}), \quad n = 1, 2, \ldots\ldots \\
y_n &= (1-\gamma_n)z_n \\
&\quad +\gamma_n\Big\{z_n - g(z_n) + \big[g(z_n) - \rho E\big(\mathcal{T}(z_n), \eta(\nu, z_n)\big)\big]\Big\}, \\
t_n &= (1-\beta_n)y_n \\
&\quad +\beta_n\Big\{y_n - h(y_n) + \Big[g(y_n) - \rho E\big(\mathcal{T}(y_n), \eta(\nu, y_n)\big)\Big]\Big\} \\
w_n &= (1-\zeta)t_n \\
&\quad +\zeta_n\Big\{t_n - g(t_n) \\
&\quad +\Big[g(t_n) - \rho E\big(\mathcal{T}(t_n), \eta(\nu, t_n)\big)\Big]\Big\} \\
\mu_{n+1} &= (1-\alpha_n)w_n \\
&\quad +\alpha_n\Big\{w_n - g(w_n) + \Big[g(\mu_n) - \rho E\big(\mathcal{T}(w_n), \eta(\nu, w_n)\big)\Big]\Big\},
\end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \zeta_n, \theta_n \in [0,1], \quad \forall n \geq 1.$

For $g = I$, Algorithm 21 collapses to the following multi step inertial hybrid method for solving the general preinvex equilibrium equations (3).

Algorithm 22. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}
z_n &= \mu_n - \theta_n(\mu_n - \mu_{n-1}), \quad n = 1, 2, \ldots\ldots \\
y_n &= (1-\gamma_n)z_n + \gamma_n\Big\{z_n - \rho E\big(\mathcal{T}(z_n), \eta(\nu, z_n)\big)\Big\}, \\
t_n &= (1-\beta_n)y_n + \beta_n\Big\{y_n - \rho E\big(\mathcal{T}(y_n), \eta(\nu, y_n)\big)\Big\} \\
w_n &= (1-\zeta)t_n + \zeta_n\Big\{t_n - \rho E\big(\mathcal{T}(t_n), \eta(\nu, t_n)\big)\Big\} \\
\mu_{n+1} &= (1-\alpha_n)w_n + \alpha_n\Big\{w_n - \rho E\big(\mathcal{T}(w_n), \eta(\nu, w_n)\big)\Big\},
\end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \zeta_n, \theta_n \in [0,1], \quad \forall n \geq 1.$

Remark 2. *Algorithms 21 and Algorithm 22 are quite general and include the Noor (three-step) iterations, Ishikawa (two-step) iterations, Mann iteration and Picard method for solving the variational inequalities, fixed point problems, complementarity problems and preinvex equilibrium equations as special cases. For appropriate and suitable choice of the operators $\mathcal{T}, h, g$, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving fixed point problem and related optimization problems. Using the techniques and ideas of Noor et al. [32,33,42], one can discuss the convergence analysis of the proposed methods*

In this section, we have tried to convey the idea that the alternative equivalent fixed point formulation of the general preinvex equilibrium equations, can be written several ways to suggest and investigate several extragradient type computational methods. It is expected to play an important role in developing numerically efficient methods for variational inequalities, equilibrium problems and nonconvex optimization problems.

5 Auxiliary principle technique

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can be applied for suggesting the iterative methods for solving the several nonlinear variational inequalities and equilibrium problems. To overcome these drawbacks, one usually applies the auxiliary principle technique, which is mainly due to Glowinski et al. [12] as developed in [25,30,33,45,49,52,53,57]. Noor [25] has modified the auxiliary principle technique involving an arbitrary function M. For the readers convenience, we recall some basic properties of the modified distance functions, which were introduced by Noor [25] in 1992 and discussed in Patrikson [57]. For an arbitrary operator M, we define the distance function as

$$\mathcal{M}(\nu, \mu) = \langle M(\nu) - M(\mu), \nu - \mu \rangle, \quad \forall \mu, \nu \in \mathcal{H}. \tag{44}$$

If the operator M is strongly monotone with constant $\zeta \geq 0$, then

$$\mathcal{M}(\nu, \mu) = \langle M(\nu) - M(\mu), \nu - \mu \rangle \geq \zeta \|\nu - \mu\|^2, \quad \mu, \nu \in \mathcal{H}.$$

It is important to emphasize that various types of function M give different modified distance functions. For example, if $M(\mu) = E'(\mu)$,then the modified distance function $\mathcal{M}$ reduces to Bregman distance function, see [57]. We

give the following important examples of some practically important types of operator M and their corresponding distance functions.

Examples

1. If $\Phi(\nu) = \|\nu\|^2$, then $\mathcal{M}(\nu,\mu) = \|\nu - \mu\|$, which is the squared Euclidean distance (SE).

2. If $\Phi(\nu) = \sum_{i=1}^{n} a_i \log(\nu_i)$, which is known as the Shannon entropy, then its corresponding $\mathcal{M}$ distance function is given as

$$\mathcal{M}(\nu,\mu) = \sum_{i=1}^{n} \left(v_i \log(\frac{\nu_i}{\mu_i}) + \mu_i - \nu_i \right).$$

   This distance is called Kullback-Leibler distance (KL) and plays a very important tool in several areas of applied mathematics such as machine learning and data sciences.

3. If $\Phi(\nu) = -\sum_{i=1}^{n} \log(\nu_i)$, which is called the Burg entropy, then its corresponding modified distance function is given as

$$\mathcal{M}(\nu,\mu) = \sum_{i=1}^{n} \left(\log(\frac{\nu_i}{\mu_i}) + \frac{\nu_i}{\mu_i} - 1 \right).$$

   This is called the Itakura-Saito distance (IS), which is very important in information theory and machine learning. It is a challenging problem to explore the applications of modified $\mathcal{M}$ distance functions for other types of nonconvex functions as biconvex, k-convex functions, general convex functions and harmonic functions.

We apply the modified auxiliary principle technique involving an arbitrary operator for finding the approximate solution of the preinvex equilibrium problem (1).

For a given $\mu \in \mathcal{H}$ satisfying (1), find $w \in \mathcal{H}$ such that

$$\begin{aligned} &\rho E\big(\mathcal{T}(w + \zeta(\mu - w)), \eta(\nu, w + \zeta(\mu - w))\big) \\ &\quad + \langle M(w) - M(\mu), \nu - w\rangle = 0, \quad \forall \nu \in \mathcal{H}, \end{aligned} \tag{45}$$

where $\rho > 0, \eta \in [0,1]$ are constants and M is an arbitrary operator. The inequality (45) is called the auxiliary preinvex equilibrium equation.

If $w = \mu$, then w is a solution of (1). This simple observation enables us to suggest the following iterative method for solving (1).

Algorithm 23. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}\rho E\big(\mathcal{T}(\mu_{n+1}+\zeta(\mu_n-\mu_{n+1})),\eta(\nu,\mu_{n+1}+\eta(\mu_n-\mu_{n+1}))\big)\\ +\langle M(\mu_{n+1})-M(\mu_n),\nu-\mu_{n+1}\rangle=0,\quad \forall\nu\in\mathcal{H}.\end{aligned}\tag{46}$$

Algorithm 23 is called the hybrid proximal point algorithm for solving the general preinvex equilibrium equations (1).

Special Cases

(I). For $\zeta=0$, Algorithm 23 reduces to:

Algorithm 24. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho E\big(\mathcal{T}(\mu_{n+1}),\eta(\nu,\mu_{n+1})\big)\\ &\quad+\langle M(\mu_{n+1})-M(\mu_n),\nu-\mu_{n+1}\rangle=0,\quad \forall\nu\in\mathcal{H},\end{aligned}\tag{47}$$

is called the implicit iterative methods for solving the preinvex equilibrium problem (1).

(II). For $\zeta=\frac{1}{2}$, Algorithm 23 becomes:

Algorithm 25. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho E\big(\mathcal{T}(\frac{\mu_{n+1}+(\mu_n)}{2}),\eta(\nu,\frac{\mu_n+\mu_{n+1}}{2})\big)\\ &\quad+\langle M(\mu_{n+1})-M(\mu_n),\nu-\mu_{n+1}\rangle\geq 0,\quad \forall\nu\in\mathcal{H},\end{aligned}$$

is known as the mid-point proximal method for solving the problem (1).

(III). For $\zeta=1$, Algorithm 23 reduces to:

Algorithm 26. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho E\big(\mathcal{T}(\mu_n),\eta(\nu,\mu_n)\big)\\ &\quad+\langle M(\mu_{n+1})-M(\mu_n),\nu-\mu_{n+1}\rangle=0,\quad \forall\nu\in\mathcal{H},\end{aligned}$$

is called the explicit iterative methods for solving the problem (1).

(IV). If $M=I$, the identity operator, Algorithm 23 reduces to:

Algorithm 27. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}\rho E\big(\mathcal{T}(\mu_{n+1}+\zeta(\mu_n-\mu_{n+1})),\eta(\nu,\mu_{n+1}+\zeta(\mu_n-\mu_{n+1})\big)\\ +\langle \mu_{n+1}-\mu_n,\nu-\mu_{n+1}\rangle=0,\quad \forall \nu\in\mathcal{H},\end{aligned}\tag{48}$$

which appears to be a new method for solving the preinvex equilibrium equations (1). For suitable and appropriate choice of the parameter η one, can obtain different approximate methods for solving the preinvex equilibrium equations (1) and their variant forms.

For the convergence analysis of Algorithm 24, we need the following concepts.

Definition 8. *The bifunction $E(\mathcal{T}(.),.)$ is said to be pseudomonotone, if*

$$E(\mathcal{T}(\mu),\eta(\nu,\mu))=0\longrightarrow -F(\mathcal{T}(\nu),\eta(\mu,\nu))=0,\quad \forall \mu,\nu\in\mathcal{H}.$$

Theorem 5. *Let the bifunction $E(\mathcal{T}(.),.)$ be a pseudo-monotone. Let the approximate solution μ_{n+1} obtained from Algorithm 24 and let μ be the exact solution $\mu\in\mathcal{H}$ of the problem (1). If the operator M is strongly monotone with constant $\xi\geq 0$ and Lipschitz continuous with constant $\delta\geq 0$, then*

$$\xi\|\mu_{n+1}-\mu_n\|\leq \delta\|\mu-\mu_n\|.\tag{49}$$

Proof. Let $\mu\in\mathcal{H}$ be a solution of the problem (1). Then

$$-E\big(\mathcal{T}(\nu),\eta(\mu,\nu)\big)=0,\tag{50}$$

since the bifunction $E(\mathcal{T}(.),.)$ is pseudomonotone.
Taking $\nu=\mu_{n+1}$ in (50), we obtain

$$-E\big(\mathcal{T}(\mu_{n+1}),\eta(\mu,\mu_{n+1})\big)=0.\tag{51}$$

Setting $\nu=\mu$ in (47), we have

$$\begin{aligned}\rho E\big(\mathcal{T}(\mu_{n+1}),\eta(\mu,\mu_{n+1})\big)\\ +\langle M(\mu_{n+1})-M(\mu_{n+1}),\mu-\mu_{n+1}\rangle=0,\quad \forall \nu\in\mathcal{H},\end{aligned}\tag{52}$$

Combining (52) and (51), we have

$$\begin{aligned}\langle M(\mu_{n+1})-M(\mu_n),\mu-\mu_{n+1}\rangle\\ =-E\big((\mathcal{T}(\mu_{n+1})),\eta(\nu,\mu_{n+1})\big)=0.\end{aligned}\tag{53}$$

From the equation (53), we have

$$\begin{aligned} 0 &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n + \mu_n - u_{n+1}\rangle \\ &= \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle + \langle M(\mu_{n+1} - M(\mu_n), \mu_n - \mu_{n+1}\rangle, \end{aligned}$$

which implies that

$$\langle M(\mu_{n+1} - M(\mu_n), \mu_{n+1} - \mu_n\rangle = \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle.$$

Now using the strong monotonicity with constant $\xi > 0$ and Lipschitz continuity with constant δ of the operator M, we obtain

$$\xi\|\mu_{n+1} - \mu_n\|^2 \leq \delta\|\mu_{n+1} - \mu_n\|\|\mu_n - \mu\|.$$

Thus

$$\xi\|\mu_n - \mu_{n+1}\| \leq \delta\|\mu_n - \mu\|,$$

the required result (49). □

Theorem 6. *Let $\mathcal{H}$ be a finite dimensional space and all the assumptions of Theorem 5 hold. Then the sequence $\{\mu_n\}_0^\infty$ given by Algorithm 24 converges to the exact solution $\mu \in \mathcal{H}$ of (1).*

Proof. Let $\mu \in \mathcal{H}$ be a solution of (1). From (49), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\xi \sum_{n=0}^{\infty} \|\mu_{n+1} - \mu_n\| \leq \delta\|\mu_n - \mu\|,$$

which implies that

$$\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0. \tag{54}$$

Let $\hat{\mu}$ be the limit point of $\{\mu_n\}_0^\infty$; whose subsequence $\{\mu_{n_j}\}_1^\infty$ of $\{\mu_n\}_0^\infty$ converges to $\hat{\mu} \in \Omega$. Replacing w_n by μ_{n_j} in (47), taking the limit $n_j \longrightarrow \infty$ and using (54), we have

$$g(\hat{\mu}) = g(\hat{\mu}) - \rho E\big(\mathcal{T}(\hat{\mu}), \eta(\nu, \hat{u})\big), \quad \forall \nu \in \mathcal{H},$$

which implies that $\hat{u}$ solves the problem (1) and

$$\|\mu_{n+1} - \mu\| \leq \|\mu_n - \mu\|.$$

Thus, it follows from the above inequality that $\{\mu_n\}_1^{\infty}$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty}(\mu_n) = \hat{\mu}.$$

the required result. □

We again apply the modified auxiliary principle approach involving an arbitrary nonlinear operator to suggest some hybrid inertial proximal point schemes for solving the general preinvex equilibrium equations (1).

For a given $\mu \in \mathcal{H}$ satisfying (1), find $w \in \mathcal{H}$ such that

$$\begin{aligned}&\rho E(\mathcal{T}(w+\zeta(\mu-w)), \eta(\nu, w+\zeta(\mu-w))\\ &\quad +\langle M(w)-M(\mu)+\alpha(\mu-\mu), \nu-w\rangle = 0, \quad \forall \nu \in \mathcal{H}, \end{aligned} \tag{55}$$

where $\rho > 0, \eta, \alpha \in [0,1]$ are constants and M is a nonlinear operator. Clearly $w = \mu$ implies that w is a solution of (1). This simple observation enables us to suggest the following iterative method for solving (1).

Algorithm 28. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho E(\mathcal{T}(\mu_{n+1})+\zeta(\mu_n-\mu_{n+1}), \eta(\nu, \mu_{n+1}+\zeta(\mu_n-\mu_{n+1}))\\ &+\langle M(\mu_{n+1})-M(\mu_n)+\alpha(\mu_n-\mu_{n-1}), \nu-\mu_{n+1}\rangle = 0, \quad \forall \nu \in \mathcal{H}.\end{aligned}$$

Algorithm 28 is called the hybrid proximal point algorithm for solving the preinvex equilibrium equation (1). For $\alpha = 0$, Algorithm 28 is exactly the Algorithm 27.

If $M = 0$, then Algorithm 28 reduces to:

Algorithm 29. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\rho E(\mathcal{T}(\mu_{n+1}+\zeta(\mu_n-\mu_{n+1}), \eta(\nu, \mu_{n+1}+\zeta(\mu_n-\mu_{n+1}))\\ &\quad +\langle \alpha(\mu_n-\mu_{n-1}), \nu-\mu_{n+1}\rangle = 0, \quad n = 1, 2, \ldots\ldots \quad \forall \nu \in \mathcal{H},\end{aligned}$$

which is called the inertial iterative method for solving the preinvex equilibrium equations (1).

Remark 3. *For different and suitable choices of the parameters ρ, η, α, operators g, M, bifunction $E(.,.)$ and convex sets, one can recover new and known iterative methods for solving general bi-equilibrium equations, Lax-Milgram lemma and related optimization problems. Using the technique and ideas of Theorem 5 and Theorem 6, one can analyze the convergence of Algorithm 28 and its special cases.*

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6 Sensitivity analysis

In recent years variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, operations research and engineering sciences. The behavior of such problems as a result of changes in the problem data is always of concern, which is called sensitivity analysis. Dafermos [9] considered the sensitivity analysis considered the sensitivity of the variational inequalities using essentially the projection method. These results were extended for quasi variational inequalities by Noor [26] and Noor et al. [43,47,48,52,53]. We like to mention that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied can stimulate new ideas and techniques for problem solving the problems due to these and other reasons. In this section, we study the sensitivity analysis of the system of general preinvex equilibrium equations.

We now consider the parametric versions of the problem (2). To formulate the problem, let M be an open subset of $\mathcal{H}$ in which the parameter λ takes values. Let $g(\mu, \lambda)$ be given identity operator defined on $\mathcal{H} \times \mathcal{H} \times M$ and take value in $\mathcal{H} \times \mathcal{H}$. From now onward, we denote $g_\lambda(.) \equiv g(., \lambda)$ and $E_\lambda(.) \equiv E(., \lambda)$, respectively, unless otherwise specified.

The parametric general preinvex equilibrium equation is to find $\mu \in \mathcal{H}$ such that For the bifunction $E(.,.) = \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ and arbitrary operator $g : \mathcal{H} \Longrightarrow \mathcal{H}$, the parametric system of general preinvex equilibrium equation problem is to find $\mu \in \mathcal{H}$ such that

$$g_\lambda(\mu) = g_\lambda(\mu) - \rho E_\lambda\big(\mathcal{T}_\lambda(\mu), \eta(\nu, \mu)\big), \quad \forall \nu \in \mathcal{H}, \tag{56}$$

is called the system of general preinvex equilibrium equation, where ρ is a parameter.

We also assume that, for some $\overline{\lambda} \in M$, the problem (56) has a unique solution $\overline{\mu}$. The parametric system of general preinvex equilibrium equation

is equivalent to the fixed point problem:

$$g_\lambda(\mu) = \big[g_\lambda(\mu) - \rho E_\lambda\big(\mathcal{T}_\lambda(\mu), \eta(\nu,\mu)\big)\big],$$

or equivalently

$$\mu = \mu - g_\lambda(\mu) + \big[g_\lambda(\mu) - \rho E_\lambda\big(\mathcal{T}_\lambda(\mu), \eta(\nu,\mu)\big)\big].$$

We now define the mapping F_λ associated with the problem (56) as

$$\begin{aligned} F_\lambda(\mu) &= \mu - g_\lambda(\mu) \\ &\quad + [g_\lambda(\mu) - \rho E_\lambda\big(\mathcal{T}_\lambda(\mu), \eta(\nu,\mu)\big)], \quad \forall (\mu,\lambda) \in \mathcal{H} \times M. \end{aligned} \tag{57}$$

We use this equivalence to study the sensitivity analysis of the system of general preinvex equilibrium equations. We assume that for some $\overline{\lambda} \in M$, problem (56) has a solution $\overline{\mu}$ and X is a closure of a ball in $\mathcal{H}$ centered at $\overline{\mu}$. We want to investigate those conditions under which, for each λ in a neighborhood of $\overline{\lambda}$, problem (28) has a unique solution $\mu(\lambda)$ near $\overline{u}$ and the function $u(\lambda)$ is (Lipschitz) continuous and differentiable.

Definition 9. *Let $E_\lambda(.)$ be a bifunction on $X \times M$. Then, the bifunction $E_\lambda(\mathcal{T}_\lambda(.),.)$ is said to :*

(a) Locally strongly monotone with constant $\sigma > 0$, if

$$\langle E_\lambda\big(\mathcal{T}_\lambda(\mu), \eta(\nu,\mu)\big) - E_\lambda\big(\mathcal{T}_\lambda(\zeta), \eta(\nu,\zeta)\big), \nu\rangle \geq \sigma\|\mu - \zeta\|^2, \quad \forall \lambda \in M, \zeta, \mu, \nu \in X.$$

(b) Locally Lipschitz continuous with constant $\delta > 0$, if

$$\|E_\lambda(\mathcal{T}_\lambda(\mu), \eta(\nu,\mu)) - E_\lambda\big(\mathcal{T}_\lambda(\zeta), \eta(\nu,\zeta)\big)\| \leq \delta\|\mu - \zeta\|, \quad \forall \lambda \in M, \zeta, \mu, \nu \in X.$$

Definition 10. *An operator $\mathcal{T}_\lambda : \mathcal{H} \to \mathcal{H}$ is said to be:*

1. *locally strongly monotone, if there exists a constant $\alpha > 0$, such that*

$$\langle \mathcal{T}_\lambda(\mu) - \mathcal{T}_\lambda(\nu), \mu - \nu\rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}.$$

2. *locally Lipschitz continuous, if there exists a constant $\beta > 0$, such that*

$$\|\mathcal{T}_\lambda(\mu) - \mathcal{T}_\lambda(\nu)\| \leq \beta\|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$

3. *locally monotone, if*

$$\langle \mathcal{T}_\lambda(\mu) - \mathcal{T}_\lambda(\nu), \mu - \nu\rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

We consider the case, when the solutions of the parametric general preinvex equilibrium equations (56) lie in the interior of X. Following the ideas of Noor et al. [43,47,48,52,53], we consider the map $F_\lambda(\mu)$ as defined by (57). We have to show that the map $F_\lambda(\mu)$ has a fixed point, which is a solution of the parametric general preinvex equilibrium equations (56). First of all, we prove that the map $F_\lambda(\mu)$, defined by (57), is a contraction map with respect to μ uniformly in $\lambda \in M$.

Lemma 2. *Let $g_\lambda(.)$ be a locally strongly monotone with constants $\sigma > 0$ and locally Lipschitz continuous with constants $\delta > 0$ respectively. If the bifunction $E_\lambda(.,.)$ is locally Lipchitz continuous with constants $\beta > 0$, we have*

$$\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| \leq \theta\|\mu_1 - \mu_2\|,$$

for

$$\rho < \frac{1-k}{\beta}, \quad k < 1, \tag{58}$$

where

$$\theta = \left\{\sqrt{1 - 2\sigma + \delta^2} + \delta + \rho\beta\right\} = \{k + \rho\beta\} \tag{59}$$

and

$$k = \sqrt{1 - 2\sigma + \delta^2} + \delta. \tag{60}$$

Proof. In order to prove the existence of a solution of (28), it is enough to show that the mapping $F_\lambda(\mu)$, defined by (57), is a contraction mapping. For $\mu_1 \neq \mu_2 \in \mathcal{H}$, we have

$$\begin{aligned}
\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| &\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&\quad + \|[g_\lambda(\mu_1) - \rho E_\lambda\big(\mathcal{T}_\lambda(\mu_1), \eta(\nu, \mu_1)\big)] - [g_\lambda(\mu_2) \\
&\quad - \rho E_\lambda\big(\mathcal{T}_\lambda(\mu_2), \eta(\nu, \mu_2)\big)]\| \\
&\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&\quad + \|g_\lambda(\mu_) - g_\lambda(\mu_2) \\
&\quad - \rho(E_\lambda\big(\mathcal{T}_\lambda(\mu_1), \eta(\nu, \mu_1)\big) - E_\lambda(\mathcal{T}_\lambda(\mu_2), \eta(\nu, \mu_2))\| \\
&\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&\quad + \rho\|E_\lambda\big(\mathcal{T}_\lambda(\mu_1), \eta(\nu, \mu_1)\big) - E_\lambda\big(\mathcal{T}_\lambda(\mu_2), \eta(\nu, \mu_2)\big)\| \\
&\quad + \|g_\lambda(\mu_1) - g_\lambda(\mu_2)\| \\
&\quad + \rho\|E_\lambda\big(\mathcal{T}_\lambda(\mu_1), \eta(\nu, \mu_1)\big) - E_\lambda\big(\mathcal{T}_\lambda(\mu_2), \eta(\nu, \mu_2)\big)\| \\
&\leq \|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2))\| \\
&\qquad + \delta\|\mu - \nu\| + \rho\beta\|\mu_1 - \mu_2)\|.
\end{aligned} \tag{61}$$

Since the operator g_λ is a locally strongly monotone with constant $\sigma > 0$ and locally Lipschitz continuous with constant $\delta > 0$, it follows that

$$\begin{aligned}
&\|\mu_1 - \mu_2 - (g_\lambda(\mu_1) - g_\lambda(\mu_2)\|^2 \\
&\leq \|u_1 - u_2\|^2 - 2\langle g_\lambda(\mu_1) - g_\lambda(\mu_2), \mu_1 - \mu_2\rangle + \|g_\lambda(\mu_1) - g_\lambda(\mu_2)\|^2 \\
&\leq (1 - 2\sigma + \delta^2)\|\mu_1 - \mu_2\|^2. \qquad (62)
\end{aligned}$$

From (60), (61), (62) and using the locally Lipschitz continuity of the operators $\mathcal{T}, g_\lambda$, we have

$$\begin{aligned}
\|F_\lambda(\mu_1) - F_\lambda(\mu_2)\| &\leq \left\{\delta + \sqrt{(1 - 2\sigma + \delta^2)} + \rho\beta\right\}\|\mu_1 - \mu_2\| \\
&= \theta\|\mu_1 - \mu_2\|,
\end{aligned}$$

where

$$\theta = k + \rho\beta.$$

From (58), it follows that $\theta < 1$. Thus it follows that the mapping $F_\lambda(\mu)$, defined by (57), is a contraction mapping and consequently it has a fixed point, which belongs to $\mathcal{H}$ satisfying the system of parametric general preinvex equilibrium equations (28), the required result. □

Remark 4. *From Lemma 2, we see that the map $F_\lambda(\mu)$ defined by (57) has a unique fixed point $\mu(\lambda)$, that is, $\mu(\lambda) = F_\lambda(\mu)$. Also, by assumption, the function $\overline{\mu}$, for $\lambda = \overline{\lambda}$ is a solution of the parametric preinvex equilibrium equations (56). Again using Lemma 2, we see that $\overline{\mu}$, for $\lambda = \overline{\lambda}$, is a fixed point of $F_\lambda(\mu)$ and it is also a fixed point of $F_{\overline{\lambda}}(\mu)$. Consequently, we conclude that*

$$\mu(\overline{\lambda}) = \overline{\mu} = F_{\overline{\lambda}}(\mu(\overline{\lambda})).$$

Using Lemma 2, we can prove the continuity of the solution $\mu(\lambda)$ of the parametric preinvex equilibrium equations (56) using the technique of Noor et al. [43, 47, 52, 53].

Lemma 3. *Assume that the bifunction E_λ and the operator h_λ are locally Lipschitz continuous with respect to the parameter λ. If the operator $g_\lambda(.)$ is Locally Lipschitz continuous and the map $\lambda \to J_A$ is continuous (or Lipschitz continuous), then the function $u(\lambda)$ satisfying (57) is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

We now state and prove the main result of this paper and is the motivation our next result.

Theorem 7. *Let $\overline{\mu}$ be the solution of the parametric preinvex equilibrium equations (56) for $\lambda = \overline{\lambda}$. Let $E_\lambda, h_\lambda(\mu)$ be the locally strongly monotone Lipschitz continuous operators for all $\mu, \nu \in X$. If the map $\lambda \to J_{A_\mu}$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$ and the operator g_λ is locally strongly monotone Lipschitz continuous, then there exists a neighborhood $N \subset M$ of $\overline{\lambda}$ such that for $\lambda \in N$, the parametric preinvex equilibrium equation (57) has a unique solution $\mu(\lambda)$ in the interior of $X, u(\overline{\lambda}) = \overline{u}$ and $u(\lambda)$ is (Lipschitz) continuous at $\lambda = \overline{\lambda}$.*

Proof. Its proof follows from Lemma 2, Lemma 3 and Remark 4. □

7 Conclusion

In this paper, we have introduced and studied some new classes of the preinvex equilibrium equations. We have proved that the general preinvex equilibrium equations are equivalent to the fixed point problem. The alternative equivalent formulation is used to suggest and propose some new multi-step explicit and implicit iterative methods for solving the preinvex equilibrium equations along with convergence criteria. Dynamical system technique for solving the preinvex equilibrium equations is exploited to discuss the asymptotic stability of the solution in the sense of Lyapunov. Second order and third order initial dynamical systems associated with the preinvex equilibrium equations are considered to propose and suggest a wide class of iterative methods for finding the approximate solution, which is entirely new and novel idea. These new methods include Noor(three step) iterations, Ishikawa (two step) iterations, Mann Iteration and Picard method as special cases. Exploring the technique and ideas of Ashish et. al. [2–4], Natarajan et al. [17], Negi et al. [18], Suantai et al. [60] and Yadav et al. [66], one can explore the applications of these Algorithms in the fixed point theory, financial mathematical sciences [1], fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics, transportation, solar penal optimization [17], sensing signal [55, 56, 60], epidemiology [3], graphics signal recovery, logistic maps [66], data science [18], complex problems in artificial intelligence and computer aided designs [9]. This is an open problem, which deserves further research efforts. It is worth mentioning these methods can be extended for harmonic-like absolute value equations, difference of two operators equations and hemi-nonlinear equilibrium equations with suitable and appropriate choice of the operators. We have only considered the theoretical aspects of the preinvex equilibrium equations. The imple-

mentable and developments of numerical efficient methods for solving such type of nonlinear bifunction equations need further research efforts. These are some interesting open problems for future research.

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References

[1] F. Al-Azemi and O. Calin, Asian options with harmonic average, *Appl. Math. Inform. Sci.* 9 (2015), 1-9.

[2] K. Ashish, M. Rani and R. Chugh, Julia sets and Mandelbrot sets in Noor orbit, *Appl. Math. Comput.* 228 (2014), 615-631.

[3] K. Ashish, J. Cao and M.A. Noor, Stabilization of fixed points in chaotic maps using Noor orbit with applications in cardiac arrhythmia, *J. Appl. Anal. Comput.* 13 (2023), 2452-2470.

[4] K. Ashish, R. Chugh and M. Rani, *Fractals and Chaos in Noor Orbit: a Four-Step Feedback Approach,* Lap Lambert Academic Publishing, Saarbrucken, Germany, 2021.

[5] E. Blum and W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student,* 63 (1994), 123-145.

[6] R.L. Burden and J.D. Faires, *Numerical Analysis*, 9th Edition, Brooks/Cole, Boston, MA USA, 2018.

[7] C. Chairatsiripong and T. Thianwan, Novel Noor iterations technique for solving nonlinear equations, *AIMS Math.* 7 (2022), 10958-10976.

[8] G. Cristescu and L. Lupsa, *Non-Connected Convexities and Applications,* Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[9] S. Dafermos, Sensitivity analysis in variational inequalities, *Math. Oper. Res.* 13 (1988), 421-434.

[10] P. Dupuis and A. Nagurney, Dynamical systems and variational inequalities, *Ann. Oper. Res.* 44 (1993), 7-42.

[11] F. Giannessi, A. Maugeri and M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models,* Kluwer Academic Publishers, Dordrecht, Holland, 2001.

[12] R. Glowinski, J.L. Lions and R. Tremolieres, *Numerical Analysis of Variational Inequalities,* NorthHolland, Amsterdam, 1981.

[13] N.K. Gupta, S. Varghese, M.A. Siddiqui and M.A. Noor, General mixed quasi equilibrium variational inequalities, *Global J. Pure Appl. Math.* 21 (2025), 407-418,

[14] A.S. Householder, *The Numerical Treatment of a Single Nonlinear Equation,* McGraw-Hill, New York, USA, 1970.

[15] M.N.A. Khan, M. Rashid, A. Kalsoom, N. Saleem and M. De La Sene, Approximation Theorems for G-nonexpansive mappings convex metric spaces by three step iterations, *Alexandria Eng. J.* 102 (2024), 1–9.

[16] A. Nagurney and D. Zhang, *Projected Dynamical Systems and Variational Inequalities with Applications,* Kluwer Academic Publishers, Boston, Dordrecht, London 1996.

[17] S.K. Natarajan and D. Negi, *Green innovations utiling fractal and power for solar panel optimization,* In: Green Innovations for Industrial Development and Business Sustainability (R. Sharma, G. Rana and S. Agarwal (Eds)), CRC Press, Florida, Boca Raton, USA, 2024, pp.146-152.

[18] D. Negi, R. Kumar, V. Kumar and A. Dhaka, Secure data transmission through fractal-based cryptosystem: a Noor iteration approach, *Sci. Rep.* 15 (2025), 22206.

[19] C.P. Niculescu and L.E. Persson, *Convex Functions and Their Applications,* Springer-Verlag, New York, 2018.

[20] K.I. Noor and M.A. Noor, A generalization of the Lax-Milgram lemma, *Can. Math. Bull.* 23 (1980), 179-184.

[21] K.I. Noor and M.A. Noor, New approach for solving systems of general equations, *FACTA UNIV(NIS), Ser. Math. Inform.* 41 (2026), 65-84.

[22] K.I. Noor, M.A. Noor and A. Hamdi, Harmonic-like quasi bifunction equilibrium inclusions, *Int. J. Appl. Anal. Appl.* 24 (2026), 107.

[23] M.A. Noor, *On Variational Inequalities,* PhD Thesis, Brunel University, London, 1975.

[24] M.A. Noor, General variational inequalities, *Appl. Math. Lett.* 1 (1988), 119-121.

[25] M.A. Noor, General algorithm for variational inequalities, *J. Optim. Theory Appl.* 73 (1992), 409-413.

[26] M.A. Noor, Sensitivity analysis for quasi variational inequalities, *J. Optim. Theory Appl.* 95 (1997), 399-407.

[27] M.A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.* 251 (2000), 217-230.

[28] M.A. Noor, Stability of the modified projected dynamical systems, *Comput. Math. Appl.* 44 (2002), 1-5.

[29] M.A. Noor, Multivalued general equilibrium problems, *J. Math. Anal. Appl.* 283 (2003), 140-149.

[30] M.A. Noor, Auxiliary principle technique for equilibrium problems, *J. Optim. Theory Appl.* 122 (2004), 373-386.

[31] M.A. Noor, Some dvelopments in general variational inequalites, *Appl. Math. Comput.* 152 (2004), 199-277.

[32] M.A. Noor, Invex equilibrium problems, *J. Math. Anal. Appl.* 302 (2005), 463-475.

[33] M.A. Noor, Fundamentals of equilbrium problems, *Math. Inequal. Appl.* 9 (2006), 529-566.

[34] M.A. Noor and K.I. Noor, On equilibrium problems, *Appl. Math. E-Notes* 4 (2004), 125-132.

[35] M.A. Noor, New family of iterative methods for nonlinear equations, *Appl. Math. Comput.* 190 (2007), 553-558.

[36] M.A. Noor, New classes of itertaive methods for nonlinear equations, *Appl. Math. Comput.* 191 (2007), 128-131.

[37] M.A. Noor, Some iterative methods free from second derivatives for nonlinear equations, *Appl. Math. Comput.* 192 (2007) 101–106.

[38] M.A. Noor and K.I. Noor, Iterative schemes for solving nonlinear equations, *Appl. Math. Comput.* 183 (2006), 774-779.

[39] M.A. Noor and K.I. Noor, Three-step iterative methods for nonlinear equations, *Appl. Math. Comput.* 183 (2006) 322–327.

[40] M.A. Noor and K.I. Noor, Predictor–corrector Halley method for nonlinear equations, *Appl. Math. Comput.* 188 (2007), 1587–1591.

[41] M.A. Noor and K.I. Noor, Some new classes of strongly generalized preinvex functions, *TWMS J. Pure Appl. Math.* 12 (2021), 181-192.

[42] M.A. Noor and K.I. Noor, Dynamical system technique for solving quasi variational inequalities, *U.P.B. Sci. Bull., Ser. A.* 84 (2022), 55-66.

[43] M.A. Noor and K.I. Noor, Some new iterative schemes for solving general quasi variational inequalities, *Le Matematiche* 79 (2024), 327-370.

[44] M.A. Noor and K.I. Noor, Some novel aspects and applications of Noor iterations and Noor orbits, *J. Adv. Math. Stud.* 17 (2024), 276-284.

[45] M.A. Noor and K.I. Noor, Iterative schemes for solving new system of general equations, *U.P.B. Sci. Bull., Ser. A.* 84 (2022), 59-70.

[46] M.A. Noor and K.I. Noor, Some new classes of general harmonic-like nonlinear equations, *General Math.* 33 (2025), 17-35.

[47] M.A. Noor and K.I. Noor, Some new methods for general nonlinear equations, *J. Operator Theory Ineq.* 1 (2026), 14-36.

[48] M.A. Noor and K.I. Noor, New classes of general triequilibrium inclusions, *Ann. Acad. Rom. Sci., Ser. Math. Appl.* 26 (2026), 179-213.

[49] M.A. Noor and K.I. Noor, Iterative algorithms for general mixed bivariational inequalities, *U.P.B. Sci. Bull., Ser. A.* 88 (2026), 53-66.

[50] M.A. Noor and K.I. Noor, Some new iterative methods for solving harmonic nonlinear equations, *Appl. Anal. Optim.* 10 (2026), 21-38.

[51] M.A. Noor and W. Oettli, On general nonlinear complementarity problems and quasi equilibria, *Le Matematiche* 49 (1994), 313-331.

[52] M.A. Noor, K.I. Noor and M.Th. Rassias, New trends in general variational inequalities, *Acta Appl. Math.* 170 (2020), 981-1064.

[53] M.A. Noor, K.I. Noor and M.T. Rassias, *General variational inequalities and optimization,* In: Geometry and Non-Convex Optimization (Editors: P.M. Pardalos, T.M. Rassias), Springer Optimization and Its Applications ((SOIA,volume 223)), Springer Nature Switzerland, 223 (2025), 361-611.

[54] M.A. Noor, K.I. Noor, S. Treanţă and K. Nonlaopon, On three-step iterative schemes associated with general quasi-variational inclusions, *Alexandria Eng. J.* 61 (2022), 12051-12059.

[55] P. Paimsangan and T. Thianwan, Signal recovery and polynomiographic visualization of modified Noor iteration of operators with property(E), *Demonstratio Math.* 57 (2024), 2024-2070.

[56] S. Paimsang, D. Yambangwai and T. Thainwan, A novel Noor iterative method of operators with property (E) as concerns convex programming applicable in signal recovery and polynomiography, *Math. Methods Appl. Sci.* 47 (2024), 9571-9588.

[57] M. Patriksson, *Nonlinear Programming and Variational Inequalities: A Unified Approach,* Kluwer Acadamic publishers, Drodrecht, 1998.

[58] K. Rattanaseeha, S. Imnang, P. Inkrong and T. Thianwan, Novel Noor iterative methods for mixed type asmptotically nonexpansive mappings from the perpestive of convex programming in hyperbolic spaces, *Int. J. Innov. Comput. Inform. Cont.* 19 (2023), 1717-1734.

[59] P.H. Sach and L.A. Tuan, Generalized equilibria with multivalued trifunctions and applications to uncertain vector equilibrium problems, *J. Optim. Theory Appl.* 208 (2026), 51.

[60] S. Suantai, M.A. Noor, K. Kankam and P. Cholamjiak, Novel forward-backward algorithms for optimization and applications to compressive sensing and image inpainting, *Adv. Differ. Equations* 2021 (2021), 265.

[61] A. Tomar, S. Antal, M. Sajid and D.J. Prajapati, Role of s-convexity in the generation of fractals as Julia and Mandelbrot sets via three-step fixed point iteration, *AIMS Math.* 10 (2025), 26077–26105.

[62] E. Tonti, Variational formulation for every nonlinear problem, *Int. J. Eng. Sci.* 22 (1984), 1343- 1371.

[63] J.F. Traub, *Iterative Methods for Solution of Equations,* Prentice-Hall, Englewood Cliffs, NJ, USA, 1964.

[64] P.T. Vuong. X. He and D.V. Thong, Global exponential stability of a neural network for inverse variational inequalities, *J. Optim. Theory Appl.* 190 (2021), 915-930.

[65] Y. S. Xia and J. Wang, On the stability of globally projected dynamical systems, *J. Optim. Theory Appl.* 106 (2000), 129-150.

[66] A. Yadav and K. Jha, Parrondo's paradox in the Noor logistic map, *Int. J. Adv. Research Eng.Tech.* 7 (2016), 1.

[67] Y.H. Yao, M.A. Noor, S. Zainab and Y.C. Liou, Mixed equilibrium problems and optimization problems, *J. Math. Anal. Appl.* 354 (2009), 319-329.