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ON ρ-STRONG CONVERGENCE FOR DIFFERENCE SEQUENCES OF FRACTIONAL ORDER IN q-RUNG ORTHOPAIR FUZZY NORMED SPACES*

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Abstract

In this paper, we propose the concept of ρ-strong convergence for difference sequences of fractional order, denoted by Δ^α-ρ-strong convergence, in a q-rung orthopair fuzzy normed space. We establish the uniqueness of this convergence and provide its algebraic characterization. A convergence criterion for subsequences is derived, and the relationship between Δ^α-strong convergence and Δ^α-ρ-strong convergence is examined under conditions on $\liminf \frac{s}{\rho_s}$. Moreover, an inclusion result is obtained by employing a positive non-decreasing sequence μ_s satisfying $\rho_s < \mu_s$ under suitable assumptions. Finally, we introduce the notion of ρ-strongly Cauchy difference sequences of fractional order and investigate their connection with Δ^α-ρ-strong convergence.

Keywords: q-rung orthopair fuzzy normed space, t-norm, t-conorm, Δ^α-ρ-strong convergence, Δ^α-ρ-strong Cauchy sequence.

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1 Introduction and preliminaries

Since its introduction in 1965, fuzzy set theory [40] has developed into a powerful tool with applications in artificial intelligence, computer science, medicine, control engineering, decision theory, management science, operations research, pattern recognition, and robotics. Unlike classical sets with binary membership, fuzzy sets allow gradual degrees of membership, making them well-suited for modeling imprecision in human reasoning. Formally, for a nonempty set $\mathcal{U}$, a fuzzy set $\mathcal{A}$ is defined as

$$\mathcal{A} = \{\langle \vartheta, \Phi_{\mathcal{A}}(\vartheta)\rangle : \vartheta \in \mathcal{U}\},$$

where $\Phi_{\mathcal{A}} : \mathcal{U} \to [0, 1]$ is the membership function.

To overcome the limitation that non-membership is not always the complement of membership, Atanassov [1] introduced intuitionistic fuzzy sets (IFS). An IFS $\mathcal{A}$ in $\mathcal{U}$ is given by

$$\mathcal{A} = \{\langle \vartheta, \Phi_{\mathcal{A}}(\vartheta), \Psi_{\mathcal{A}}(\vartheta)\rangle : \vartheta \in \mathcal{U}\},$$

where $\Phi_{\mathcal{A}}$ and $\Psi_{\mathcal{A}}$ denote, respectively, membership and non-membership functions satisfying

$$0 \leq \Phi_{\mathcal{A}}(\vartheta) + \Psi_{\mathcal{A}}(\vartheta) \leq 1.$$

The hesitation margin is defined as

$$\pi_{\mathcal{A}}(\vartheta) = 1 - \Phi_{\mathcal{A}}(\vartheta) - \Psi_{\mathcal{A}}(\vartheta),$$

which quantifies the degree of uncertainty.

Yager [33] extended this idea to Pythagorean fuzzy sets (PFS), where membership and non-membership degrees satisfy

$$(\Phi_{\mathcal{A}}(\vartheta))^2 + (\Psi_{\mathcal{A}}(\vartheta))^2 \leq 1.$$

This provides greater flexibility in handling uncertainty compared to IFS. To generalize further, Yager [34] introduced q-rung orthopair fuzzy sets (q-ROFS), characterized by the condition

$$(\Phi_{\mathcal{A}}(\vartheta))^q + (\Psi_{\mathcal{A}}(\vartheta))^q \leq 1,$$

where $q \geq 1$. Clearly, IFS and PFS appear as special cases for $q = 1$ and $q = 2$, respectively. Increasing q enlarges the admissible domain, allowing a richer representation of fuzzy information.

This framework has been further advanced in functional analysis and topology. Yager and Alajlan [35] developed approximate reasoning techniques within q-ROFS, while Turkarslan et al. [30] introduced q-rung orthopair fuzzy topological spaces. Parimala et al. [23] studied their supra-topological applications, and Saeed and Ibrahim [28] investigated n, m^{th} power root fuzzy sets. Uluçay introduced q-rung orthopair fuzzy normed spaces and established new notions of statistical convergence, statistical Cauchy sequences, and completeness criteria, thereby deepening the understanding of convergence behavior in this setting.

1.1 Identified research gaps and underlying motivation

The concept of difference sequence spaces was first introduced by Kızmaz [15] and later generalized by Et et al. [13, 14] as follows:

$$\Delta^m(\mathcal{X}) = \{\hbar = (\hbar_k) : (\Delta^m \hbar_k) \in \mathcal{X}\},$$

where $\mathcal{X}$ is any sequence space, $m \in \mathbb{N}$,

$$\Delta^0 \hbar = (\hbar_k), \quad \Delta \hbar = (\hbar_k - \hbar_{k+1}), \quad \Delta^m \hbar = (\Delta^m \hbar_k) = \left(\Delta^{m-1}\hbar_k - \Delta^{m-1}\hbar_{k+1}\right).$$

Thus,

$$\Delta^m \hbar_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} \hbar_{k+v}.$$

If $\hbar \in \Delta^m(\mathcal{X})$, then there exists a unique sequence $\vartheta = (\vartheta_k) \in \mathcal{X}$ such that $\vartheta_k = \Delta^m \hbar_k$ and

$$\hbar_k = \sum_{v=1}^{k-m} (-1)^m \binom{k-v-1}{m-1} \vartheta_v = \sum_{v=1}^{k} (-1)^m \binom{k+m-v-1}{m-1} \vartheta_{v-m}, \quad (1)$$

with $\vartheta_{1-m} = \vartheta_{2-m} = \cdots = \vartheta_0 = 0$ for sufficiently large k (e.g., $k > 2m$). Since then, several properties of difference sequence spaces have been studied in [2, 3, 14, 17, 26, 29].

For a proper fraction α, the *fractional difference operator* $\Delta^\alpha : \mathcal{W} \to \mathcal{W}$ is defined by

$$\Delta^\alpha(\hbar_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)} \hbar_{k+i}. \quad (2)$$

In particular,

$$\begin{aligned}
\Delta^{1/2}\hbar_k &= \hbar_k - \tfrac{1}{2}\hbar_{k+1} - \tfrac{1}{8}\hbar_{k+2} - \tfrac{1}{16}\hbar_{k+3} - \tfrac{5}{128}\hbar_{k+4} - \cdots,\\
\Delta^{-1/2}\hbar_k &= \hbar_k + \tfrac{1}{2}\hbar_{k+1} + \tfrac{3}{8}\hbar_{k+2} + \tfrac{5}{16}\hbar_{k+3} + \tfrac{35}{128}\hbar_{k+4} + \cdots,\\
\Delta^{1/3}\hbar_k &= \hbar_k - \tfrac{1}{3}\hbar_{k+1} - \tfrac{1}{9}\hbar_{k+2} - \tfrac{5}{81}\hbar_{k+3} - \tfrac{10}{243}\hbar_{k+4} - \cdots,\\
\Delta^{2/3}\hbar_k &= \hbar_k - \tfrac{2}{3}\hbar_{k+1} - \tfrac{1}{9}\hbar_{k+2} - \tfrac{4}{81}\hbar_{k+3} - \tfrac{7}{243}\hbar_{k+4} - \cdots.
\end{aligned}$$

Here, $\Gamma(r)$ denotes the Gamma function for real numbers

$$r \notin \{0, -1, -2, -3, \dots\},$$

defined by the improper integral

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1}\, dt.$$

Its well-known properties include:

- For any natural number n, $\Gamma(n+1) = n!$,
- For real $n \notin \{0, -1, -2, \dots\}$, $\Gamma(n+1) = n\Gamma(n)$,
- Special cases: $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2!$, $\Gamma(4) = 3!$, etc.

By definition, the series in (2) is convergent. Moreover, if α is a positive integer, the infinite sum in (2) reduces to a finite sum:

$$\sum_{i=0}^{\alpha} (-1)^i \frac{\Gamma(\alpha+1)}{i!\,\Gamma(\alpha-i+1)} \hbar_{k+i}.$$

Thus, the operator Δ^α generalizes the classical difference operator introduced by Et and Çolak [13].

Recently, using this fractional difference operator Δ^α, Baliarsingh et al. [7, 8, 22] introduced the sequence space

$$\Delta^\alpha(\mathcal{X}) = \{\hbar = (\hbar_k) : (\Delta^\alpha \hbar_k) \in \mathcal{X}\},$$

where $\mathcal{X}$ is any sequence space.

The concept of ρ-statistical convergence was first introduced by Çakallı [10] where $\rho = (\rho_s)$ is a non-decreasing sequence of positive reals tending to ∞ such that $\limsup_s \frac{\rho_s}{s} < \infty$, $\Delta\rho_s = O(1)$ and $\Delta\rho_s = \rho_{s+1} - \rho_s$ for each positive integer s. Several authors have extensively investigated convergence theory through the use of the sequence $\{\rho_s\}$, including Barlak [9], Debnath and Debnath [12], Kandemir [19], among others. In recent years, the study of fractional order difference sequence spaces has emerged as an active and significant area of research. Notable contributions in this field include the works of Yaying and Hazarika [38], Kadak [18], Srivastava and Mahato [27], Yaying [39], and many others, where investigations have been carried out in various settings such as neutrosophic normed spaces [4] and n-normed spaces [11]. Furthermore, for the development of difference sequence spaces of fractional order, we refer the readers to [36, 37], while a comprehensive account of sequence convergence theory can be found in [5, 20, 21]. These works form part of the fundamental concepts that underpin and enrich our present study. Notably, our study draws its primary motivation from the seminal work of Aral et al. [6].

Although the concepts of convergence based on the sequence $\{\rho_s\}$ and difference sequences of fractional order have been extensively studied by numerous researchers, there remains a strong need to explore these notions within broader and more abstract frameworks, particularly in the setting of q-rung orthopair fuzzy normed spaces. Despite several notable contributions such as investigations into statistical convergence, the study of summability theory and sequence convergence in q-rung orthopair fuzzy normed spaces is still in its early stages. In particular, the extension of statistical convergence to generalized forms, such as ρ-statistical convergence for difference sequences of fractional order, remains largely unexplored. This indicates a significant gap in understanding the interplay between ρ-strong convergence for difference sequences of fractional order under appropriate conditions. Moreover, there exists ample scope to extend this line of research by incorporating modulus functions in a q-rung orthopair fuzzy normed space.

1.2 Key contributions

The main contributions of this paper are as follows. We introduce the notion of ρ-strong convergence for difference sequences of fractional order (denoted by Δ^α-ρ-strong convergence) in the framework of q-rung orthopair fuzzy normed spaces, thereby extending classical convergence concepts into a broader and more abstract setting. We establish the uniqueness of Δ^α-ρ-strong convergence and provide its algebraic characterization, along with a criterion for the Δ^α-ρ-strong convergence of subsequences, ensuring stability of convergence under subsequential analysis. By imposing suitable conditions on $\liminf \frac{s}{\rho_s}$, we further examine the relationship between Δ^α-strong convergence and Δ^α-ρ-strong convergence. In addition, we present a significant inclusion result involving Δ^α-ρ-strong convergence by employing a positive non-decreasing sequence (μ_s) with $\rho_s < \mu_s$ under appropriate assumptions. Finally, we introduce and investigate the concept of ρ-strongly Cauchy difference sequences of fractional order in q-rung orthopair fuzzy normed spaces, and explore its connection with Δ^α-ρ-strong convergence.

To proceed, we first revisit essential definitions concerning q-rung orthopair fuzzy normed spaces. Unless otherwise stated, $\mathbb{N}$ and $\mathbb{R}$ will denote the sets of natural and real numbers, respectively.

Definition 1. *[24] A mapping $\boxtimes$, named as binary operation, from $\mathcal{O} \times \mathcal{O}$ to $\mathcal{O}$, where $\mathcal{O} = [0,1]$, is referred to as a continuous t-norm if for each $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathcal{O}$, the conditions listed below are met:*

1. *$\boxtimes$ exhibits both associativity and commutativity;*
2. *$\boxtimes$ exhibits continuous behavior;*
3. *$\vartheta_1 \boxtimes 1 = \vartheta_1$, $\forall\ \vartheta_1 \in \mathcal{O}$;*
4. *$\vartheta_1 \boxtimes \vartheta_2 \leq \vartheta_3 \boxtimes \vartheta_4$ whenever $\vartheta_1 \leq \vartheta_3$ and $\vartheta_2 \leq \vartheta_4$.*

Definition 2. *[24] A mapping $\otimes$, named as binary operation, from $\mathcal{O} \times \mathcal{O}$ to $\mathcal{O}$, where $\mathcal{O} = [0,1]$, is referred to as a continuous t-conorm if for each $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathcal{O}$, the conditions listed below are met:*

1. *$\otimes$ exhibits both associativity and commutativity;*
2. *$\otimes$ exhibits continuous behavior;*
3. *$\vartheta_1 \otimes 0 = \vartheta_1$, $\forall\ \vartheta_1 \in \mathcal{O}$;*
4. *$\vartheta_1 \otimes \vartheta_2 \leq \vartheta_3 \otimes \vartheta_4$ whenever $\vartheta_1 \leq \vartheta_3$ and $\vartheta_2 \leq \vartheta_4$.*

Example 1. *[16] Some illustrations of $\boxtimes$ and $\otimes$ are: $\vartheta_1 \boxtimes \vartheta_2 = \min\{\vartheta_1, \vartheta_2\}$ and $\vartheta_1 \boxtimes \vartheta_2 = \vartheta_1.\vartheta_2$. $\vartheta_1 \otimes \vartheta_2 = \max\{\vartheta_1, \vartheta_2\}$ and $\vartheta_1 \otimes \vartheta_2 = \vartheta_1 + \vartheta_2 - \vartheta_1.\vartheta_2$.*

Lemma 1. *[25] If $\boxtimes$ is a continuous t-norm, $\otimes$ is a continuous t-conorm, $\vartheta_i \in (0,1)$ and $1 \leq i \leq 7$, then the following statements hold:*

1. *If $\vartheta_1 > \vartheta_2$, there are $\vartheta_3, \vartheta_4 \in (0,1)$ such that $\vartheta_1 \boxtimes \vartheta_3 \geq \vartheta_2$ and $\vartheta_1 \geq \vartheta_2 \otimes \vartheta_4$*
2. *If $\vartheta_5 \in (0,1)$, there are $\vartheta_6, \vartheta_7 \in (0,1)$ such that $\vartheta_6 \boxtimes \vartheta_6 \geq \vartheta_5$ and $\vartheta_5 \geq \vartheta_7 \otimes \vartheta_7$.*

Definition 3. *[40] A set $\mathcal{C}$ of the form*

$$\mathcal{C} = \{(\vartheta, \mathcal{R}_{\mathcal{C}}(\vartheta)) : \vartheta \in \mathcal{S}\}$$

is considered to be a fuzzy set on a non-empty set $\mathcal{S}$. For each element $\vartheta \in \mathcal{S}$, the function $\mathcal{R}_{\mathcal{C}}(\vartheta)$ represents the membership degree of ϑ in $\mathcal{C}$, where $\mathcal{R}_{\mathcal{C}}(\vartheta) \in [0, 1]$.

Definition 4. *Let $\mathcal{C}$ be a set defined as:*

$$\mathcal{C} = \{(\vartheta, \mathcal{R}_{\mathcal{C}}(\vartheta), \mathcal{V}_{\mathcal{C}}(\vartheta)) : \vartheta \in \mathcal{S}\}$$

where $\mathcal{S}$ is a non-empty set. Depending on the constraints imposed on the membership $\mathcal{R}_{\mathcal{C}}(\vartheta)$ and non-membership $\mathcal{V}_{\mathcal{C}}(\vartheta)$ functions, $\mathcal{C}$ can be classified as follows:

1. *Intuitionistic fuzzy set [1]: when $0 \leq \mathcal{R}_{\mathcal{C}}(\vartheta) + \mathcal{V}_{\mathcal{C}}(\vartheta) \leq 1$.*
2. *Pythagorian fuzzy set [32]: when $0 \leq \mathcal{R}^2_{\mathcal{C}}(\vartheta) + \mathcal{V}^2_{\mathcal{C}}(\vartheta) \leq 1$.*
3. *q-rung orthopair fuzzy set [34]: when $0 \leq \mathcal{R}^q_{\mathcal{C}}(\vartheta) + \mathcal{V}^q_{\mathcal{C}}(\vartheta) \leq 1$.*

for each $\vartheta \in \mathcal{S}$, $\mathcal{R}_{\mathcal{C}}(\vartheta)$ represents the membership function, while $\mathcal{V}_{\mathcal{C}}(\vartheta)$ denotes the non-membership function of the set $\mathcal{C}$.

Remark 1. *q-rung orthopair fuzzy set turns into intuitionistic fuzzy set whenever $q = 1$ while it coincides with the notion of Pythagorian fuzzy set whenever $q = 2$.*

Definition 5. *[31] Let $\mathcal{Q}$ be a vector space, $\boxtimes$ be a continuous t-norm and $\otimes$ be a continuous t-conorm. Also, let $\mathcal{R}$ and $\mathcal{V}$ be two fuzzy sets on $\mathcal{Q} \times (0, \infty)$ and $q \geq 1$ be a real number. Then, the 5-tuple $(\mathcal{Q}, \mathcal{R}, \mathcal{V}, \boxtimes, \otimes)$ is termed a q-rung orthopair fuzzy normed space (abbreviated q-ROFNS) if for every $\vartheta, \kappa \in \mathcal{Q}$ and $\zeta, \gamma > 0$ the conditions listed below are met:*

1. $\mathcal{R}^q(\vartheta, \zeta) + \mathcal{V}^q(\vartheta, \zeta) \leq 1$,
2. $\mathcal{R}(\vartheta, \zeta) > 0$,
3. $\mathcal{R}^q(\vartheta, \zeta) = 1$ *if and only if* $\vartheta = \theta$,
4. $\mathcal{R}^q(\alpha\vartheta, \zeta) = \mathcal{R}^q\left(\vartheta, \frac{\zeta}{|\alpha|}\right)$, *for each* $\alpha \neq 0$,
5. $\mathcal{R}^q(\vartheta, \zeta) \boxtimes \mathcal{R}^q(\kappa, \gamma) \leq \mathcal{R}^q(\vartheta + \kappa, \zeta + \gamma)$,

6. $\mathcal{R}(\vartheta, \cdot) : (0, \infty) \to [0, 1]$ *is continuous,*

7. $\lim_{\zeta\to\infty} \mathcal{R}(\vartheta, \zeta) = 1$ *and* $\lim_{\zeta\to 0} \mathcal{R}(\vartheta, \zeta) = 0$,

8. $\mathcal{V}(\vartheta, \zeta) < 1$,

9. $\mathcal{V}^q(\vartheta, \zeta) = 0$ *if and only if* $\vartheta = \theta$,

10. $\mathcal{V}^q(\alpha\vartheta, \zeta) = \mathcal{V}^q\left(\vartheta, \frac{\zeta}{|\alpha|}\right)$, *for each* $\alpha \neq 0$,

11. $\mathcal{V}^q(\vartheta, \zeta) \otimes \mathcal{V}^q(\kappa, \gamma) \geq \mathcal{V}^q(\vartheta + \kappa, \zeta + \gamma)$,

12. $\mathcal{V}(\vartheta, \cdot) : (0, \infty) \to [0, 1]$ *is continuous,*

13. $\lim_{\zeta\to\infty} \mathcal{V}(\vartheta, \zeta) = 0$ *and* $\lim_{\zeta\to 0} \mathcal{V}(\vartheta, \zeta) = 1$.

In this case, $(\mathcal{R}, \mathcal{V})$ is termed a q-rung orthopair fuzzy norm (abbreviated q-ROFN). When $q = 2$, this structure corresponds to a Pythagorean fuzzy normed space, whereas for $q = 1$, it simplifies to an intuitionistic fuzzy normed space. In the sequel $\mathcal{Q}$ will stands for the 5-tuple $(\mathcal{Q}, \mathcal{R}, \mathcal{V}, \boxtimes, \otimes)$ for the sake of abbreviation.

Notably, every intuitionistic fuzzy normed space inherently qualifies as a q-rung orthopair fuzzy normed space. However, the reverse does not always hold true. The following example illustrates this distinction.

Example 2. *[31] Let $\mathcal{Q} = \mathbb{R}$ on which usual norm is imposed. Consider t-norm and t-conorm as $\vartheta_1 \boxtimes \vartheta_2 = \vartheta_1\vartheta_2$ and $\vartheta_1 \otimes \vartheta_2 = \min\{\vartheta_1 + \vartheta_2, 1\}$, $\forall\vartheta_1, \vartheta_2 \in [0, 1]$. define $\mathcal{R}_0(\vartheta, \zeta) = \sqrt[q]{\frac{\zeta}{\zeta+|\vartheta|}}$ and $\mathcal{V}_0(\vartheta, \zeta) = \sqrt[q]{\frac{|\vartheta|}{\zeta+|\vartheta|}}$. Then $(\mathbb{R}, \mathcal{R}_0, \mathcal{V}_0, \boxtimes, \otimes)$ becomes q-rung orthopair fuzzy normed space but not intuitionistic fuzzy normed space.*

Definition 6. *[31] Consider $\{\vartheta_i\}$ to be a sequence in a q-ROFNS $\mathcal{Q}$. Then, $\{\vartheta_i\}$ is termed convergent to $\varrho \in \mathcal{Q}$ if $\mathcal{R}^q(\vartheta_i - \varrho, \zeta) \to 1$ and $\mathcal{V}^q(\vartheta_i - \varrho, \zeta) \to 0$ whenever $i \to \infty$ for every $\zeta > 0$.*

We now introduce the concept of convergence for difference sequences of fractional order with respect to $(\mathcal{R}, \mathcal{V})$.

Definition 7. *Consider a sequence $\hbar_k$ in a q-rung orthopair fuzzy normed space $\mathcal{Q}$. Then, $\hbar_k$ is referred to as Δ^α-convergent to $\tau \in \mathcal{Q}$ in relation to $(\mathcal{R}, \mathcal{V})$ (briefly, $(\mathcal{R}, \mathcal{V})_q(\Delta^\alpha)$-convergence) if, for every $\lambda > 0$ and $\varpi \in (0, 1)$, there can be found $k_0 \in \mathbb{N}$ such that*

$$\mathcal{R}^q(\Delta^\alpha\hbar_k - \tau, \lambda) > 1 - \varpi \quad \text{and} \quad \mathcal{V}^q(\Delta^\alpha\hbar_k - \tau, \lambda) < \varpi,\ \forall k \geq k_0.$$

In this scenario, we express $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})_q(\Delta^\alpha)} \tau$.

Lemma 2. *[31] A sequence* $\{\vartheta_i\}$ *in q-ROFNS* $(\mathbb{R}, \mathcal{R}_0, \mathcal{V}_0, \boxtimes, \otimes)$ *is convergent if and only if it is convergent in* $(\mathbb{R}, |\cdot|)$.

2 Main results

This section is devoted to presenting our main results. Throughout this section, $\mathcal{Q}$ denotes a q-rung orthopair fuzzy normed space, and α represents a proper fraction, unless specified otherwise. Henceforth, Δ^α is regarded as closed linear operator on the sequence space $\mathcal{Q}$, ensuring that both $\Delta^\alpha \hbar_k$ and the difference $\Delta^\alpha \hbar_k - \tau$ are well defined.

Definition 8. *Consider a sequence* $\{\hbar_k\}$ *in* $\mathcal{Q}$. *Then,* $\{\hbar_k\}$ *is referred to as* $\Delta^\alpha - \rho$*-strongly convergent to* $\tau \in \mathcal{Q}$ *in relation to* $(\mathcal{R}, \mathcal{V})$ *if for every* $\lambda > 0$ *and* $\varpi \in (0,1)$ *there can be found* $s_0 \in \mathbb{N}$ *such that*

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda) > 1 - \varpi \quad \textit{and} \quad \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q(\Delta^\alpha \hbar_k - \tau, \lambda) < \varpi, \ \forall s \geq s_0.$$

In this scenario, we express $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha, \rho)$. *When* $\rho_s = s$, *it follows that* $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha)$.

Next, we present an example illustrating the sequence of Δ^α-ρ-strong convergence for difference sequences of fractional order in a q-rung orthopair fuzzy normed space $\mathcal{Q}$.

Example 3. *Let* $\mathcal{Q} = \mathbb{R}$ *equipped with the usual norm. Define the t-norm and t-conorm, respectively, by*

$$\vartheta_1 \boxtimes \vartheta_2 = \vartheta_1 \vartheta_2, \quad \vartheta_1 \otimes \vartheta_2 = \min\{\vartheta_1 + \vartheta_2, 1\} \quad \forall \vartheta_1, \vartheta_2 \in [0,1].$$

We consider the q-ROFN given as in Example 2. With this structure, $\mathcal{Q}$ *becomes a q-ROFNS. Now, define a sequence* $\{\hbar_k\}$ *by*

$$\Delta^\alpha \hbar_k = \begin{cases} 1, & \textit{if } k = b^2, \ b \in \mathbb{N} \\ 0, & \textit{otherwise.} \end{cases}$$

For any $\lambda > 0$ *and* $0 < \varpi < 1$, *set*

$$\mathcal{B} = \{k \leq s : \mathcal{R}^q(\Delta^\alpha \hbar_k, \lambda) > 1 - \varpi \quad \textit{and} \quad \mathcal{V}^q(\Delta^\alpha \hbar_k, \lambda) < \varpi\}.$$

We now compute:

$$\begin{aligned}\mathcal{B} &= \{k \le s : \mathcal{R}^q(\Delta^\alpha \hbar_k, \lambda) > 1-\varpi \quad \textit{and} \quad \mathcal{V}^q(\Delta^\alpha \hbar_k, \lambda) < \varpi\} \\ &= \left\{k \le s : \frac{\lambda}{\lambda + |\Delta^\alpha \hbar_k|} > 1-\varpi \quad \textit{and} \quad \frac{|\Delta^\alpha \hbar_k|}{\lambda + |\Delta^\alpha \hbar_k|} < \varpi\right\} \\ &= \left\{k \le s : |\Delta^\alpha \hbar_k| < \frac{\varpi\lambda}{1-\varpi}\right\} \\ &\subseteq \{k \le s : |\Delta^\alpha \hbar_k| = 1\} \\ &= \{k \le s : k = b^2\}.\end{aligned}$$

Hence,

$$\left\{s \in \mathbb{N} : \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q(\Delta^\alpha \hbar_k, \lambda) > 1-\varpi \quad \textit{and} \quad \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q(\Delta^\alpha \hbar_k, \lambda) < \varpi\right\}$$

is a finite set. Therefore, we conclude that $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} 0(\Delta^\alpha, \rho)$.

In the following theorem, we determine the limit of Δ^α-ρ-strong convergence in a q-rung orthopair fuzzy normed space.

Theorem 1. *Consider a sequence* $\{\hbar_k\}$ *in a* q*-rung orthopair fuzzy normed spaces* $\mathcal{Q}$*. If* $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha, \rho)$*, then the limit* τ *is uniquely determined.*

Proof. Let $\varpi \in (0,1)$ be fixed. Select $\gamma \in (0,1)$ satisfying

$$(1-\gamma) \boxtimes (1-\gamma) > 1-\varpi \quad \gamma \otimes \gamma < \varpi.$$

Suppose that

$$\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau_1(\Delta^\alpha, \rho) \quad \text{and} \quad \hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau_2(\Delta^\alpha, \rho)$$

with $\tau_1 \neq \tau_2$. By the definition of convergence, for any $\lambda > 0$ and $\gamma \in (0,1)$, there exist $s_1, s_2 \in \mathbb{N}$ such that

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q\left(\Delta^\alpha \hbar_k - \tau_1, \frac{\lambda}{2}\right) > 1-\gamma \quad \text{and} \quad \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q\left(\Delta^\alpha \hbar_k - \tau_1, \frac{\lambda}{2}\right) < \gamma,$$

for all $s \ge s_1$, and

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q\left(\Delta^\alpha \hbar_k - \tau_2, \frac{\lambda}{2}\right) > 1-\gamma \quad \text{and} \quad \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q\left(\Delta^\alpha \hbar_k - \tau_2, \frac{\lambda}{2}\right) < \gamma,$$

for all $s \geq s_2$. Let $s_0 = \max\{s_1, s_2\}$. Then, for all $s \geq s_0$, there exists a positive integer t such that

$$\begin{aligned}\mathcal{R}^q(\tau_1 - \tau_2, \lambda) &\geq \mathcal{R}^q\left(\Delta^\alpha \hbar_t - \tau_1, \frac{\lambda}{2}\right) \boxtimes \mathcal{R}^q\left(\Delta^\alpha \hbar_t - \tau_2, \frac{\lambda}{2}\right)\\ &> (1-\gamma) \boxtimes (1-\gamma)\\ &> 1 - \varpi\end{aligned}$$

and similarly,

$$\begin{aligned}\mathcal{V}^q(\tau_1 - \tau_2, \lambda) &\leq \mathcal{V}^q\left(\Delta^\alpha \hbar_t - \tau_1, \frac{\lambda}{2}\right) \otimes \mathcal{V}^q\left(\Delta^\alpha \hbar_t - \tau_2, \frac{\lambda}{2}\right)\\ &< \gamma \otimes \gamma\\ &< \varpi.\end{aligned}$$

Since $\varpi > 0$ is arbitrary, it follows that $\mathcal{R}^q(\tau_1 - \tau_2, \lambda) = 1$ and $\mathcal{V}^q(\tau_1 - \tau_2, \lambda) = 0$. This implies that $\tau_1 = \tau_2$, contradicting our initial assumption. Thus, the proof stands established. □

Next, we turn to the algebraic characterization of Δ^α-ρ-strong convergence for sequences in $\mathcal{Q}$.

Theorem 2. *Let $\{\hbar_k\}$ and $\{\eta_k\}$ be sequences in a q-rung orthopair fuzzy normed space $\mathcal{Q}$ over the field $\mathbb{R}$. Then, the following properties hold:*

1. *If $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau_1(\Delta^\alpha, \rho)$ and $\eta_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau_2(\Delta^\alpha, \rho)$, then $\hbar_k + \eta_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau_1 + \tau_2(\Delta^\alpha, \rho)$.*

2. *If $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha, \rho)$, then for any scalar $\kappa \in \mathbb{R} \setminus \{0\}$, $\kappa\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \kappa\tau(\Delta^\alpha, \rho)$.*

Proof. Since the proof is routine, we omit the details. □

We now proceed to establish the Δ^α-ρ-strong convergence criterion for subsequences of a given sequence.

Theorem 3. *If $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha, \rho)$, then there exists a subsequence $\Delta^\alpha \hbar_{j_k}$ of $\Delta^\alpha \hbar_k$ such that $\hbar_{j_k} \xrightarrow{(\mathcal{R},\mathcal{V})_q(\Delta^\alpha)} \tau$.*

Proof. Suppose that $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha, \rho)$. Then, for every $\lambda > 0$ and $\varpi \in (0, 1)$ there exists $s_0 \in \mathbb{N}$ such that

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda) > 1 - \varpi \quad \text{and} \quad \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q(\Delta^\alpha \hbar_k - \tau, \lambda) < \varpi,\ \forall s \geq s_0.$$

It follows that for each $s \geq s_0$, we may choose $j_k \leq s$ such that

$$\mathcal{R}^q(\Delta^\alpha \hbar_{j_k} - \tau, \lambda) > \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda) > 1 - \varpi$$
$$\mathcal{V}^q(\Delta^\alpha \hbar_{j_k} - \tau, \lambda) < \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q(\Delta^\alpha \hbar_k - \tau, \lambda) < \varpi.$$

By virtue of Definition 7, the subsequence $\{\hbar_{j_k}\}$ adheres to the $(\mathcal{R}, \mathcal{V})_q(\Delta^\alpha)$-convergence condition sharing the same limit τ. Consequently,

$$\hbar_{j_k} \xrightarrow{(\mathcal{R},\mathcal{V})_q(\Delta^\alpha)} \tau$$

. This completes the proof. □

In the sequel, depending on the condition of $\frac{s}{\rho_s}$, we establish the connection between Δ^α-strong convergence and Δ^α-ρ-strong convergence in $\mathcal{Q}$.

Theorem 4. *Let $\{\hbar_k\}$ be a sequence in $\mathcal{Q}$ with $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha)$. Then, $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha, \rho)$ whenever $\liminf_s \frac{s}{\rho_s} > 1$.*

Proof. Suppose that

$$\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha), \tag{3}$$

and

$$\liminf_{s\to\infty} \frac{s}{\rho_s} > 1. \tag{4}$$

We aim to show that this implies

$$\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha, \rho).$$

For any $\lambda > 0$ and $0 < \varpi < 1$, consider

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda).$$

This expression can be rewritten as

$$\frac{1}{\rho_s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k-\tau,\lambda)=\frac{s}{\rho_s}\cdot\frac{1}{s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k-\tau,\lambda). \tag{5}$$

From (3), we know that $\hbar_k$ converges to τ in the $(\mathcal{R},\mathcal{V})$-sense with respect to Δ^α. This means that for sufficiently large s,

$$\frac{1}{s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k-\tau,\lambda)>1-\varpi.$$

By assumption (4), for sufficiently large s we have $\frac{s}{\rho_s}\geq 1$. Substituting into (5), we obtain

$$\frac{1}{\rho_s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k-\tau,\lambda)\geq\frac{1}{s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k-\tau,\lambda)>1-\varpi.$$

Similarly, using (3), for large s we have $\frac{1}{s}\sum_{k=1}^{s}\mathcal{V}^q(\Delta^\alpha \hbar_k-\tau,\lambda)<\varpi$. Since $\frac{s}{\rho_s}\geq 1$, it follows that $\frac{1}{\rho_s}\sum_{k=1}^{s}\mathcal{V}^q(\Delta^\alpha \hbar_k-\tau,\lambda)<\varpi$. Thus, under the conditions (3) and (4), we conclude that $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha,\rho)$. □

Remark 2. *Condition (4) ensures that the ratio $\frac{s}{\rho_s}$ does not fall below 1 in the limit. In other words, the sequence $\{\rho_s\}$ does not reduce the effect of the terms in the summation. Since convergence already holds in the (Δ^α)-sense, this condition guarantees that the same convergence persists in the more generalized form (Δ^α,ρ)-convergence. The above result shows that (Δ^α)-convergence implies (Δ^α,ρ)-convergence whenever $\liminf\frac{s}{\rho_s}>1$. Thus, the (Δ^α,ρ)-convergence can be viewed as a natural extension of the standard (Δ^α)-convergence under a suitable condition imposed on ρ_s.*

Now, we investigate the relationship between Δ^α-ρ-strong convergence and Δ^α-μ-strong convergence with respect to two non-decreasing positive sequences (ρ_s) and (μ_s) where $\rho_s<\mu_s$ for all $s\in\mathbb{N}$. The inclusion and equality of these convergence classes depend on how the ratio $\frac{\mu_s}{\rho_s}$ behaves when s becomes very large.

Theorem 5. *Let $\{\rho_s\}$ and $\{\mu_s\}$ be two sequences such that $\rho_s<\mu_s$ for all $s\in\mathbb{N}$, and let $\mathcal{B}\subseteq\mathcal{Q}$ be a q-rung orthopair fuzzy bounded set. Then, every sequence that is Δ^α-ρ-strongly convergent is also Δ^α-μ-strongly convergent, provided that*

$$\lim_{s\to\infty}\frac{\mu_s}{\rho_s}>0. \tag{6}$$

Proof. Since $\hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha,\rho)$ and $\mathcal{B}$ is q-rung orthopair fuzzy bounded, there exists some $\lambda > 0$ such that for each $(\Delta^\alpha \hbar_k - \tau) \in \mathcal{B}$, we have

$$\frac{1}{\rho_s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda) > 1 - \varpi \quad \text{and} \quad \frac{1}{\rho_s}\sum_{k=1}^{s}\mathcal{V}^q(\Delta^\alpha \hbar_k - \tau, \lambda) < \varpi,$$

for some $0 < \varpi < 1$.

Now, since $\rho_s \leq \mu_s$ for all $s \in \mathbb{N}$, we may rewrite the first inequality as

$$\frac{1}{\rho_s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda) = \frac{\mu_s}{\rho_s} \cdot \frac{1}{\mu_s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda).$$

By assumption (6), we know that

$$\lim_{s\to\infty} \frac{\mu_s}{\rho_s} > 0,$$

which guarantees that $\frac{\mu_s}{\rho_s}$ remains strictly positive for sufficiently large s. Hence, the inequality

$$\frac{1}{\rho_s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda) > 1 - \varpi$$

directly implies

$$\frac{1}{\mu_s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda) > 1 - \varpi.$$

A similar argument applies to the second inequality involving $\mathcal{V}$:

$$\frac{1}{\rho_s}\sum_{k=1}^{s}\mathcal{V}^q(\Delta^\alpha \hbar_k - \tau, \lambda) < \varpi \quad \Longrightarrow \quad \frac{1}{\mu_s}\sum_{k=1}^{s}\mathcal{V}^q(\Delta^\alpha \hbar_k - \tau, \lambda) < \varpi.$$

Thus, both conditions required for Δ^α-μ-strong convergence are satisfied. Therefore, we conclude that $\{\hbar_k\}$ is also Δ^α-μ-strongly convergent. □

Remark 3. *The condition $\rho_s < \mu_s$ means that (μ_s) increases faster than (ρ_s). In simple terms, dividing by the larger sequence μ_s makes the expressions*

$$\frac{1}{\mu_s}\sum_{k=1}^{s}\mathcal{R}^q(\Delta^\alpha \hbar_k - \tau, \lambda), \quad \frac{1}{\mu_s}\sum_{k=1}^{s}\mathcal{V}^q(\Delta^\alpha \hbar_k - \tau, \lambda)$$

represent weaker (less strict) conditions compared to using ρ_s. Thus, if a sequence satisfies the stronger requirement of being in the class of Δ^α-ρ-strongly convergent sequences, it will automatically satisfy the weaker requirement of being in the class of Δ^α-μ-strongly convergent sequences provided the ratio $\frac{\mu_s}{\rho_s}$ does not vanish (i.e., $\lim_{s\to\infty} \frac{\mu_s}{\rho_s} > 0$).

The following class is defined as:

$$(\mathcal{R},\mathcal{V})_\rho[\Delta^\alpha] = \left\{ \{\hbar_k\} : \exists\ \tau \in \mathcal{Q} : \hbar_k \xrightarrow{(\mathcal{R},\mathcal{V})} \tau(\Delta^\alpha, \rho) \right\}.$$

Corollary 1. *Let (ρ_s) and (μ_s) be two non-decreasing sequences of positive real numbers such that $\rho_s < \mu_s$ for all $s \in \mathbb{N}$. If $\lim_{s\to\infty} \frac{\mu_s}{\rho_s} = c > 0$, c is a finite positive constant and if $\mathcal{B} \subseteq \mathcal{Q}$ is q-rung orthopair fuzzy bounded, then $(\mathcal{R},\mathcal{V})_\rho[\Delta^\alpha] = (\mathcal{R},\mathcal{V})_\mu[\Delta^\alpha]$.*

Proof. From Theorem 5, we already have

$$(\mathcal{R},\mathcal{V})_\rho[\Delta^\alpha] \subseteq (\mathcal{R},\mathcal{V})_\mu[\Delta^\alpha].$$

Now, interchanging the roles of ρ and μ, and noting that

$$\lim_{s\to\infty} \frac{\rho_s}{\mu_s} = \frac{1}{c} > 0,$$

we obtain the reverse inclusion

$$(\mathcal{R},\mathcal{V})_\mu[\Delta^\alpha] \subseteq (\mathcal{R},\mathcal{V})_\rho[\Delta^\alpha].$$

Therefore,

$$(\mathcal{R},\mathcal{V})_\rho[\Delta^\alpha] = (\mathcal{R},\mathcal{V})_\mu[\Delta^\alpha].$$

□

Remark 4. *An interesting question naturally arises: what can be said about the inclusion relations between the above convergence classes in the case when $\lim_{s\to\infty} \frac{\mu_s}{\rho_s} = \infty$? We leave this as an open problem for the reader to explore.*

We now introduce the notion of a Δ^α-ρ-strongly Cauchy sequence in a q-rung orthopair fuzzy normed space, and investigate its relationship with Δ^α-strong convergence.

Definition 9. *Let $\{\hbar_k\}$ be a sequence in a q-rung orthopair fuzzy normed space $\mathcal{Q}$. The sequence $\{\hbar_k\}$ is said to be a Δ^α-ρ-strongly Cauchy sequence with respect to $(\mathcal{R},\mathcal{V})$ if, for every $\lambda > 0$ and $\varpi \in (0,1)$, there exist $s_0, t \in \mathbb{N}$ such that*

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q(\Delta^\alpha \hbar_k - \Delta^\alpha \hbar_t, \lambda) > 1 - \varpi \quad \textit{and} \quad \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q(\Delta^\alpha \hbar_k - \Delta^\alpha \hbar_t, \lambda) < \varpi,$$

for all $s \geq s_0$.

Theorem 6. *Let $\{\hbar_k\}$ be a sequence in a q-rung orthopair fuzzy normed space $\mathcal{Q}$. If $\{\hbar_k\}$ is Δ^α-ρ-strongly convergent, then it is Δ^α-ρ-strongly Cauchy sequence.*

Proof. Let $\varpi \in (0,1)$ be given. Choose $\gamma \in (0,1)$ such that

$$(1-\gamma) \boxtimes (1-\gamma) > 1-\varpi \quad \text{and} \quad \gamma \otimes \gamma < \varpi.$$

Suppose that $\{\hbar_k\}$ is Δ^α-ρ-strongly convergent to $\tau \in \mathcal{Q}$. Then, for every $\lambda > 0$, there exists $s_0 \in \mathbb{N}$ such that

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q\left(\Delta^\alpha \hbar_k - \tau, \frac{\lambda}{2}\right) > 1-\gamma \quad \text{and} \quad \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q\left(\Delta^\alpha \hbar_k - \tau, \frac{\lambda}{2}\right) < \gamma,$$

for all $s \geq s_0$. For $s \geq s_0$, one can select $t \in \mathbb{N}$ such that

$$\mathcal{R}^q\left(\Delta^\alpha \hbar_t - \tau, \frac{\lambda}{2}\right) > \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q\left(\Delta^\alpha \hbar_k - \tau, \frac{\lambda}{2}\right) > 1-\gamma$$

and

$$\mathcal{V}^q\left(\Delta^\alpha \hbar_t - \tau, \frac{\lambda}{2}\right) < \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q\left(\Delta^\alpha \hbar_k - \tau, \frac{\lambda}{2}\right) < \gamma.$$

We now prove that

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q(\Delta^\alpha \hbar_k - \Delta^\alpha \hbar_t, \lambda) > 1-\varpi \quad \text{and} \quad \frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q(\Delta^\alpha \hbar_k - \Delta^\alpha \hbar_t, \lambda) < \varpi,$$

$\forall\ s \geq s_0$. Indeed, we have

$$\begin{aligned}
\mathcal{R}^q(\Delta^\alpha \hbar_k - \Delta^\alpha \hbar_t, \lambda) &\geq \mathcal{R}^q\left(\Delta^\alpha \hbar_k - \tau, \frac{\lambda}{2}\right) \boxplus \mathcal{R}^q\left(\Delta^\alpha \hbar_t - \tau, \frac{\lambda}{2}\right) \\
&> (1-\gamma) \boxtimes (1-\gamma) \\
&> 1-\varpi.
\end{aligned}$$

This leads to

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{R}^q(\Delta^\alpha \hbar_k - \Delta^\alpha \hbar_t, \lambda) > 1-\varpi. \tag{7}$$

And,

$$\begin{aligned}\mathcal{V}^q(\Delta^\alpha \hbar_k - \Delta^\alpha \hbar_t, \lambda) &\leq \mathcal{V}^q\left(\Delta^\alpha \hbar_k - \tau, \frac{\lambda}{2}\right) \otimes \mathcal{V}^q\left(\Delta^\alpha \hbar_t - \tau, \frac{\lambda}{2}\right) \\ &< \gamma \otimes \gamma \\ &< \varpi.\end{aligned}$$

We deduce that

$$\frac{1}{\rho_s}\sum_{k=1}^{s} \mathcal{V}^q(\Delta^\alpha \hbar_k - \Delta^\alpha \hbar_t, \lambda) < \varpi. \tag{8}$$

Hence, from (7) and (8), it follows that the sequence $\{\hbar_k\}$ is Δ^α-ρ-strongly Cauchy sequence. Thus, the proof stands established. □

Conclusion and future scope

In this paper, we introduced the novel concept of Δ^α-ρ-strong convergence in a q-rung orthopair fuzzy normed space. We established the uniqueness of the Δ^α-ρ-strong convergence of a sequence and provided an algebraic characterization of this convergence. Further, we derived the Δ^α-ρ-strong convergence criterion for subsequences of a given sequence. Depending on the conditions imposed on $\liminf \frac{s}{\rho_s}$, we also examined the relationship between Δ^α-strong convergence and Δ^α-ρ-strong convergence. Moreover, we presented a significant inclusion result involving Δ^α-ρ-strong convergence using an alternative a positive non-decreasing sequence μ_s and the inequality $\rho_s < \mu_s$, under suitable assumptions. In addition, we introduced the notion of a Δ^α-ρ-strong Cauchy sequence in a q-rung orthopair fuzzy normed space and explored its relationship with Δ^α-ρ-strong convergence.

Since research on sequence convergence in q-rung orthopair fuzzy normed spaces is still at an early stage, we believe this work provides a solid foundation for further developments. Future research directions include extending these ideas in connection with modulus functions and double sequences of order $0 < \alpha \leq 1$ within the setting of q-rung orthopair fuzzy normed spaces. Another promising direction would be the construction of new sequence spaces based on this convergence concept in association with Orlicz functions, followed by an investigation of their topological properties. These advancements could lead to powerful tools for addressing a wide range of convergence related problems across mathematics, applied sciences, and engineering disciplines.

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