Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 18, No. 2/2026

ON BEST PROXIMITY POINTS OF CYCLIC ORBITAL PROXIMAL CONTRACTIONS*

Kanagajothi Dharumaraj† Selvaraj Chellachi Premila‡
Prabavathy Magadevan§ Saravanan Karpagam¶

Communicated by G. Moroşanu

DOI 10.56082/annalsarscimath.2026.2.263

Abstract

Let A and B be non-empty subsets of a metric space (X, d). Let $T : A \cup B \to A \cup B$ be a map such that $T(A) \subseteq B$ and $T(B) \subseteq A$ satisfying a certain contractive condition called cyclic orbital proximal contraction. We give the necessary conditions for the existence of a unique point $\xi \in A$ such that $d(\xi, T\xi)$ is equal to the distance between A and B. Our main result generalizes the main result of [A.A. Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl., 323 (2006), 1001-1006].

Keywords: cyclic map, best proximity point, orbital contractions, uniformly convex Banach space.

MSC: 47H10, 54H25.

*Accepted for publication on March 18, 2026

†kanagajothi82@gmail.com, Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R and D Institute of Science and Technology, Chennai, Tamilnadu, India

‡premilasc77@gmail.com, Department of Mathematics, Saveetha Engineering College(Autonomous), Thandalam, Chennai, India

§prabavathy09@gmail.com, m.prabavathy@adjadmc.ac.in, PG and Research Department of Mathematics, A.D.M College for Women(Autonomous), Nagapattinam - 611 001, Tamilnadu, India

¶karpagam.saravanan@gmail.com, Department of Mathematics, Bhaktavatsalam Memorial College for Women, Chennai, Tamilnadu, India

1 Introduction and preliminaries

Many problems of practical interest are formulated as fixed point equations. The most commonly used fixed point theorem is given in [2]. The classical Banach contraction theorem given in [2] states that "Let T be a self mapping of a complete metric space (X, d) such that for some k, $0 < k < 1$, $d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$. Then for any $x \in X$, the sequence $\{T^n x\}$ converges to a unique fixed point of T in X. " The mapping T is called the Banach Contraction. Since the contraction condition is strong, numerous attempts were made to weaken the condition on the mapping. One such attempt was made in [3], in which an approximation to the fixed point was given. In [3], a notion of best proximity point was introduced. If there does not exist a fixed point, the best proximity point gives an approximate solution to the fixed point problem. If (X, d) is a metric space and A and B are non-empty subsets of X, a map $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$ is introduced. We call such a map the Cyclic map. A contraction condition is imposed on the cyclic map as follows:

Definition 1. *(see [3]) Let (X, d) be a metric space. Let A and B be non-empty subsets of X. Let $T : A \cup B \to A \cup B$ be a cyclic map. If for some k, $0 < k < 1$,*

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)dist(A, B), \quad x \in A, \quad y \in B, \tag{1}$$

where $dist(A, B) = inf\{d(x, y) : x \in A, y \in B\}$, then T is said to be a Cyclic Contraction map.

In [3], a best proximity point is defined for a cyclic map as a point $x \in A$ such that $d(x, Tx) = dist(A, B)$. Note that if $dist(A, B) = 0$, then equation (1) reduces to a Banach contraction and the obtained best proximity point becomes a fixed point. In [3], a best proximity point for a cyclic contraction map is obtained in a uniformly convex Banach space settings as follows:

Theorem 1. *(see [3]) Let X be a uniformly convex Banach space. Let A and B be non-empty, closed and convex subsets of X. Let $T : A \cup B \to A \cup B$ be a cyclic contraction map. Then for any $x \in A$, the sequence $\{T^{2n} x\}$ converges to a unique best proximity point of T in A.*

In [6], a notion of cyclic orbital contraction was introduced.

Definition 2. *(see [6]) Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic map such that for some $x \in A$,* *there exists a* $k \in (0,1)$ *such that*

$$d(T^{2n}x, Ty) \leq kd(T^{2n-1}x, y), \quad n \in \mathbb{N}, \quad y \in A. \tag{2}$$

Then T *is called a Cyclic Orbital contraction.*

In [6], the following fixed point theorem is obtained.

Theorem 2. *(see [6]) Let* A *and* B *be non-empty and closed subsets of a metric space* (X, d) *and* $T : A \cup B \to A \cup B$ *be a cyclic orbital contraction. Then* $A \cap B$ *is non-empty and* T *has a unique fixed point.*

In this paper, we attempt to combine Theorem 1 and Theorem 2 and obtain a new best proximity point result by introducing a notion of Cyclic Orbital Proximal Contraction map. For further generalizations of Banach contraction theorem, one may refer to [1], [4], [5], [7], [10], [11], [13] and other papers found in the literature.

In [3], the following lemmas are proved which are useful to prove the main result.

Lemma 1. *(see [3]) Let* A *be a non-empty, closed and convex subset and* B *be a non-empty, closed subset of a uniformly convex Banach space. Let* $\{x_n\}$ *and* $\{z_n\}$ *be sequences in* A *and* $\{y_n\}$ *be a sequence in* B *satisfying*
(1) $\| z_n - y_n \| \longrightarrow dist(A, B)$,
(2) for every $\epsilon > 0$, *there exists* $N_0 \in \mathbb{N}$ *such that for all* $m > n \geq N_0$, $\| x_m - y_n \| \leq dist(A, B) + \epsilon$.
Then for every $\epsilon > 0$, *there exists* $N_1 \in \mathbb{N}$, *such that for all* $m > n \geq N_1$, $\| x_m - z_n \| \leq \epsilon$.

Lemma 2. *(see [3]) Let* A *be a non-empty, closed and convex subset and* B *be a non-empty, closed subset of a uniformly convex Banach space. Let* $\{x_n\}$ *and* $\{z_n\}$ *be sequences in* A *and* $\{y_n\}$ *be a sequence in* B *satisfying*
(1) $\| x_n - y_n \| \longrightarrow dist(A, B)$,
(2) $\| z_n - y_n \| \longrightarrow dist(A, B)$.
Then $\| x_n - z_n \| \longrightarrow 0$.

2 Main results

We now give a notion of cyclic orbital proximal contraction.

Definition 3. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic map such that for some $x \in A$, there exists a $k \in (0, 1)$ such that*

$$d(T^{2n}x, Ty) \leq kd(T^{2n-1}x, y) + (1-k)dist(A, B), \quad n \in \mathbb{N}, \quad y \in A. \qquad (3)$$

Then T is called a Cyclic Orbital Proximal contraction.

We note that if $dist(A, B) = 0$ then a cyclic orbital proximal contraction becomes a cyclic orbital contraction. In that case, $A \cap B$ is non-empty and we obtain a unique fixed point, which is stated in Theorem 2.

Definition 4. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic map such that for some $x \in A$,*

$$d(T^{2n}x, Ty) \leq d(T^{2n-1}x, y), \quad n \in \mathbb{N}, \quad y \in A. \qquad (4)$$

Then T is called a cyclic orbital nonexpansive map.

Note that if T is a cyclic orbital proximal contraction then it is a cyclic orbital nonexpansive map. Now we prove a useful convergence result.

Proposition 1. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic orbital proximal contraction with an $x \in A$ satisfying equation (3). Then, for all $y \in A$, we have the following:*

$$d(T^{2n}x, T^{2n+1}y) \longrightarrow dist(A, B) \quad as \quad n \longrightarrow \infty.$$

Proof. Consider the following:

$$\begin{aligned} d(T^{2n}x, T^{2n+1}y) &\leq kd(T^{2n-1}x, T^{2n}y) + (1-k)dist(A, B) \\ &\leq k(kd(T^{2n-2}x, T^{2n-1}y) + (1-k)dist(A, B)) \\ &\qquad\qquad + (1-k)dist(A, B) \\ &= k^2 d(T^{2n-2}x, T^{2n-1}y) + (1-k^2)dist(A, B). \end{aligned}$$

Inductively, we have

$$d(T^{2n}x, T^{2n+1}y) \leq k^{2n}d(x, Ty) + (1-k^{2n})dist(A, B).$$

Since $0 < k < 1$, as $n \longrightarrow \infty$, we have the desired result. □

We use the following Proposition in proving the main result.

Proposition 2. *Let A and B be non-empty and closed subsets of a metric space (X,d) and $T : A \cup B \to A \cup B$ be a cyclic orbital proximal contraction with an $x \in A$ satisfying equation (3). If $\{T^{2n}x\}$ converges to $\xi \in A$, then ξ is a best proximity point of T in A.*

Proof. Given $T^{2n}x \longrightarrow \xi$ as $n \longrightarrow \infty$. Since T is cyclic orbital proximal contraction, by Proposition 1, $d(T^{2n}x, T^{2n+1}y) \longrightarrow dist(A,B)$. Now $dist(A,B) \leq d(T^{2n}x, T\xi) \leq d(T^{2n-1}x, \xi)$ which tends to $dist(A,B)$ as $n \longrightarrow \infty$. Therefore, $\lim_{n\to\infty} d(T^{2n}x, T\xi) = dist(A,B)$. That is, $d(\xi, T\xi) = dist(A,B)$. Hence ξ is a best proximity point of T in A. □

Now we obtain a best proximity point for a cyclic orbital proximal contraction, which is the main result.

Theorem 3. *Let A and B be non-empty, closed and convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a cyclic orbital proximal contraction with an $x \in A$ satisfying equation (3). Then the sequence $\{T^{2n}x\}$ converges to a best proximity point ξ in A. Moreover, if $y \in A$, $y \neq x$ satisfies equation (3), then the sequence $\{T^{2n}y\}$ converges to the same best proximity point ξ in A.*

Proof. If $dist(A,B) = 0$, then there exists a unique fixed point in $A \cap B$ as given in Theorem 2. Therefore, assume $dist(A,B) > 0$. By Proposition 1,

$$\| T^{2n}x - T^{2n+1}x \| \longrightarrow dist(A,B) \ \ and \ \ \| T^{2n+1}x - T^{2n+2}x \| \longrightarrow dist(A,B).$$

By Lemma 2, we have $\| T^{2n}x - T^{2n+2}x \| \longrightarrow 0$. We now show that for $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $m > n \geq N_0$,

$$\| T^{2m}x - T^{2n+1}x \| \leq dist(A,B) + \epsilon.$$

Suppose not, then there exists $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $m_k > n_k \geq k$ for which

$$\| T^{2m_k}x - T^{2n_k+1}x \| \geq dist(A,B) + \epsilon.$$

This m_k can be chosen such that it is the least integer greater than n_k to satisfy the above inequality. Now, we obtain

$$\begin{aligned} dist(A,B) + \epsilon &\leq \| T^{2m_k}x - T^{2n_k+1}x \| \\ &\leq \| T^{2m_k}x - T^{2m_k-2}x \| + \| T^{2m_k-2}x - T^{2n_k+1}x \| . \end{aligned}$$

Since $\| T^{2m_k}x - T^{2m_k-2}x \| \longrightarrow 0$ and $\| T^{2m_k-2}x - T^{2n_k+1}x \| \leq dist(A,B) + \epsilon$ as $k \longrightarrow \infty$, we have

$$\lim_{k \to \infty} \| T^{2m_k}x - T^{2n_k+1}x \| = dist(A,B) + \epsilon.$$

Consequently,

$$\begin{aligned} \| T^{2m_k}x - T^{2n_k+1}x \| &\leq \| T^{2m_k}x - T^{2m_k+2}x \| + \| T^{2m_k+2}x - T^{2n_k+3}x \| \\ &+ \| T^{2n_k+3}x - T^{2n_k+1}x \| \\ &\leq \| T^{2m_k}x - T^{2m_k+2}x \| + K^2 \| T^{2m_k}x - T^{2n_k+1}x \| \\ &+ (1-k^2) dist(A,B) + \| T^{2n_k+3}x - T^{2n_k+1}x \|, \end{aligned}$$

$$\begin{aligned} dist(A,B) + \epsilon &\leq k^2 dist(A,B) + \epsilon + (1-k^2) dist(A,B) \\ &= dist(A,B) + k^2\epsilon, \end{aligned}$$

which is a contradiction. Therefore, $\| T^{2m}x - T^{2n+1}x \| \leq dist(A,B) + \epsilon$. Already we have $\| T^{2m}x - T^{2n+1}x \| \longrightarrow dist(A,B)$ as $n \longrightarrow \infty$. Therefore, by Lemma 1 we have for given $\epsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$\| T^{2m}x - T^{2n}x \| \leq \epsilon \quad for \quad m > n > n_1.$$

Hence $\{T^{2n}x\}$ is a Cauchy sequence and converges to a $\xi \in A$. By Proposition 2, ξ is a best proximity point of T in A. Now, let $y \in A$, $y \neq x$, satisfy the inequality (3). By what we have proved now, the sequence $\{T^{2n}y\}$ converges to an $\eta \in A$, such that η is a best proximity point. That is $\| \eta - T\eta \| = dist(A,B)$. Let us show that $\eta = \xi$. Suppose that $\eta \neq \xi$. Now, we get

$$\| T^2\xi - T\xi \| = \lim_n \| T^{2n+2}x - T^{2n+1}x \| = dist(A,B),$$

$\| \xi - T\xi \| = dist(A,B)$ implies that $T^2\xi = \xi$. Similarly, we have $T^2\eta = \eta$. Also,

$$\| T\xi - \eta \| = \lim_n \| T^{2n+1}x - T^{2n+2}y \| \leq \lim_n \| T^{2n}x - T^{2n+1}y \| = \| \xi - T\eta \|$$

$$\| T\eta - \xi \| = \lim_n \| T^{2n+1}y - T^{2n+2}x \| \leq \lim_n \| T^{2n}y - T^{2n+1}x \| = \| \eta - T\xi \|,$$

which implies $\| T\xi - \eta \| = \| T\eta - \xi \|$. Now, $\eta \neq \xi \Rightarrow \| T\eta - \xi \| > dist(A,B)$. Now,

$$\| T\eta - \xi \| = \| T\eta - T^2\xi \| = \lim_n \| T^{2n+1}y - T^{2n+2}x \| = dist(A,B),$$

which is a contradiction. Hence, $\eta = \xi$. Hence, the theorem.

□

The following example illustrates the main result.

Example 1. *Consider the metric space* $(R), |.|)$. *Let* $A = [0, 1]$ *and* $B = [2, 3]$ *be the closed intervals. Define* $T : A \cup B \to A \cup B$ *as follows:* $T(0) = 3$, $T(3) = 0$.

$$T(x) = (5 - x)/2, \quad \forall x \in (0, 1] \quad and \quad T(y) = (4 - y)/2, \quad \forall y \in [2, 3).$$

Clearly $T(A) \subseteq B$ *and* $T(B) \subseteq A$ *and* $dist(A, B) = 1$. *It is an easy exercise to see that for all* $x \in (0, 1]$*, the following inequality is satisfied:*

$$|T^{2n}x - Ty| \leq (1/2)|T^{2n-1}x - y| + (1/2)dist(A, B).$$

Thus, T *is a cyclic orbital proximal contraction with* $k = 1/2$. *We find that for all* $x \in (0, 1]$*, the sequence* $\{T^{2n}x\}$ *converges to 1, such that* $|1 - T(1)| = dist(A, B)$*,which is the unique best proximity point of* T *in* A.

3 Conclusion

In this paper we have introduced a new map called cyclic orbital proximal contraction (see Definition 3) and obtained a unique best proximity point of the map, where the best proximity point is the limit of Picard type iterative sequence of a point satisfying equation (3). This contractive condition weakens the contractive condition given in definition 1 (see equation (1)). Moreover, the new map also generalizes the notion of cyclic orbital contraction. Hence, the main result of this paper generalizes the main result of [3] and also generalizes Theorem 2 of [6]. We have also illustrated our main result by giving an example.

References

[1] M. Al-Thagafi and N. Shahzad, Convergence and existence results for best proximity points, *Nonlinear Anal.* 70 (2009), 3665-3671.

[2] S. Banach, Sur les opérations dans les ensembles abstrait et leur application aux équations integrales, *Fund. Math.* 3 (1922), 133-181.

[3] A.A. Eldred and P. Veeramani, Existence and convergence of best proximity points, *J. Math. Anal. Appl.* 323 (2006), 1001-1006.

[4] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[5] T. Gunasekar, S. Karpagam and B. Zlatanov, On p-cyclic orbital M-K contractions in a Partial Metric Space, *Mathematics* 6 (2018), 116.

[6] S. Karpagam and S. Agrawal, Best proximity point theorems for cyclic orbital Meir - Keeler contraction maps, *Nonlinear Anal.* 74 (2011), 1040-1046.

[7] S. Karpagam and S. Agrawal, Best Proximity point for p-cyclic Meir - Keeler contractions, *Fixed Point Theory Appl.* 2009 (2009), 197308.

[8] S. Karpagam and B. Zlatanov, Best proximity points of p-cyclic orbital Meir - Keeler contraction maps, *Nonlinear Anal. Model. Control* 21 (2016), 790-806.

[9] S. Karpagam and B. Zlatanov, A note on p-summing orbital Meir - Keeler contraction maps, *Int. J. Pure Appl. Math.* 107 (2016), 225-243.

[10] W.A. Kirk, S. Reich, and P. Veeramani, Proximal retracts and best proximity pair theorems, *Numer. Funct. Anal. Optim.* 24 (2003), 851-862.

[11] W.A. Kirk, P.S. Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, *Fixed Point Theory* 4 (2003), 79-89.

[12] M. Petric and B. Zlatanov, Best proximity points and Fixed points for p-summing maps, *Fixed Point Theory Appl.* 2012 (2012), 86.

[13] T. Suzuki, M. Kikkawa and C. Vetro, The existence of best proximity points in metric spaces with the property UC, *Nonlinear Anal.*, 71 (2009), 2918-2926.