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SPECTRAL PROPERTIES OF POSITIVE LINEAR OPERATORS UNDER LACUNARY STATISTICALLY RELATIVELY UNIFORM CONVERGENCE*

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Abstract

This paper introduces the spectral properties of positive linear operators under the framework of lacunary statistically relatively uniform convergence. We define the concept of lacunary statistically relatively uniform and establish inclusion and stability results for sequences of positive linear operators. Furthermore, we examine the convergence of spectral radii and provide illustrative examples using classical operators. This work extends both Korovkin-type approximation theory and spectral theory into the lacunary statistical relatively uniform setting.

Keywords: statistical convergence, lacunary convergence, spectral radius, point spectrum.

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1 Introduction

A classical milestone in the study of positive linear operators and their approximation properties is the well-known Korovkin theorem, which characterizes the uniform convergence of sequences of positive linear operators through the convergence of their action on test functions [1, 19]. Subsequently, many generalizations of the Korovkin theorem have been obtained by relaxing the mode of convergence, leading to statistical approximation [2, 15, 18, 22], lacunary approximation [16, 17], and other extensions. In this direction, Demirci and Orhan [9] studied *statistical relative approximation on modular spaces*. Later, Demirci, Dirik, and Yıldız [10] discussed *approximation through statistical relative uniform convergence of sequences of functions using the power series method*, while Demirci, Orhan, and Kolay [11] examined *weighted statistical relative approximation by positive linear operators*.

On the other hand, spectral theory provides a powerful framework for analyzing the behavior of linear operators through their spectra and spectral radii [7, 20]. Spectral properties are not only essential in pure operator theory but also in approximation processes, stability analysis, and ergodic theory. In particular, investigating the spectrum of sequences of positive linear operators under various modes of convergence has emerged as a natural and significant question.

A different line of generalization began with the introduction of relatively uniform convergence, first studied by Moore [21] and Chittenden [4, 5]. In this mode, convergence is measured relative to a scale function, thus providing more flexibility compared to ordinary uniform convergence. Later, statistical convergence, originally introduced by Fast [15] and Steinhaus [22], was extended to function spaces and approximation theory [13, 14].

Recently, Demirci and Orhan [8] unified these two approaches by introducing the concept of *statistically relatively uniform convergence*. They applied this new framework to Korovkin-type approximation theorems and demonstrated that it provides strictly stronger results than both statistical and relatively uniform convergence.

In this paper, we aim to extend this framework further by investigating the *spectral properties of positive linear operators under lacunary statistically relatively uniform convergence*. The lacunary approach, introduced by Freedman, Sember, and Raphael [16], and later studied by Fridy [17], refines statistical convergence by using lacunary intervals. Combining it with relatively uniform convergence provides a rich setting for both approximation and operator theory. We define the notion of the *lacunary statistically* *relatively uniform spectrum*, study spectral inclusion and spectral radius convergence theorems, and illustrate our results with classical operators such as Bernstein and Baskakov operators.

2 Preliminaries

Let $X \subset \mathbb{R}$ be a non-empty compact set and denote by $C(X)$ the space of all real-valued continuous functions on X, which becomes a Banach space under the supremum norm

$$\|f\|_{C(X)} = \sup_{x \in X} |f(x)|, \qquad f \in C(X).$$

A sequence $\{T_n\}$ of operators on $C(X)$ is called a sequence of *positive linear operators* if each $T_n : C(X) \to C(X)$ is linear, i.e. $T_n(af + bg) = aT_n(f) + bT_n(g)$ for all $f, g \in C(X)$ and $a, b \in \mathbb{R}$, and positive, i.e. $f(x) \geq 0$ on X implies $(T_n f)(x) \geq 0$ on X. In classical approximation theory one studies whether $T_n(f) \to f$ uniformly for every $f \in C(X)$ as $n \to \infty$; a standard example is the sequence of Bernstein polynomials $B_n(f; x)$, which converge uniformly to $f \in C[0, 1]$.

2.1 Lacunary statistically relatively uniform convergence

Let $\theta = (k_r)$ be a lacunary sequence with $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$, and intervals

$$I_r = (k_{r-1}, k_r], \qquad r \geq 1.$$

For a subset $A \subset \mathbb{N}$, the lacunary density of A with respect to θ is defined as

$$\delta_\theta(A) = \lim_{r \to \infty} \frac{1}{h_r} |A \cap I_r|,$$

whenever the limit exists.

Let $\sigma : X \to (0, \infty)$ be a continuous scale function and endow $C(X)$ with the weighted sup–norm

$$\|f\|_\sigma = \sup_{x \in X} \frac{|f(x)|}{\sigma(x)}.$$

Definition 1. *A sequence (f_n) in $C(X)$ is said to converge to $f \in C(X)$ in the* lacunary statistically relatively uniform sense *(abbreviated lac-SRU, with respect to σ) if for every $\varepsilon > 0$,*

$$\delta_\theta\Big(\{n \in \mathbb{N} : \ \|f_n - f\|_\sigma \geq \varepsilon\}\Big) = 0.$$

We denote this convergence by

$$f_n \xrightarrow[\theta]{\text{lac-st}} f \quad (X;\sigma).$$

This definition reduces to ordinary relative uniform convergence when θ is the trivial sequence $k_r = r$, and to lacunary statistical convergence when $\sigma \equiv 1$.

2.2 Basic properties

The following properties are straightforward consequences of the definition.

Proposition 1 (Linearity). *If $f_n \xrightarrow[\theta]{\text{lac-st}} f$ and $g_n \xrightarrow[\theta]{\text{lac-st}} g$ in $C(X)$, then for all scalars $a, b \in \mathbb{C}$,*

$$af_n + bg_n \xrightarrow[\theta]{\text{lac-st}} af + bg.$$

Proposition 2 (Domination). *If $f_n \xrightarrow[\theta]{\text{lac-st}} f$ and (g_n) is a sequence in $C(X)$ such that $\|g_n - f_n\|_\sigma \to 0$ uniformly in n, then $g_n \xrightarrow[\theta]{\text{lac-st}} f$.*

Proposition 3 (Implication from uniform convergence). *If $f_n \to f$ uniformly in $\|\cdot\|_\sigma$, then $f_n \xrightarrow[\theta]{\text{lac-st}} f$ for every lacunary sequence θ.*

Proposition 4 (Non-converse). *Lac-SRU convergence does not imply uniform convergence.*

The following example illustrates the above.

Example 1. *Consider the sequence of functions*

$$f_n(x) = \sin(nx), \qquad x \in [0, 2\pi],$$

with the constant scale function $\sigma(x) \equiv 1$.

First observe that (f_n) does not converge uniformly on $[0, 2\pi]$. Indeed, if there were a uniform limit f, then for each x we would need

$$\lim_{n\to\infty} \sin(nx) = f(x).$$

However, the sequence $\sin(nx)$ oscillates between -1 and 1 for most x, and no pointwise limit exists except possibly at special points (e.g., $x = k\pi$). In particular, since $\sup_{x\in[0,2\pi]} |f_n(x) - f_m(x)| = 2$ for infinitely many pairs (n, m), uniform convergence fails.

On the other hand, lacunary statistical relative uniform (Lac-SRU) convergence may still hold. Choose a lacunary sequence (k_r)*, for instance* $k_r = 2^r$*. By lacunary averaging, we examine the means*

$$\frac{1}{h_r}\sum_{n\in I_r}|f_n(x)|, \qquad I_r = (k_{r-1}, k_r],$$

where $h_r = k_r - k_{r-1}$*. Since* $\sin(nx)$ *is oscillatory and symmetric, its average value over such lacunary intervals tends to zero for each* x*. Consequently, we obtain*

$$f_n(x) \xrightarrow[\textit{Lac-SRU}]{} 0 \quad \textit{on } [0, 2\pi].$$

Thus, (f_n) *is Lac-SRU convergent to the zero function, but not uniformly convergent. This shows that Lac-SRU convergence does not imply uniform convergence.*

These results establish lac-SRU convergence as a natural weakening of uniform convergence, while still retaining enough structure (linearity, stability) to allow spectral analysis of operator sequences.

2.3 Operators acting on $C(X)$

In our setting, we apply lac-SRU convergence to operator sequences. That is, for a sequence $\{T_n\}$ of positive linear operators on $C(X)$, we say

$$T_n(f) \overset{\text{lac-st}}{\underset{\theta}{\Longrightarrow}} f \quad (X;\sigma), \qquad \forall f \in C(X),$$

whenever for every $\varepsilon > 0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k \in I_r : \sup_{x\in X}\frac{|T_k(f)(x) - f(x)|}{\sigma(x)} \geq \varepsilon\right\}\right| = 0.$$

This generalizes the usual notion of approximation by positive linear operators in uniform or statistical sense. In particular:

- If $\sigma(x) \equiv 1$, we recover lacunary statistical uniform convergence.
- If $\theta = (n)$ (i.e., no lacunarity), we recover ordinary statistically relatively uniform convergence.
- If $\theta = (n)$ and $\sigma(x) \equiv 1$, we recover ordinary statistical uniform convergence.

3 Main results

Theorem 1 (Spectral Inclusion)**.** *If* $T_n(f) \xrightarrow[\theta]{lac\text{-}st} f\ (X;\sigma)$ *for all* $f \in C(X)$, *then*

$$\limsup_{n\to\infty,\theta} \sigma(T_n) \subseteq \sigma(I) = \{1\}.$$

Proof. Let X be compact and let $\sigma \in C(X)$ satisfy $\sigma(x) > 0$ for all $x \in X$. On $C(X)$ consider the (equivalent) weighted norm

$$\|f\|_\sigma := \sup_{x\in X} \frac{|f(x)|}{\sigma(x)}.$$

(The equivalence with the usual sup–norm follows from compactness of X and continuity/positivity of σ, so $0 < m \le \sigma \le M < \infty$.)

Write $S_n := T_n - I$. The hypothesis says: for every $f \in C(X)$ and every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \Big| \big\{ k \in I_r : \ \|S_k f\|_\sigma \ge \varepsilon \big\} \Big| = 0. \tag{1}$$

First we show the **equiboundedness** of (T_n) in $\|\cdot\|_\sigma$).

It is well known that for a positive linear operator T on $C(X)$ one has

$$\|T\|_{\infty\to\infty} = \|T\mathbf{1}\|_\infty,$$

where $\mathbf{1}$ denotes the constant function $\mathbf{1}(x) = 1$.

By assumption, $T_n(\mathbf{1}) \xrightarrow[\theta]{\text{lac-st}} \mathbf{1}$ in the relatively uniform sense. Hence, there exists $r_0 \in \mathbb{N}$ such that for all $r \ge r_0$, and for all but a subset of I_r with lacunary density tending to zero, we have

$$\|T_k\mathbf{1} - \mathbf{1}\|_\infty \le 1, \qquad (k \in I_r).$$

This inequality immediately gives,

$$\|T_k\mathbf{1}\|_\infty \le 2, \qquad \text{for lacunarily many } k.$$

Consequently,

$$\|T_k\|_{\infty\to\infty} \le 2 \qquad \text{for all } k \text{ in a set of lacunary density 1.}$$

Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|_\sigma$ are equivalent on $C(X)$ (because σ is continuous and strictly positive on compact X), there exists a constant $C \ge 1$ such that

$$\|T_k\|_{\sigma\to\sigma} \le C \qquad \text{for } k \text{ in a set of lacunary density 1.}$$

Finally, by discarding a subset of indices of lacunary density zero, we may assume without loss of generality that

$$\sup_k \|T_k\|_{\sigma\to\sigma} \le C < \infty. \tag{2}$$

Next, we show the transition from point-wise lacunary RU convergence to operator–norm lacunary statistically RU convergence.

Fix $\varepsilon > 0$. Let $B := \{f \in C(X) : \|f\|_\sigma \le 1\}$ be the unit ball in $(C(X), \|\cdot\|_\sigma)$. Choose a finite $\varepsilon/(4C)$–net $\{f^{(1)}, \dots, f^{(N)}\} \subset B$ (existence follows from separability of $C(X)$). For each j, by (1) there is r_j such that for all $r \ge r_j$,

$$\frac{1}{h_r}\Big|\big\{k \in I_r : \; \|S_k f^{(j)}\|_\sigma \ge \varepsilon/2\big\}\Big| \le \varepsilon/2.$$

Let $r_* := \max_j r_j$. Then for $r \ge r_*$, outside a subset $E_r \subset I_r$ with $\frac{|E_r|}{h_r} \le \varepsilon/2$, we have $\|S_k f^{(j)}\|_\sigma < \varepsilon/2$ for all $j = 1, \dots, N$.

Now take any $f \in B$ and pick j with $\|f - f^{(j)}\|_\sigma \le \varepsilon/(4C)$. For $k \in I_r \setminus E_r$, using linearity and (2),

$$\|S_k f\|_\sigma \le \|S_k f^{(j)}\|_\sigma + \|S_k(f - f^{(j)})\|_\sigma$$
$$\le \frac{\varepsilon}{2} + (\|T_k\|_{\sigma\to\sigma} + 1)\cdot\frac{\varepsilon}{4C} \le \frac{\varepsilon}{2} + \frac{C+1}{4C}\varepsilon \le \varepsilon.$$

Therefore,

$$\frac{1}{h_r}\Big|\big\{k \in I_r : \; \|S_k\|_{\sigma\to\sigma} \ge \varepsilon\big\}\Big|$$
$$\le \frac{1}{h_r}\Big|\big\{k \in I_r : \; \exists f \in B, \; \|S_k f\|_\sigma \ge \varepsilon\big\}\Big| \le \frac{|E_r|}{h_r} \le \frac{\varepsilon}{2}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude

$$\lim_{r\to\infty}\frac{1}{h_r}\Big|\big\{k \in I_r : \; \|T_k - I\|_{\sigma\to\sigma} \ge \varepsilon\big\}\Big| = 0. \tag{3}$$

That is, $T_k \to I$ in operator norm in the lacunary statistical sense.

Further, we check for the resolvent invertibility via Neumann series for $\lambda \ne 1$. Fix $\lambda \in \mathbb{C}$ with $\lambda \ne 1$ and set $\delta := |1 - \lambda| > 0$. By (3), for any $\eta \in (0, \delta)$ the set

$$G_\eta := \big\{k : \; \|T_k - I\|_{\sigma\to\sigma} < \eta\big\}$$

has lacunary density 1. For $k \in G_\eta$, write

$$T_k - \lambda I = (1 - \lambda)\Big[I + \frac{T_k - I}{1 - \lambda}\Big].$$

Since $\left\|\frac{T_k - I}{1-\lambda}\right\|_{\sigma\to\sigma} < \eta/\delta < 1$, the bracket is invertible by the Neumann series, hence $T_k - \lambda I$ is invertible. Thus, for every $\lambda \neq 1$, the set $\{k : \lambda \in \sigma(T_k)\}$ has lacunary density 0.

By the previous steps, no $\lambda \neq 1$ can belong to the lacunary lim sup of the spectra. Since $\sigma(I) = \{1\}$, we obtain

$$\limsup_{n\to\infty,\theta} \sigma(T_n) \subseteq \sigma(I) = \{1\},$$

as claimed. □

Theorem 2 (Spectral Radius Convergence). *If* (T_n) *is a sequence of positive linear operators such that* $T_n(f) \xrightarrow[\theta]{lac\text{-}st} f$ *for all* $f \in C(X)$, *then*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} r(T_k) = 1.$$

Proof. Let X be compact, $\sigma > 0$ continuous, and equip $C(X)$ with the equivalent weighted sup–norm

$$\|f\|_\sigma := \sup_{x\in X} \frac{|f(x)|}{\sigma(x)}.$$

As in the proof of the Spectral Inclusion Theorem 3.1, the assumption

$$T_n(f) \xrightarrow[\theta]{\text{lac-st}} f \qquad (f \in C(X))$$

implies that

$$\lim_{r\to\infty} \frac{1}{h_r} \Big| \{k \in I_r : \ \|T_k - I\|_{\sigma\to\sigma} \geq \varepsilon\} \Big| = 0, \tag{4}$$

for every $\varepsilon > 0$. That is, $T_k \to I$ in operator norm in the lacunary statistical sense.

For any bounded operator S, the spectral radius satisfies

$$r(S) \leq \|S\|.$$

Therefore,

$$|r(T_k) - 1| \leq \|T_k - I\|_{\sigma\to\sigma},$$

since $r(I) = 1$ and $r(\cdot)$ is 1-Lipschitz with respect to operator norm perturbations.

Fix $\varepsilon > 0$ and for sufficiently large r,

$$\frac{1}{h_r}\Big|\{k \in I_r : \ |r(T_k) - 1| \geq \varepsilon\}\Big| \leq \frac{1}{h_r}\Big|\{k \in I_r : \ \|T_k - I\|_{\sigma\to\sigma} \geq \varepsilon\}\Big| < \varepsilon.$$

Hence, for large r,

$$\begin{aligned}\left|\frac{1}{h_r}\sum_{k\in I_r} r(T_k) - 1\right| &\leq \frac{1}{h_r}\sum_{k\in I_r} |r(T_k) - 1| \\ &\leq \varepsilon \cdot 2 + \varepsilon,\end{aligned}$$

since on at most εh_r indices we may have deviation up to 2 (boundedness from equiboundedness of T_k), and on the rest $|r(T_k) - 1| < \varepsilon$. Thus the whole average deviation is bounded by 3ε.

As $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} r(T_k) = 1.$$

□

Theorem 3 (Point Spectrum Approximation)**.** *Suppose each T_n admits an eigenvalue λ_n with eigenfunction f_n. If $f_n \xrightarrow[\theta]{\text{lac-st}} f \neq 0$, then $\lambda_n \to 1$ in lac-SRU sense.*

Proof. Let (T_n) be a sequence of positive linear operators on $C(X)$. By assumption, for each n there exists $\lambda_n \in \mathbb{C}$ and $f_n \in C(X) \setminus \{0\}$ such that

$$T_n f_n = \lambda_n f_n.$$

We are also given that

$$f_n \xrightarrow[\theta]{\text{lac-st}} f \neq 0,$$

in the relatively uniform sense with respect to σ. That is, for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r : \ \|f_k - f\|_\sigma \geq \varepsilon\right\}\right| = 0.$$

First we start with the approximation property of T_n. Since $T_n(g) \xrightarrow[\theta]{\text{lac-st}} g$ for all $g \in C(X)$, applying this with $g = f$ gives

$$T_n f \xrightarrow[\theta]{\text{lac-st}} f.$$

Now, for any $k \in I_r$,

$$\|T_k f_k - T_k f\|_\sigma \leq \|T_k\|_{\sigma\to\sigma}\,\|f_k - f\|_\sigma.$$

From the Spectral Inclusion Theorem, the operators are equibounded in $\|\cdot\|_\sigma$, say $\|T_k\|_{\sigma\to\sigma} \leq C$ for lacunarily many k. Thus, whenever $\|f_k - f\|_\sigma$ is small, so is $\|T_k f_k - T_k f\|_\sigma$.

But $T_k f_k = \lambda_k f_k$, so

$$\lambda_k f_k - f = (T_k f_k - T_k f) + (T_k f - f).$$

Taking $\|\cdot\|_\sigma$, and using the triangle inequality,

$$\|\lambda_k f_k - f\|_\sigma \leq \|T_k f_k - T_k f\|_\sigma + \|T_k f - f\|_\sigma.$$

By approximation of T_n, the second term tends to 0 lacunary statistically. The first term tends to 0 lacunary statistically as well (since $f_k \to f$ lac-st RU and $\{T_k\}$ is equibounded). Hence,

$$\lambda_k f_k \xrightarrow[\theta]{\text{lac-st}} f.$$

On the other hand, $f_k \xrightarrow[\theta]{\text{lac-st}} f$. Therefore, both $\lambda_k f_k$ and f_k converge to the same nonzero limit f in the lac-SRU sense.

Thus, for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{k \in I_r : \ \|\lambda_k f_k - f_k\|_\sigma \geq \varepsilon\right\}\right| = 0.$$

We see that,

$$\|\lambda_k f_k - f_k\|_\sigma = |\lambda_k - 1|\,\|f_k\|_\sigma.$$

Since $f_k \xrightarrow[\theta]{\text{lac-st}} f \neq 0$, there exists $\delta > 0$ and a set of lacunary density 1 such that $\|f_k\|_\sigma \geq \delta$ for all those k.

Therefore, on a set of indices of lacunary density 1 we have

$$|\lambda_k - 1| \leq \frac{\|\lambda_k f_k - f_k\|_\sigma}{\|f_k\|_\sigma}.$$

But the numerator converges to 0 in lac-SRU sense, and the denominator stays bounded away from zero on a density–one subsequence. Hence

$$\lambda_k \to 1 \quad \text{in lacunary statistically relatively uniform sense.}$$

□

4 Examples

We illustrate the abstract results with some classical positive linear operators from approximation theory. These operators are well-known to approximate the identity in the sense of Korovkin's theorem, and we show that they also exhibit the spectral properties described in the preceding theorems which we studied under lacunary statistically relatively uniformly convergent.

- **Bernstein operators.** For $f \in C[0,1]$ and $x \in [0,1]$, the n-th Bernstein polynomial is defined by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

It is classical that $B_n f \to f$ uniformly on $[0,1]$ for every $f \in C[0,1]$. Consequently, along any lacunary sequence $\theta = (k_r)$ we have

$$B_n f \xrightarrow[\theta]{\text{lac-st}} f,$$

so Bernstein operators satisfy the lac-SRU convergence. Spectrally, since $B_n(\mathbf{1}) = \mathbf{1}$, we have $1 \in \sigma(B_n)$ for each n, and the spectral radius satisfies $r(B_n) = 1$. By the Spectral Radius Convergence Theorem, we recover

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} r(B_k) = 1.$$

- **Baskakov operators.** On $C[0,\infty)$, the n-th Baskakov operator is given by

$$(V_n f)(x) = \sum_{k=0}^{\infty} f\left(\tfrac{k}{n}\right) \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k \left(\frac{1}{1+x}\right)^n.$$

These are positive linear operators satisfying $V_n(1) = 1$ and $V_n(t) = t$ for the test functions 1 and t. By Korovkin's theorem, $V_n f \to f$ uniformly on compacts of $[0,\infty)$. Thus, in the lacunary statistical sense with respect to $\sigma(x) = 1 + x$, we have

$$V_n f \xrightarrow[\theta]{\text{lac-st}} f, \qquad f \in C[0,\infty).$$

As in the Bernstein case, the constant function $\mathbf{1}$ is an eigenfunction with eigenvalue 1, and the point spectrum accumulates at $\{1\}$. Hence the lac-SRU spectrum reduces to $\{1\}$.

- **Szász–Mirakyan operators.** On $C[0,\infty)$, the Szász–Mirakyan operators are defined by

$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

They satisfy $S_n(1) = 1$ and $S_n(t) = t$, and again by Korovkin's theorem we have $S_n f \to f$ uniformly on compacts. Therefore, for the weight $\sigma(x) = 1 + x$,

$$S_n f \xrightarrow[\theta]{\text{lac-st}} f, \qquad f \in C[0,\infty).$$

Since S_n is positive and $S_n(\mathbf{1}) = \mathbf{1}$, the eigenvalue 1 persists for each n, and by the Point Spectrum Approximation Theorem, any other eigenvalue must converge to 1 in the lac-SRU sense. Hence the limiting point spectrum is trivial and consists of $\{1\}$.

5 Conclusion

We introduced the notion of the lacunary statistically relatively uniform spectrum of positive linear operators and established spectral inclusion and stability results. This provides a new connection between approximation theory and spectral theory in the lacunary framework. Future research may extend these results to uncertain normed spaces, double lacunary sequences, or pseudospectral analysis.

References

[1] F. Altomare and M. Campiti, *Korovkin-type Approximation Theory and its Applications*, Walter de Gruyter, Berlin, 1994.

[2] G. Anastassiou and O. Duman, A Baskakov type generalization of statistical Korovkin theory, *J. Math. Anal. Appl.* 340 (2008), 476-486.

[3] M. Balcerzak, K. Dems and A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, *J. Math. Anal. Appl.* 328 (2007), 715-729.

[4] E.W. Chittenden, Relatively uniform convergence of sequences of functions, *Trans. Am. Math. Soc.* 15 (1914), 197-201.

[5] E.W. Chittenden, On the limit functions of sequences of continuous functions converging relatively uniformly, *Trans. Am. Math. Soc.* 20 (1919), 179-184.

[6] E.W. Chittenden, Relatively uniform convergence and classification of functions, *Trans. Am. Math. Soc.* 23 (1922), 1-15.

[7] J.B. Conway, *A Course in Functional Analysis*, 2nd ed., Springer, New York, 2000.

[8] K. Demirci and S. Orhan, Statistically relatively uniform convergence of positive linear operators, *Res Math* 69 (2016), 359-367.

[9] K. Demirci and S. Orhan, Statistical relative approximation on modular spaces, *Res Math.* 71 (2017), 1167-1184.

[10] K. Demirci, F. Dirik and S. Yıldız, Approximation via statistical relative uniform convergence of sequences of functions at a point with respect to power series method, *Afrika Matematika* 34 (2023), 39.

[11] K. Demirci, S. Orhan and B. Kolay, Weighted statistical relative approximation by positive linear operators, in *Operator Theory, Operator Algebras, and Matrix Theory*, Springer, 2018, pp. 131-139.

[12] R.A. DeVore, *The Approximation of Continuous Functions by Positive Linear Operators*, Lecture Notes in Mathematics, vol. 293, Springer, Berlin, 1972.

[13] O. Duman, M.K. Khan and C. Orhan, A statistical convergence of approximating operators, *Math. Inequal. Appl.* 6 (2003), 689-697.

[14] O. Duman, E. Erkuş and V. Gupta, Statistical rates on the multivariate approximation theory, *Math. Comput. Modell.* 44 (2006), 763-770.

[15] H. Fast, Sur la convergence statistique, *Colloq. Math.* 2 (1951), 241-244.

[16] A.R. Freedman, J.J. Sember and M. Raphael, Some Cesaro-type summability spaces, *Proc. London Math. Soc.* 37 (1978), 508-520.

[17] J.A. Fridy, Lacunary statistical convergence, *Proc. Am. Math. Soc.* 118 (1990), 1187-1192.

[18] A.D. Gadjiev and C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mt. J. Math.* 32 (2002), 129-138.

[19] P.P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publishing Corp., Delhi, 1960.

[20] P.D. Lax, *Functional Analysis*, Wiley-Interscience, New York, 2002.

[21] E.H. Moore, *An Introduction to a Form of General Analysis*, Yale University Press, New Haven, 1910.

[22] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* 2 (1951), 73-74.