

SOLVING COMMON FIXED POINT PROBLEMS WITH SUMMABLE ERRORS*

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Abstract

We study iterative methods for solving common fixed point problems in the presence of summable computational errors.

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1 Introduction

For more than sixty-five years now, there has been a lot of research activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) and contractive mappings. See, for example, [1, 6–10, 13–15, 17, 18] and references cited therein. In the present paper, we study iterative methods for solving common fixed point problems in the presence of summable computational errors.

Let (X, ρ) be a complete metric space. For each point $x \in X$ and each number $r > 0$, set

$$B(x, r) := \{y \in X : \rho(x, y) \leq r\}.$$

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In his seminal paper [12], Ostrowski established the following result.

Theorem 1. *Assume that $\gamma \in (0,1)$, the mapping $T : X \rightarrow X$ satisfies*

$$\rho(T(x), T(y)) \leq \gamma \rho(x, y), \quad x, y \in X,$$

and that a sequence $\{x_i\}_{i=0}^{\infty}$ satisfies

$$\sum_{i=0}^{\infty} \rho(x_{i+1}, T(x_i)) < \infty.$$

Then the sequence $\{x_i\}_{i=0}^{\infty}$ converges to a fixed point of T .

In [3], the following generalization of Ostrowski's theorem was obtained.

Theorem 2. *Assume that a mapping $T : X \rightarrow X$ satisfies*

$$\rho(T(x), T(y)) \leq \rho(x, y), \quad x, y \in X,$$

and that for each point $x \in X$, the sequence $\{T^i(x)\}_{i=1}^{\infty}$ converges to a fixed point of T . Then each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying

$$\sum_{i=0}^{\infty} \rho(x_{i+1}, T(x_i)) < \infty$$

converges to a fixed point of T .

This result became the starting point of the superiorization methodology, where it found many applications [2, 4, 5]. It means that if all exact iterates of a nonexpansive mapping converge, then the same holds true for all its inexact iterates with summable errors.

Let $T : X \rightarrow X$ be a self-mapping of X such that

$$\rho(T(x), T(y)) \leq \rho(x, y) \text{ for all } x, y \in X.$$

Set $T^0(x) := x$ for all $x \in X$. Since the existence of a fixed point of T is not guaranteed in general, we are also interested in approximate fixed points.

A point $x \in X$ is called a (γ) -approximate fixed point of the operator T , where $\gamma > 0$, if $\rho(x, T(x)) \leq \gamma$. Note that if C is a bounded, closed and convex subset of a Banach space and T is a nonexpansive self-mapping of C , then a (γ) -approximate fixed point of T does exist for each $\gamma > 0$. As a matter of fact, there are also unbounded sets C with this property. For more information regarding this issue see, for instance, [16], [8] and [11].

In the following theorem, which was obtained in [19], we assume that for an arbitrary number $\epsilon > 0$ and each point $x \in X$, the iterates $T^n(x)$ are (ϵ) -approximate fixed points of the operator T for all sufficiently large n . Under this assumption it is shown that for each $\epsilon > 0$ and each sequence $\{x_n\}_{n=0}^\infty$ of inexact iterates of the operator T with summable errors, x_n is an (ϵ) -approximate fixed point of T for all sufficiently large natural numbers n . Therefore, inexact iterates of T with summable errors have the same asymptotic behavior as its exact iterates.

Theorem 3. *Assume that for every point $x \in X$ and every number $\epsilon > 0$, there exists a natural number n_0 such that for all natural numbers $n \geq n_0$, we have*

$$\rho(T^{n+1}(x), T^n(x)) \leq \epsilon.$$

Let a sequence of numbers $\{\epsilon_i\}_{i=1}^\infty \subset (0, \infty)$ satisfy

$$\sum_{i=1}^{\infty} \epsilon_i < \infty$$

and let a sequence of points $\{x_i\}_{i=0}^\infty \subset X$ satisfy

$$\rho(x_{i+1}, T(x_i)) \leq \epsilon_{i+1} \text{ for each integer } i \geq 0.$$

Then for each positive number ϵ , there exists a natural number n_0 such that $\rho(x_i, T(x_i)) \leq \epsilon$ for each natural number $i \geq n_0$.

It was shown in [19] that the assumption (and hence the conclusion) of this result holds if

$$F := \{z \in X : T(z) = z\} \neq \emptyset$$

and

$$\rho(z, T(x))^2 + \bar{c}\rho(x, T(x))^2 \leq \rho(z, x)^2,$$

for all $z \in F$ and all $x \in X$, where the constant $\bar{c} \in (0, 1]$. (Note that these properties hold if T is the nearest point projection onto a closed and convex subset of a Hilbert space.)

In the present paper, we extend Theorem 3 to common fixed point problems.

2 Common fixed point problems

Assume that (X, ρ) is a complete metric space, $\bar{N} \geq m$ are natural numbers and that for each $i \in \{1, \dots, m\}$, the operator $T_i : X \rightarrow X$ satisfies

$$\rho(T_i(x), T_i(y)) \leq \rho(x, y), \quad x, y \in X. \quad (1)$$

Let the mapping

$$r : \{0, 1, \dots\} \rightarrow \{1, \dots, m\}$$

satisfy, for each integer $i \geq 0$,

$$\{1, \dots, m\} \subset r(\{i, \dots, i + \bar{N} - 1\}). \quad (2)$$

We consider the following problem:

$$\text{Find } x \in X \text{ such that } T_i(x) = x, \quad i = 1, \dots, m.$$

Since in general the existence of a solution to this problem is not guaranteed, in this paper we are interested in approximate solutions to this problem, that is, points $y \in X$ which satisfy

$$\rho(y, T_i(y)) \leq \gamma, \quad i = 1, \dots, m,$$

where γ is a small positive number. In order to obtain such an approximate solution we first choose a point $x \in X$ and then define

$$x_0 = x,$$

$$x_{i+1} = T_{r(i)}(x_i), \quad i = 0, 1, \dots$$

We assume that this iterative process produces approximate solutions to our common fixed point problem. This fact indeed holds for many important common fixed point problems [20]. Namely, we assume that the following property holds.

(P) For each $x \in X$ and each integer $k \geq 0$,

$$\lim_{n \rightarrow \infty} \rho\left(\prod_{i=0}^n T_{r(i+k)}(x), \prod_{i=0}^{n+1} T_{r(i+k)}(x)\right) = 0.$$

Note that property (P) indeed holds if for each $i \in \{1, \dots, m\}$ and each $x \in X$, we have

$$\rho(z, T_i(x))^2 + \bar{c}\rho(x, T_i(x))^2 \leq \rho(z, x)^2,$$

where z is a common fixed point and the constant $\bar{c} \in (0, 1]$ [20].

In the present paper, we establish the following result.

Theorem 4. Assume that $\{x_n\}_{n=0}^\infty \subset X$, $\{\epsilon_n\}_{n=0}^\infty \subset (0, \infty)$,

$$\sum_{n=0}^{\infty} \epsilon_n < \infty \quad (3)$$

and that for each integer $n \geq 0$, we have

$$\rho(x_{n+1}, T_{r(n)}(x_n)) \leq \epsilon_n. \quad (4)$$

Then

$$\lim_{n \rightarrow \infty} \rho(x_n, x_{n+1}) = 0$$

and for each $i \in \{1, \dots, m\}$,

$$\lim_{n \rightarrow \infty} \rho(x_n, T_i(x_n)) = 0.$$

Proof. Let $\epsilon > 0$ be given. By (3), there exists a natural number k such that

$$\sum_{n=k}^{\infty} \epsilon_n < \epsilon/4. \quad (5)$$

Property (P) implies that

$$\lim_{n \rightarrow \infty} \rho\left(\prod_{i=0}^n T_{r(i+k)}(x_k), \prod_{i=0}^{n+1} T_{r(i+k)}(x_k)\right) = 0. \quad (6)$$

In view of (6), there exists a natural number n_1 such that for each integer $n \geq n_1$, we have

$$\rho\left(\prod_{i=0}^n T_{r(i+k)}(x_k), \prod_{i=0}^{n+1} T_{r(i+k)}(x_k)\right) < \epsilon/4. \quad (7)$$

We now estimate

$$\rho\left(\prod_{i=0}^{n-1} T_{r(i+k)}(x_k), x_{k+n}\right), \quad n = 1, 2, \dots$$

It follows from (4) that

$$\rho(x_{k+1}, T_{r(k)}(x_k)) \leq \epsilon_k. \quad (8)$$

We claim that for each integer $n \geq 1$,

$$\rho\left(\prod_{i=0}^{n-1} T_{r(i+k)}(x_k), x_{k+n}\right) \leq \sum_{i=k}^{k+n-1} \epsilon_i. \quad (9)$$

In view of (8), inequality (9) holds for $n = 1$.

Assume that $n \geq 1$ is an integer and that (9) holds. Inequalities (1), (4) and (9) imply that

$$\begin{aligned} \rho\left(\prod_{i=0}^n T_{r(i+k)}(x_k), x_{k+n+1}\right) &\leq \rho(x_{k+n+1}, T_{r(n+k)}(x_{n+k})) \\ &\quad + \rho(T_{r(n+k)}(x_{n+k}), \prod_{i=0}^n T_{r(i+k)}(x_k)) \\ &\leq \epsilon_{k+n} + \rho(x_{n+k}, \prod_{i=0}^{n-1} T_{r(i+k)}(x_k)) \leq \sum_{i=k}^{k+n} \epsilon_i \end{aligned}$$

and so (9) holds for $n + 1$ too. Thus, we have shown by mathematical induction that inequality (9) holds for each integer $n \geq 1$, as claimed. It now follows from (5) and (9) that for each integer $n \geq 1$, we have

$$\rho\left(\prod_{i=0}^{n-1} T_{r(i+k)}(x_k), x_{k+n}\right) < \epsilon/4. \quad (10)$$

By (7) and (10), for each integer $n \geq n_1 + 1$,

$$\begin{aligned} \rho(x_{n+k}, x_{n+k+1}) &\leq \rho(x_{n+k}, \prod_{i=0}^{n-1} T_{r(i+k)}(x_k)) \\ &\quad + \rho\left(\prod_{i=0}^{n-1} T_{r(i+k)}(x_k), \prod_{i=0}^n T_{r(i+k)}(x_k)\right) + \rho\left(\prod_{i=0}^n T_{r(i+k)}(x_k), x_{k+n+1}\right) \\ &< \epsilon/4 + \rho\left(\prod_{i=0}^{n-1} T_{r(i+k)}(x_k), \prod_{i=0}^n T_{r(i+k)}(x_k)\right) + \epsilon/4 \leq 3\epsilon/4. \end{aligned}$$

Since ϵ is an arbitrary positive number, we conclude that

$$\lim_{n \rightarrow \infty} \rho(x_n, x_{n+1}) = 0. \quad (11)$$

Fix $p \in \{1, \dots, m\}$. Next, we claim that

$$\lim_{n \rightarrow \infty} \rho(x_n, T_p(x_n)) = 0.$$

Let $\epsilon > 0$. By (3), (4) and (11), there exists a natural number n_0 such that for each integer $n \geq n_0$,

$$\epsilon_n < \epsilon(2\bar{N} + 2)^{-1} \quad (12)$$

and

$$\rho(x_n, x_{n+1}) < \epsilon(2\bar{N} + 2)^{-1}. \quad (13)$$

Assume that $n \geq n_0$ is an integer. By (13), for each $i \in \{n, \dots, n + \bar{N}\}$,

$$\rho(x_n, x_i) \leq \bar{N}\epsilon(2\bar{N} + 2)^{-1}, \quad (14)$$

In view of (2), there is a natural number

$$j \in \{n, \dots, n + \bar{N} - 1\}$$

such that

$$r(j) = p. \quad (15)$$

By (4), (12), (13) and (15),

$$\begin{aligned} \rho(x_j, T_p(x_j)) &= \rho(x_j, T_{r(j)}(x_j)) \leq \rho(x_j, x_{j+1}) + \rho(x_{j+1}, T_{r(j)}(x_j)) \\ &\leq \epsilon(2\bar{N} + 2)^{-1} + \epsilon_j \leq \epsilon(\bar{N} + 1)^{-1}. \end{aligned} \quad (16)$$

In view of (1), (14) and (16), we have

$$\begin{aligned} \rho(x_n, T_p(x_n)) &\leq \rho(x_n, x_j) + \rho(x_j, T_p(x_j)) + \rho(T_p(x_j), T_p(x_n)) \\ &\leq 2\rho(x_n, x_j) + \epsilon(\bar{N} + 1)^{-1} \leq 2\bar{N}\epsilon(2\bar{N} + 2)^{-1} + \epsilon(\bar{N} + 1)^{-1} \leq \epsilon. \end{aligned}$$

Thus, for each integer $n \geq n_0$, we have

$$\rho(x_n, T_p(x_n)) \leq \epsilon.$$

Since ϵ is an arbitrary positive number, we infer that

$$\lim_{n \rightarrow \infty} \rho(x_n, T_p x_n) = 0.$$

This completes the proof of Theorem 4. □

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