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NEW PROPERTIES AND IDENTITIES FOR BICOMPLEX FIBONACCI FINITE OPERATOR SEQUENCES*

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Abstract

In this paper, we construct a new class of bicomplex sequences whose components are Fibonacci finite operator sequences. A systematic study of their structural properties is carried out within the framework of the idempotent decomposition of bicomplex numbers. We derive explicit recurrence relations, polynomial representations, and matrix formulations for these bicomplex Fibonacci finite operator sequences. Several identities are established using the associated matrix representation.

Keywords: bicomplex number, Fibonacci number, finite operator, Fibonacci polynomial, matrix representation.

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1 Introduction and preliminaries

In recent years, the theory of bicomplex numbers has become an evolving area of mathematical research, leading to substantial developments and the

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emergence of new research directions. In the historical development of bicomplex numbers, Segre [17] is credited with their initial introduction. Subsequently, Price [15] carried out a systematic study of derivatives, integrals, and their higher-dimensional generalizations in the bicomplex setting.

A bicomplex number is defined as

$$z = z_1 + jz_2 = x + iy + ju + ijv,$$

where $z_1 = x + iy$ and $z_2 = u + iv$, with $z_1, z_2 \in \mathbb{C}$ and $x, y, u, v \in \mathbb{R}$. Here, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. The imaginary units i and j satisfy

$$i^2 = j^2 = -1, \qquad ij = ji.$$

We denote the set of bicomplex numbers by $\mathbb{BC}$. The algebra of $\mathbb{BC}$ possesses two distinguished idempotent zero divisors

$$e_1 = \frac{1 + ij}{2}, \qquad e_2 = \frac{1 - ij}{2}.$$

Every bicomplex number $z = z_1 + jz_2$ can be uniquely expressed in the idempotent form

$$z = \beta_1 e_1 + \beta_2 e_2,$$

where $\beta_1 = z_1 - iz_2$ and $\beta_2 = z_1 + iz_2$. This representation is known as the idempotent decomposition of a bicomplex number.

For $z = \beta_1 e_1 + \beta_2 e_2$ and $w = \gamma_1 e_1 + \gamma_2 e_2$, we have

$$z \pm w = (\beta_1 \pm \gamma_1)e_1 + (\beta_2 \pm \gamma_2)e_2,$$

$$zw = (\beta_1\gamma_1)e_1 + (\beta_2\gamma_2)e_2.$$

The Euclidean norm on $\mathbb{BC}$ is defined by

$$\begin{aligned} \|z\|_{\mathbb{BC}} &= \sqrt{|z_1|^2 + |z_2|^2} \\ &= \sqrt{\frac{|\beta_1|^2 + |\beta_2|^2}{2}}. \end{aligned}$$

The theory of bicomplex numbers has been further developed and applied by several authors including Beg and Dutta [1], Sager and Sağır [16]. Various classes of bicomplex sequence spaces have been introduced and studied by Srivastava and Srivastava [18], Wagh [22], and Bera and Tripathy [2–4], and many others.

The Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, \qquad n \geq 0, \tag{1}$$

with initial conditions $F_0 = 0$ and $F_1 = 1$.

The Binet formula for the Fibonacci sequence is given by

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

The Fibonacci polynomial $F_n(x)$ defined by

$$F_{n+1}(x) = \begin{cases} 1, & n = 0, \\ x, & n = 1, \\ xF_n(x) + F_{n-1}(x), & n \geq 2. \end{cases}$$

For the details of Fibonacci Number and Fibonacci polynomial, one may refer to Koshy [13], Vajda [20], Verner and Hoggatt [21], Dunlap [6], and many others.

Kızılateş [8] introduced the concept of the Fibonacci finite operator sequence and established its Binet-type representation. Let $\mathcal{F} = \left\{\mathcal{F}_n^{(k)}\right\}$ denote the k-th Fibonacci finite operator sequence defined by the recurrence relation

$$\mathcal{F}_{n+1}^{(k)} = \mathcal{F}_n^{(k)} + \mathcal{F}_{n-1}^{(k)}, \tag{2}$$

where $n,\ k \geq 1$.

The corresponding Binet-type representation of the sequence is given by

$$\mathcal{F}_n^{(k)} = \mathcal{F}_1^{(k)} F_n + \mathcal{F}_0^{(k)} F_{n-1}, \tag{3}$$

where $\{F_n\}$ denotes the classical Fibonacci sequence. For the Fibonacci finite operator sequence one may refer to [8] and the refer ences therein.

The bicomplex Fibonacci number [7] is defined by

$$\mathbb{BC}F_n = F_n + iF_{n+1} + jF_{n+2} + ijF_{n+3}. \tag{4}$$

Its Binet-type formula is

$$\mathbb{BC}F_n = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\sqrt{5}}, \tag{5}$$

where

$$\hat{\alpha} = 1 + i\alpha + j\alpha^2 + ij\alpha^3, \quad \hat{\beta} = 1 + i\beta + j\beta^2 + ij\beta^3.$$

The idempotent representation of $\mathbb{BC}F_n$ is given by

$$\mathbb{BC}F_n = \bar{\beta}_n e_1 + \beta_n e_2,$$

where $\beta_n = z_n + iz'_n$, $\bar{\beta}_n = z_n - iz'_n$,

$$z_n = F_n + iF_{n+1}, \qquad z'_n = F_{n+2} + iF_{n+3}.$$

For further details of bicomplex Fibonacci numbers and their further generalizations one may refer to [5, 9, 10, 14] and the references therein.

The extension of classical Fibonacci-type sequences to hypercomplex number systems has attracted considerable attention. Various generalizations have been obtained by embedding Fibonacci numbers into quaternionic, octonionic, sedenionic, and bicomplex frameworks, enriching both algebraic and structural properties.

Kızılateş and Kone [10] developed higher-order Fibonacci 2^m-ions and derived recurrence relations, Binet formulas, generating functions, and classical identities. Kızılateş [11] introduced incomplete Fibonacci and Lucas quaternions and obtained their fundamental properties.

In the bicomplex setting. In the context of hypercomplex generalizations, Kızılateş et. al. [12] introduced higher-order Fibonacci quaternions using higher-order Fibonacci numbers, extending the classical quaternionic framework studied by A. F. Horadam and Ş. Halıcı. They established recurrence relations, Binet formulas, generating functions, and several algebraic properties.

Halıcı [7] introduced bicomplex Fibonacci numbers and derived Binet-type formulas and matrix representations. Kızılateş et al. [9] studied generalized bicomplex Tribonacci quaternions. Terzioğlu et al. [19] investigated Fibonacci finite operator quaternions and established several identities. Moreover, Çürük and Halıcı [5] studied bicomplex Fibonacci numbers via idempotent decomposition.

Despite these developments, most works analyze bicomplex Fibonacci-type sequences component-wise. A systematic investigation of bicomplex Fibonacci finite operator sequences explicitly formulated through the idempotent representation has not yet been fully developed.

Motivated by this observation, the present paper establishes new recurrence relations, polynomial representations, and matrix formulations of bicomplex Fibonacci finite operator sequences via the idempotent basis. This approach simplifies derivations and reveals structural identities that are not immediately visible in the classical representation.

2 Main results

Throughout this section, we work entirely in the bicomplex idempotent framework. The idempotent decomposition enables a component-wise analysis of bicomplex Fibonacci finite operator sequences, thereby simplifying the derivation of structural properties. This approach provides a systematic method to obtain recurrence relations, polynomial expressions, and matrix representations, and reveals identities that are not directly observable in the standard bicomplex form.

2.1 Algebraic properties of bicomplex Fibonacci finite operators and their new attributes

In this section we study bicomplex Fibonacci finite operator sequences and established some of their recurrence relations.

Definition 1. *For each $n \geq 0$, the bicomplex Fibonacci finite operator $\mathbb{BC}\mathcal{F}_n^{(k)}$ is defined by*

$$\mathbb{BC}\mathcal{F}_n^{(k)} = \mathcal{F}_n^{(k)} + i\,\mathcal{F}_{n+1}^{(k)} + j\,\mathcal{F}_{n+2}^{(k)} + ij\,\mathcal{F}_{n+3}^{(k)}, \tag{6}$$

where $\mathcal{F}_n^{(k)}$ denotes the k-th finite operator sequence.

The bicomplex Fibonacci finite operator can be written as

$$\mathbb{BC}\mathcal{F}_n^{(k)} = \mathcal{F}_n^{(k)} + u,$$

where $u = i\mathcal{F}_{n+1}^{(k)} + j\mathcal{F}_{n+2}^{(k)} + ij\mathcal{F}_{n+3}^{(k)}$.
The conjugate of the bicomplex Fibonacci finite operator, $\mathbb{BC}\mathcal{F}_n^{(k)}$ is denoted by $\left(\mathbb{BC}\mathcal{F}_n^{(k)}\right)^*$ where

$$\left(\mathbb{BC}\mathcal{F}_n^{(k)}\right)^* = \mathcal{F}_n^{(k)} - u. \tag{7}$$

For the bicomplex Fibonacci finite operator, we may obtain

$$\mathbb{BC}\mathcal{F}_n^{(k)} + \left(\mathbb{BC}\mathcal{F}_n^{(k)}\right)^* = 2\mathcal{F}_n^{(k)}.$$

Remark 1. *The idempotent representation of bicomplex Fibonacci finite operator sequence is given as*

$$\mathbb{BC}\mathcal{F}_n^{(k)} = \bar{\beta}_n^{(k)} e_1 + \beta_n^{(k)} e_2,$$

where $\bar{\beta}_n^k = p_n - ip'_n$, $\beta_n^k = p_n + ip'_n$, $p_n = \mathcal{F}_n^{(k)} + i\mathcal{F}_{n+1}^{(k)}$ *and* $p'_n = \mathcal{F}_{n+2}^{(k)} + i\mathcal{F}_{n+3}^{(k)}$

Theorem 1. *For each integer $n \geq 2$, the bicomplex Fibonacci finite operator $\mathbb{BC}\mathcal{F}_n^{(k)}$ admits the representation*

$$\mathbb{BC}\mathcal{F}_n^{(k)} = \left(\bar{\beta}_{n-1}^{(k)} + \bar{\beta}_{n-2}^{(k)}\right)e_1 + \left(\beta_{n-1}^{(k)} + \beta_{n-2}^{(k)}\right)e_2. \tag{8}$$

Proof. Using equation (2) we have

$$\begin{aligned}
\mathbb{BC}\mathcal{F}_n^{(k)} =& \mathcal{F}_n^{(k)} + i\mathcal{F}_{n+1}^{(k)} + j\mathcal{F}_{n+2}^{(k)} + ij\mathcal{F}_{n+3}^{(k)} \\
=& \big(\mathcal{F}_{n-1}^{(k)} + \mathcal{F}_{n-2}^{(k)}\big) + i\big(\mathcal{F}_n^{(k)} + \mathcal{F}_{n-1}^{(k)}\big) + j\big(\mathcal{F}_{n+1}^{(k)} + \mathcal{F}_n^{(k)}\big) \\
& + ij\big(\mathcal{F}_{n+2}^{(k)} + \mathcal{F}_{n+1}^{(k)}\big) \\
=& \big(\mathcal{F}_{n-1}^{(k)} + i\mathcal{F}_n^{(k)} + j\mathcal{F}_{n+1}^{(k)} + ij\mathcal{F}_{n+2}^{(k)}\big) \\
& + \big(\mathcal{F}_{n-2}^{(k)} + i\mathcal{F}_{n-1}^{(k)} + j\mathcal{F}_n^{(k)} + ij\mathcal{F}_{n+1}^{(k)}\big) \\
=& \mathbb{BC}\mathcal{F}_{n-1}^{(k)} + \mathbb{BC}\mathcal{F}_{n-2}^{(k)} \\
=& \left(\bar{\beta}_{n-1}^{(k)} + \bar{\beta}_{n-2}^{(k)}\right)e_1 + \left(\beta_{n-1}^{(k)} + \beta_{n-2}^{(k)}\right)e_2.
\end{aligned}$$

□

Remark 2. *The above expression also can be written as*

$$\bar{\beta}_n^{(k)}e_1 + \beta_n^{(k)}e_2 = \left(\bar{\beta}_{n-1}^{(k)} + \bar{\beta}_{n-2}^{(k)}\right)e_1 + \left(\beta_{n-1}^{(k)} + \beta_{n-2}^{(k)}\right)e_2. \tag{9}$$

Theorem 2. *Let $\{\mathbb{BC}\mathcal{F}_n^{(k)}\}_{n\geq 0}$ be the bicomplex Fibonacci finite operator sequence. Then, for each $n \geq 1$, the sequence admits the following Binet-type representation:*

$$\mathbb{BC}\mathcal{F}_n^{(k)} = \frac{\hat{\alpha}\,\alpha^{n-1}\left(\alpha\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\right) - \hat{\beta}\,\beta^{n-1}\left(\beta\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\right)}{\sqrt{5}}, \tag{10}$$

where α and β are the roots of the characteristic equation associated with the corresponding Fibonacci-type recurrence.

Proof. Using the equation (5) and (3) we have

$$\begin{aligned}
\mathbb{BC}\mathcal{F}_n^{(k)} =& \mathcal{F}_n^{(k)} + i\mathcal{F}_{n+1}^{(k)} + j\mathcal{F}_{n+2}^{(k)} + ij\mathcal{F}_{n+3}^{(k)} \\
=& \mathcal{F}_1^{(k)}F_n + \mathcal{F}_0^{(k)}F_{n-1} + i\big(\mathcal{F}_1^{(k)}F_{n+1} + \mathcal{F}_0^{(k)}F_n\big) \\
&+ j\big(\mathcal{F}_1^{(k)}F_{n+2} + \mathcal{F}_1^{(k)}F_{n+1}\big) + ij\big(\mathcal{F}_1^{(k)}F_{n+3} + \mathcal{F}_0^{(k)}F_{n+2}\big) \\
=& \mathcal{F}_1^{(k)}(F_n + iF_{n+1} + jF_{n+2} + ijF_{n+3}) \\
&+ \mathcal{F}_0^{(k)}(F_{n-1} + iF_n + jF_{n+1} + ijF_{n+2})
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{F}_1^{(k)}\mathbb{BC}F_n + \mathcal{F}_0^{(k)}\mathbb{BC}F_{n-1} \\
&= \mathcal{F}_1^{(k)}\frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\sqrt{5}} + \mathcal{F}_0^{(k)}\frac{\hat{\alpha}\alpha^{n-1} - \hat{\beta}\beta^{n-1}}{\sqrt{5}} \\
&= \frac{\hat{\alpha}\alpha^{n-1}\left(\alpha\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\right) - \hat{\beta}\beta^{n-1}\left(\beta\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\right)}{\sqrt{5}}.
\end{aligned}$$

□

Theorem 3. *Let* $\{\mathbb{BC}\mathcal{F}_n^{(k)}\}_{n\geq 0}$ *be the bicomplex Fibonacci finite operator sequence. Then its ordinary generating function*

$$\mathbb{BC}\mathcal{F}^{(k)}(z) = \sum_{n=0}^{\infty}\mathbb{BC}\mathcal{F}_n^{(k)}z^n$$

is given by

$$\mathbb{BC}\mathcal{F}^{(k)}(z) = \frac{(\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2) + \left[(\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2) - (\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2)\right]z}{1 - z - z^2}.$$

Proof. Let $\mathbb{BC}\mathcal{F}_n^{(k)}(z)$ be the generating function of the bicomplex Fibonacci finite operator, defined as

$$\mathbb{BC}\mathcal{F}_n^{(k)}(z) = \sum_{n=0}^{\infty}\mathbb{BC}\mathcal{F}_n^{(k)}z^n.$$

Then, we have,

$$\begin{aligned}
\mathbb{BC}\mathcal{F}_n^{(k)}(z) &= \mathbb{BC}\mathcal{F}_0^{(k)} + \mathbb{BC}\mathcal{F}_1^{(k)}z + \mathbb{BC}\mathcal{F}_2^{(k)}z^2 + \cdots + \mathbb{BC}\mathcal{F}_n^{(k)}z^n + \cdots \\
-z\mathbb{BC}\mathcal{F}_n^{(k)}(z) &= -\mathbb{BC}\mathcal{F}_0^{(k)}z - \mathbb{BC}\mathcal{F}_1^{(k)}z^2 - \mathbb{BC}\mathcal{F}_2^{(k)}z^3 - \cdots - \mathbb{BC}\mathcal{F}_n^{(k)}z^{n+1} - \cdots \\
-z^2\mathbb{BC}\mathcal{F}_n^{(k)}(z) &= -\mathbb{BC}\mathcal{F}_0^{(k)}z^2 - \mathbb{BC}\mathcal{F}_1^{(k)}z^3 - \mathbb{BC}\mathcal{F}_2^{(k)}z^4 - \cdots - \mathbb{BC}\mathcal{F}_n^{(k)}z^{n+2} - \cdots
\end{aligned}$$

Using the above identities and equation (8), we have

$$\begin{aligned}\left(1-z-z^2\right)\mathbb{BC}\mathcal{F}_n^{(k)}(z) =&\mathbb{BC}\mathcal{F}_0^{(k)} + \left(\mathbb{BC}\mathcal{F}_1^{(k)} - \mathbb{BC}\mathcal{F}_0^{(k)}\right)z + \\ &\sum_{n=2}^{\infty}\left(\mathbb{BC}\mathcal{F}_n^{(k)} - \mathbb{BC}\mathcal{F}_{n-1}^{(k)} - \mathbb{BC}\mathcal{F}_{n-2}^{(k)}\right)z^n \\ \left(1-z-z^2\right)\mathbb{BC}\mathcal{F}_n^{(k)}(z) =&(\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2) + \left((\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2) - (\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2)\right)z \\ &+ \sum_{n=2}^{\infty}\Big((\bar{\beta}_n^{(k)}e_1 + \beta_n^{(k)}e_2) - (\bar{\beta}_{n-1}^{(k)}e_1 + \beta_{n-1}^{(k)}e_2) - (\bar{\beta}_{n-2}^{(k)}e_1 \\ &+ \beta_{n-2}^{(k)}e_2)\Big)z^n\end{aligned}$$

Following equation (9), we have

$$\mathbb{BC}\mathcal{F}^{(k)}(z) = \frac{(\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2) + \left[(\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2) - (\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2)\right]z}{1-z-z^2}.$$

□

Theorem 4. *The exponential generating function associated with the bicomplex Fibonacci finite operator sequence* $\{\mathbb{BC}\mathcal{F}_n^{(k)}\}_{n\geq 0}$ *is given by*

$$\sum_{n=0}^{\infty}\mathbb{BC}\mathcal{F}_n^{(k)}\frac{z^n}{n!} = \frac{\mathcal{F}_1^{(k)}\left(\hat{\alpha}e^{\alpha z} - \hat{\beta}e^{\beta z}\right) - \mathcal{F}_0^{(k)}\left(\hat{\alpha}\beta e^{\alpha z} - \hat{\beta}\alpha e^{\beta z}\right)}{\sqrt{5}},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$.

Proof. Using equation (10) we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathbb{BC}\mathcal{F}_n^{(k)}\frac{z^n}{n!} =&\sum_{n=0}^{\infty}\left(\frac{\hat{\alpha}\alpha^{n-1}\big(\alpha\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\big) - \hat{\beta}\beta^{n-1}\big(\beta\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\big)}{\sqrt{5}}\right)\frac{z^n}{n!} \\ =&\frac{1}{\sqrt{5}}\left(\frac{\hat{\alpha}\big(\alpha\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\big)}{\alpha}e^{\alpha z} - \frac{\hat{\beta}\big(\beta\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\big)}{\beta}e^{\beta z}\right) \\ =&\frac{\hat{\alpha}\beta e^{\alpha z}\big(\alpha\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\big) - \hat{\alpha}\beta e^{\alpha z}\big(\beta\mathcal{F}_1^{(k)} + \mathcal{F}_0^{(k)}\big)}{\sqrt{5}\alpha\beta} \\ =&\frac{\mathcal{F}_1^{(k)}\big(\hat{\alpha}e^{\alpha z} - \hat{\beta}e^{\beta z}\big) - \mathcal{F}_0^{(k)}\big(\hat{\alpha}\beta e^{\alpha z} - \hat{\beta}\alpha e^{\beta z}\big)}{\sqrt{5}}.\end{aligned}$$

□

Theorem 5. *Let n be a non-negative integer and let $p \in \mathbb{Z}$. Then the following identity holds:*

$$\sum_{t=0}^{n}(-1)^t\binom{n}{t}\mathbb{BC}\mathcal{F}_{2t+p}^{(k)} = (-1)^n\left(\bar{\beta}_{n+p}^{(k)}e_1 + \beta_{n+p}^{(k)}e_2\right).$$

Proof. Observing equation (10) we find that

$$\begin{aligned}
\sum_{t=0}^{n}(-1)^t\binom{n}{t}\mathbb{BC}\mathcal{F}_{2t+p}^{(k)} =& \sum_{t=0}^{n}(-1)^t\binom{n}{t}\left(\frac{\hat{\alpha}\alpha^{2t+p-1}\left(\alpha\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\right)}{\sqrt{5}}\right)\\
&-\sum_{t=0}^{n}(-1)^t\binom{n}{t}\left(\frac{\hat{\beta}\beta^{2t+p-1}\left(\beta\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\right)}{\sqrt{5}}\right)\\
=&\frac{\hat{\alpha}\alpha^{p-1}\left(\alpha\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\right)}{\sqrt{5}}\left(1-\alpha^2\right)^n\\
&-\frac{\hat{\beta}\beta^{p-1}\left(\beta\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\right)}{\sqrt{5}}\left(1-\beta^2\right)^n\\
=&\frac{\hat{\alpha}\alpha^{p-1}\left(\alpha\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\right)}{\sqrt{5}}(-\alpha)^n\\
&-\frac{\hat{\beta}\beta^{p-1}\left(\beta\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\right)}{\sqrt{5}}(-\beta)^n\\
=&(-1)^n\mathbb{BC}\mathcal{F}_{n+p}^{(k)}.\\
=&(-1)^n(\bar{\beta}_{n+p}^{(k)}e_1+\beta_{n+p}^{(k)}e_2).
\end{aligned}$$

□

Theorem 6. *Let $n \in \mathbb{N}_0$. Then*

$$\sum_{t=0}^{n}\binom{n}{t}\mathbb{BC}\mathcal{F}_t^{(k)} = \bar{\beta}_{2n}^{(k)}e_1 + \beta_{2n}^{(k)}e_2.$$

Proof. From equation (10) we have

$$\begin{aligned}\sum_{t=0}^{n}\binom{n}{t}\mathbb{BC}\mathcal{F}_t^{(k)} =&\sum_{t=0}^{n}\binom{n}{t}\left(\frac{\hat{\alpha}\alpha^{t-1}\big(\alpha\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\big)-\hat{\beta}\beta^{t-1}\big(\beta\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\big)}{\sqrt{5}}\right)\\ =&\frac{\hat{\alpha}\alpha^{-1}\big(\alpha\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\big)}{\sqrt{5}}(1+\alpha)^n-\frac{\hat{\beta}\beta^{-1}\big(\beta\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\big)}{\sqrt{5}}(1+\beta)^n\\ =&\frac{\hat{\alpha}\alpha^{2n-1}\big(\alpha\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\big)}{\sqrt{5}}-\frac{\hat{\beta}\beta^{2n-1}\big(\beta\mathcal{F}_1^{(k)}+\mathcal{F}_0^{(k)}\big)}{\sqrt{5}}\\ =&\mathbb{BC}\mathcal{F}_{2n}^{(k)}\\ =&\bar{\beta}_{2n}^{(k)}e_1+\beta_{2n}^{(k)}e_2.\end{aligned}$$

□

2.2 Polynomial representation of bicomplex Fibonacci finite operators and their new attributes

In this section we introduce bicomplex Fibonacci polynomial and established bicomplex version of some well known inequalities.

Definition 2. *The bicomplex Fibonacci polynomial for finite operator sequences denoted as*

$\mathbb{BC}\mathcal{F}_n^{(k)}(z)$ *and defined as* $\mathbb{BC}\mathcal{F}_{n+1}^{(k)}(z)=\begin{cases}1 & n=0,\\ z & n=1,\\ z\mathbb{BC}\mathcal{F}_n^{(k)}(z)+\mathbb{BC}\mathcal{F}_{n-1}^{(k)}(z) & n\geq 2.\end{cases}$

The first few bicomplex Fibonacci finite operator polynomials are

$$\begin{aligned}\mathbb{BC}\mathcal{F}_0^{(k)}(z)&=0\\ \mathbb{BC}\mathcal{F}_1^{(k)}(z)&=1\\ \mathbb{BC}\mathcal{F}_2^{(k)}(z)&=z\\ \mathbb{BC}\mathcal{F}_3^{(k)}(z)&=z^2+1\\ \mathbb{BC}\mathcal{F}_4^{(k)}(z)&=z^3+2z\\ \mathbb{BC}\mathcal{F}_5^{(k)}(z)&=z^4+3z^2+1\\ \mathbb{BC}\mathcal{F}_6^{(k)}(z)&=z^5+4z^3+3z.\end{aligned}$$

We define a new class of matrix $Q(z)$ to study the $\mathbb{BC}\mathcal{H}_{n+1}^{(k)}(z)$ polynomial where,

$$Q(z)=\begin{bmatrix}z & 1\\ 1 & 0\end{bmatrix}$$

Then

$$Q^n(z) = \begin{bmatrix} \mathbb{BC}\mathcal{F}^{(k)}_{n+1}(z) & \mathbb{BC}\mathcal{F}^{(k)}_{n}(z) \\ \mathbb{BC}\mathcal{F}^{(k)}_{n}(z) & \mathbb{BC}\mathcal{F}^{(k)}_{n-1}(z) \end{bmatrix} \tag{11}$$

where $n \geq 1$. Since $|Q| = -1, |Q^n| = (-1)^n$. Accordingly, equation (11) gives the Cassini-like formula for $\mathbb{BC}\mathcal{F}^{(k)}_{n}(z)$:

$$\mathbb{BC}\mathcal{F}^{(k)}_{n+1}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n-1}(z) - \left(\mathbb{BC}\mathcal{F}^{(k)}_{n}(z)\right)^2 = (-1)^n \tag{12}$$

It follows from equation (11) that

$$Q^{m+n}(z) = \begin{bmatrix} \mathbb{BC}\mathcal{F}^{(k)}_{m+n+1}(z) & \mathbb{BC}\mathcal{F}^{(k)}_{m+n}(z) \\ \mathbb{BC}\mathcal{F}^{(k)}_{m+n}(z) & \mathbb{BC}\mathcal{F}^{(k)}_{m+n-1}(z) \end{bmatrix}$$

$$\begin{aligned} &Q^{m+n}(z) \\ =&Q^m(z)Q^n(z) \\ =&\begin{bmatrix} \mathbb{BC}\mathcal{F}^{(k)}_{m+1}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n+1}(z) + \mathbb{BC}\mathcal{F}^{(k)}_{m}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n}(z) & \mathbb{BC}\mathcal{F}^{(k)}_{m+1}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n}(z) + \mathbb{BC}\mathcal{F}^{(k)}_{m}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n-1}(z) \\ \mathbb{BC}\mathcal{F}^{(k)}_{m}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n+1}(z) + \mathbb{BC}\mathcal{F}^{(k)}_{m-1}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n}(z) & \mathbb{BC}\mathcal{F}^{(k)}_{m}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n}(z) + \mathbb{BC}\mathcal{F}^{(k)}_{m-1}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n-1}(z) \end{bmatrix} \end{aligned}$$

Consequently,

$$\mathbb{BC}\mathcal{F}^{(k)}_{m+n}(z) = \mathbb{BC}\mathcal{F}^{(k)}_{m+1}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n}(z) + \mathbb{BC}\mathcal{F}^{(k)}_{m}(z)\mathbb{BC}\mathcal{F}^{(k)}_{n-1}(z) \tag{13}$$

In particular, let $z = 1$. This yields an identity

$$\mathbb{BC}\mathcal{F}^{(k)}_{m+n} = \mathbb{BC}\mathcal{F}^{(k)}_{m+1}\mathbb{BC}\mathcal{F}^{(k)}_{n} + \mathbb{BC}\mathcal{F}^{(k)}_{m}\mathbb{BC}\mathcal{F}^{(k)}_{n-1}.$$

The above identity is known as bicomplex version of Honsberger's identity. Since $\mathbb{BC}\mathcal{H}^{(k)}_{0}(z) = \mathbb{BC}\mathcal{F}^{(k)}_{1}(z) + \mathbb{BC}\mathcal{F}^{(k)}_{0}(z) = 1 + 0 = 1$ and $\mathbb{BC}\mathcal{H}^{(k)}_{1}(z) = \mathbb{BC}\mathcal{F}^{(k)}_{2}(z) + \mathbb{BC}\mathcal{F}^{(k)}_{1}(z) = z + 1$, it follows that $\mathbb{BC}\mathcal{H}^{(k)}_{n}(z) = \mathbb{BC}\mathcal{F}^{(k)}_{n+1}(z) + \mathbb{BC}\mathcal{F}^{(k)}_{n}(z)$;

$$\begin{aligned} \therefore (-1)^{i+1}\mathbb{BC}\mathcal{H}^{(k)}_{i}(z) =&(-1)^{i+1}\mathbb{BC}\mathcal{F}^{(k)}_{i+1}(z) - (-1)^{i}\mathbb{BC}\mathcal{F}^{(k)}_{i}(z) \\ \sum_{i=0}^{n}(-1)^{i+1}\mathbb{BC}\mathcal{H}^{(k)}_{i}(z) =&\sum_{i=0}^{n}[(-1)^{i+1}\mathbb{BC}\mathcal{F}^{(k)}_{i+1}(z) - (-1)^{i}\mathbb{BC}\mathcal{F}^{(k)}_{i}(z)] \\ =&(-1)^{n+1}\mathbb{BC}\mathcal{F}^{(k)}_{i+1}(z) \end{aligned}$$

Thus we can express the bicomplex Fibonacci polynomial $\mathbb{BC}\mathcal{F}^{(k)}_{n}(z)$ in terms of $\mathbb{BC}\mathcal{H}^{(k)}_{n}(z)$ polynomial as

$$\mathbb{BC}\mathcal{F}^{(k)}_{n+1}(z) = \sum_{i=0}^{n}(-1)^{n+i}\mathbb{BC}\mathcal{H}^{(k)}_{i}(z)$$

Now its follows from the equation (11) and (12)

$$Q^{n+1}(z) + Q^n(z) = \begin{bmatrix} \mathbb{BC}\mathcal{H}_{n+1}^{(k)}(z) & \mathbb{BC}\mathcal{H}_{n}^{(k)}(z) \\ \mathbb{BC}\mathcal{H}_{n}^{(k)}(z) & \mathbb{BC}\mathcal{H}_{n-1}^{(k)}(z) \end{bmatrix}$$

$$\begin{vmatrix} \mathbb{BC}\mathcal{H}_{n+1}^{(k)}(z) & \mathbb{BC}\mathcal{H}_{n}^{(k)}(z) \\ \mathbb{BC}\mathcal{H}_{n}^{(k)}(z) & \mathbb{BC}\mathcal{H}_{n-1}^{(k)}(z) \end{vmatrix} = \left|Q^n(z)[Q(z)+I]\right| \\ = |Q^n(z)| \cdot |Q+I|.$$

That is,

$$\mathbb{BC}\mathcal{H}_{n+1}^{(k)}(z)\mathbb{BC}\mathcal{H}_{n-1}^{(k)}(z) - \left(\mathbb{BC}\mathcal{H}_{n}^{(k)}(z)\right)^2 = z(-1)^{-n} \tag{14}$$

The above identity is again a generalization of Cassini's rule in the bicomplex setting. We have a further generalization of equation (14):

$$\begin{vmatrix} \mathbb{BC}\mathcal{H}_{n+a}^{(k)}(z) & \mathbb{BC}\mathcal{H}_{n+a+b}^{(k)}(z) \\ \mathbb{BC}\mathcal{H}_{n}^{(k)}(z) & \mathbb{BC}\mathcal{H}_{n+b}^{(k)}(z) \end{vmatrix} = (-1)^n \begin{vmatrix} \mathbb{BC}\mathcal{H}_{a}^{(k)}(z) & \mathbb{BC}\mathcal{H}_{a+b}^{(k)}(z) \\ \mathbb{BC}\mathcal{H}_{0}^{(k)}(z) & \mathbb{BC}\mathcal{H}_{b}^{(k)}(z) \end{vmatrix}.$$

Remark 3. *If we replace $\mathbb{BC}\mathcal{F}_n^k(z)$ by $(\bar{\beta}_n^{(k)}e_1 + \beta_n^k e_2)(z)$, then we get the idempotent representation of bicomplex polynomial.*

2.3 Matrix representations of bicomplex Fibonacci finite operators and their new attributes

In this section we study matrix representation of bicomplex Fibonacci finite operator sequences and established some useful theorems.

Definition 3. *The bicomplex Fibonacci matrix is defined by*

$$P = \begin{bmatrix} \mathbb{BC}F_2 & \mathbb{BC}F_1 \\ \mathbb{BC}F_1 & \mathbb{BC}F_0 \end{bmatrix}, \tag{15}$$

and satisfies

$$PQ^{n-1} = \begin{bmatrix} \mathbb{BC}F_{n+1} & \mathbb{BC}F_n \\ \mathbb{BC}F_n & \mathbb{BC}F_{n-1} \end{bmatrix}, \tag{16}$$

where

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

From the recurrence relation of the Fibonacci finite operator numbers, we can easily see the matrix relation

$$H = \begin{bmatrix} \mathcal{F}_2^{(k)} & \mathcal{F}_1^{(k)} \\ \mathcal{F}_1^{(k)} & \mathcal{F}_0^{(k)} \end{bmatrix} \Longrightarrow HQ^{n-1} = \begin{bmatrix} \mathcal{F}_{n+1}^{(k)} & \mathcal{F}_n^{(k)} \\ \mathcal{F}_n^{(k)} & \mathcal{F}_{n-1}^{(k)} \end{bmatrix}. \tag{17}$$

Considering the matrix equalities in equation (16) and (17), we have a matrix representation of the bicomplex Fibonacci finite operator as follows:

$$\big(HQ^{n-1}\big)P = P\big(HQ^{n-1}\big) = \begin{bmatrix} \mathbb{BC}\mathcal{F}_{n+2}^{(k)} & \mathbb{BC}\mathcal{F}_{n+1}^{(k)} \\ \mathbb{BC}\mathcal{F}_{n+1}^{(k)} & \mathbb{BC}\mathcal{F}_n^{(k)} \end{bmatrix}. \tag{18}$$

Let us note that equality equation (18) is held even though matrix multiplication is not commutative. Namely

$$\begin{aligned} \big(HQ^{n-1}\big)P &= \begin{bmatrix} \mathcal{F}_{n+1}^{(k)} & \mathcal{F}_n^{(k)} \\ \mathcal{F}_n^{(k)} & \mathcal{F}_{n-1}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbb{BC}F_2 & \mathbb{BC}F_1 \\ \mathbb{BC}F_1 & \mathbb{BC}F_0 \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{F}_{n+1}^{(k)}\mathbb{BC}F_2 + \mathcal{F}_n^{(k)}\mathbb{BC}F_1 & \mathcal{F}_{n+1}^{(k)}\mathbb{BC}F_1 + \mathcal{F}_n^{(k)}\mathbb{BC}F_0 \\ \mathcal{F}_n^{(k)}\mathbb{BC}F_2 + \mathcal{F}_{n-1}^{(k)}\mathbb{BC}F_1 & \mathcal{F}_n^{(k)}\mathbb{BC}F_1 + \mathcal{F}_{n-1}^{(k)}\mathbb{BC}F_0 \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{BC}F_2\mathcal{F}_{n+1}^{(k)} + \mathbb{BC}F_1\mathcal{F}_n^{(k)} & \mathbb{BC}F_1\mathcal{F}_{n+1}^{(k)} + \mathbb{BC}F_0\mathcal{F}_n^{(k)} \\ \mathbb{BC}F_2\mathcal{F}_n^{(k)} + \mathbb{BC}F_1\mathcal{F}_{n-1}^{(k)} & \mathbb{BC}F_1\mathcal{F}_n^{(k)} + \mathbb{BC}F_0\mathcal{F}_{n-1}^{(k)} \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{BC}F_2 & \mathbb{BC}F_1 \\ \mathbb{BC}F_1 & \mathbb{BC}F_0 \end{bmatrix} \begin{bmatrix} \mathcal{F}_{n+1}^{(k)} & \mathcal{F}_n^{(k)} \\ \mathcal{F}_n^{(k)} & \mathcal{F}_{n-1}^{(k)} \end{bmatrix} \\ &= P\big(HQ^{n-1}\big). \end{aligned}$$

Additionally, the Bicomplex Fibonacci finite operator has the following matrix representation:

$$L := \begin{bmatrix} \mathbb{BC}\mathcal{F}_2^{(k)} & \mathbb{BC}\mathcal{F}_1^{(k)} \\ \mathbb{BC}\mathcal{F}_1^{(k)} & \mathbb{BC}\mathcal{F}_0^{(k)} \end{bmatrix} \Longrightarrow Q^nL = LQ^n = \begin{bmatrix} \mathbb{BC}\mathcal{F}_{n+2}^{(k)} & \mathbb{BC}\mathcal{F}_{n+1}^{(k)} \\ \mathbb{BC}\mathcal{F}_{n+1}^{(k)} & \mathbb{BC}\mathcal{F}_n^{(k)} \end{bmatrix}. \tag{19}$$

Equation (19) can be expressed as

$$L := \begin{bmatrix} \bar{\beta}_2^{(k)}e_1 + \beta_2^{(k)}e_2 & \bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2 \\ \bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2 & \bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2 \end{bmatrix}$$

and

$$Q^nL = LQ^n = \begin{bmatrix} \bar{\beta}_{n+2}^{(k)}e_1 + \beta_{n+2}^{(k)}e_2 & \bar{\beta}_{n+1}^{(k)}e_1 + \beta_{n+1}^{(k)}e_2 \\ \bar{\beta}_{n+1}^{(k)}e_1 + \beta_{n+1}^{(k)}e_2 & \bar{\beta}_n^{(k)}e_1 + \beta_n^{(k)}e_2 \end{bmatrix}.$$

Theorem 7. *For $n \geq 0$, we have*

$$\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\mathbb{BC}\mathcal{F}_{n-1}^{(k)} - \left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2 = (-1)^{n-1}\Big((\bar{\beta}_2^{(k)}e_1 + \beta_2^{(k)}e_2)(\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2) - (\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2)^2\Big), \tag{20}$$

$$\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\mathbb{BC}\mathcal{F}_{n+1}^{(k)} - \left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2 = (-1)^{n-1}\Big((\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2)(\bar{\beta}_2^{(k)}e_1 + \beta_2^{(k)}e_2) - \left(\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2\right)^2\Big), \tag{21}$$

$$\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\mathbb{BC}\mathcal{F}_{n-1}^{(k)} - \left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2 = (-1)^{n-1}\big((\bar{\beta}_2e_1 + \beta_2e_2)(\bar{\beta}_0e_1 + \beta_0e_2) - (\bar{\beta}_1e_1 + \beta_1e_2)^2\big)\Big((\mathcal{F}_1^{(k)})^2 - \mathcal{F}_1^{(k)}\mathcal{F}_0^{(k)} - (\mathcal{F}_0^{(k)})^2\Big), \tag{22}$$

$$\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\mathbb{BC}\mathcal{F}_{n+1}^{(k)} - \left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2 = (-1)^{n-1}\big((\bar{\beta}_0e_1 + \beta_0e_2)(\bar{\beta}_2e_1 + \beta_2e_2) - (\bar{\beta}_1e_1 + \beta_1e_2)^2\big)\Big((\mathcal{F}_1^{(k)})^2 - \mathcal{F}_1^{(k)}\mathcal{F}_0^{(k)} - (\mathcal{F}_0^{(k)})^2\Big). \tag{23}$$

Proof. To prove equations (20) and (21), using the equation (19), we get

$$\begin{aligned}
\begin{vmatrix} \mathbb{BC}\mathcal{F}_{n+1}^{(k)} & \mathbb{BC}\mathcal{F}_{n}^{(k)} \\ \mathbb{BC}\mathcal{F}_{n}^{(k)} & \mathbb{BC}\mathcal{F}_{n-1}^{(k)} \end{vmatrix} &= \left|Q^{n-1}L\right| \\
&= |Q|^{n-1}|L| = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^{n-1} \begin{vmatrix} \mathbb{BC}\mathcal{F}_{2}^{(k)} & \mathbb{BC}\mathcal{F}_{1}^{(k)} \\ \mathbb{BC}\mathcal{F}_{1}^{(k)} & \mathbb{BC}\mathcal{F}_{0}^{(k)} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^{n-1} \begin{vmatrix} \bar{\beta}_2^{(k)}e_1 + \beta_2^{(k)}e_2 & \bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2 \\ \bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2 & \bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2 \end{vmatrix}
\end{aligned}$$

$$\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\mathbb{BC}\mathcal{F}_{n-1}^{(k)} - \left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2 = (-1)^{n-1}\Big((\bar{\beta}_2^{(k)}e_1 + \beta_2^{(k)}e_2)(\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2) - (\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2)^2\Big),$$

and

$$\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\mathbb{BC}\mathcal{F}_{n+1}^{(k)} - \left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2 = (-1)^{n-1}\Big((\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2)(\bar{\beta}_2^{(k)}e_1 + \beta_2^{(k)}e_2) - \left(\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2\right)^2\Big).$$

In the same way, equations (22) and (23) are obtained if we take determinants on both sides of the matrix equation (18). □

Theorem 8. *For integer $m, n \geq 1$, the following equalities hold:*

$$\mathcal{F}_n^{(k)}\mathbb{BC}F_{m+1} + \mathcal{F}_{n-1}^{(k)}\mathbb{BC}F_m = \bar{\beta}_{m+n}^{(k)}e_1 + \beta_{m+n}^{(k)}e_2, \tag{24}$$

$$F_n\mathbb{BC}\mathcal{F}_{m+1}^{(k)} + F_{n-1}\mathbb{BC}\mathcal{F}_m^{(k)} = \bar{\beta}_{m+n}^{(k)}e_1 + \beta_{m+n}^{(k)}e_2, \tag{25}$$

$$\begin{aligned}\mathbb{BC}\mathcal{F}_{m+1}^{(k)}\mathbb{BC}F_{n+1} + \mathbb{BC}\mathcal{F}_m^{(k)}\mathbb{BC}F_n &= (\bar{\beta}_2e_1 + \beta_2e_2)(\bar{\beta}_{m+n}^{(k)}e_1 + \beta_{m+n}^{(k)}e_2)\\ &+ (\bar{\beta}_1e_1 + (\beta_1e_2)(\bar{\beta}_{m+n-1}^{(k)}e_1 + \beta_{m+n-1}^{(k)}e_2),\end{aligned} \tag{26}$$

$$\begin{aligned}\mathbb{BC}\mathcal{F}_{m+1}^{(k)}\mathbb{BC}\mathcal{F}_{n+1}^{(k)} + \mathbb{BC}\mathcal{F}_m^{(k)}\mathbb{BC}\mathcal{F}_n^{(k)} &= (\bar{\beta}_2^{(k)}e_1 + \beta_2^{(k)}e_2)(\bar{\beta}_{m+n}^{(k)}e_1 + \beta_{m+n}^{(k)}e_2)\\ &+ (\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2)(\bar{\beta}_{m+n-1}^{(k)}e_1 + \beta_{m+n-1}^{(k)}e_2).\end{aligned} \tag{27}$$

Proof. Substituting $n \longrightarrow m+n-1$ into (18) and (19), we have

$$\begin{bmatrix}\mathbb{BC}\mathcal{F}_{m+n+1}^{(k)} & \mathbb{BC}\mathcal{F}_{m+n}^{(k)}\\ \mathbb{BC}\mathcal{F}_{m+n}^{(k)} & \mathbb{BC}\mathcal{F}_{m+n-1}^{(k)}\end{bmatrix} = (HQ^{m+n-2})P = \left(HQ^{n-1}\right)\left(Q^{m-1}P\right)$$

$$\begin{bmatrix}\mathbb{BC}\mathcal{F}_{m+n+1}^{(k)} & \mathbb{BC}\mathcal{F}_{m+n}^{(k)}\\ \mathbb{BC}\mathcal{F}_{m+n}^{(k)} & \mathbb{BC}\mathcal{F}_{m+n-1}^{(k)}\end{bmatrix} = \begin{bmatrix}\mathcal{F}_{n+1}^{(k)} & \mathcal{F}_n^{(k)}\\ \mathcal{F}_n^{(k)} & \mathcal{F}_{n-1}^{(k)}\end{bmatrix}\begin{bmatrix}\mathbb{BC}F_{m+1} & \mathbb{BC}F_m\\ \mathbb{BC}F_m & \mathbb{BC}F_{m-1}\end{bmatrix}$$

$$\begin{bmatrix}\bar{\beta}_{m+n+1}^{(k)}e_1 + \beta_{m+n+1}^{(k)}e_2 & \bar{\beta}_{m+n}^{(k)}e_1 + \beta_{m+n}^{(k)}e_2\\ \bar{\beta}_{m+n}^{(k)}e_1 + \beta_{m+n}^{(k)}e_2 & \bar{\beta}_{m+n-1}^{(k)}e_1 + \beta_{m+n-1}^{(k)}e_2\end{bmatrix} = \begin{bmatrix}\mathcal{F}_{n+1}^{(k)} & \mathcal{F}_n^{(k)}\\ \mathcal{F}_n^{(k)} & \mathcal{F}_{n-1}^{(k)}\end{bmatrix}\begin{bmatrix}\mathbb{BC}F_{m+1} & \mathbb{BC}F_m\\ \mathbb{BC}F_m & \mathbb{BC}F_{m-1}\end{bmatrix}.$$

The equation (24) is obtained by comparing the corresponding entries of the two matrix equations. From equation (19) we see that

$$\begin{bmatrix}\mathbb{BC}\mathcal{F}_{m+n+1}^{(k)} & \mathbb{BC}\mathcal{F}_{m+n}^{(k)}\\ \mathbb{BC}\mathcal{F}_{m+n}^{(k)} & \mathbb{BC}\mathcal{F}_{m+n-1}^{(k)}\end{bmatrix} = Q^{m+n-1}L = Q^{n-1}(Q^mL)$$

$$\begin{bmatrix}\mathbb{BC}\mathcal{F}_{m+n+1}^{(k)} & \mathbb{BC}\mathcal{F}_{m+n}^{(k)}\\ \mathbb{BC}\mathcal{F}_{m+n}^{(k)} & \mathbb{BC}\mathcal{F}_{m+n-1}^{(k)}\end{bmatrix} = \begin{bmatrix}F_n & F_{n-1}\\ F_{n-1} & F_{n-2}\end{bmatrix}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{m+2}^{(k)} & \mathbb{BC}\mathcal{F}_{m+1}^{(k)}\\ \mathbb{BC}\mathcal{F}_{m+1}^{(k)} & \mathbb{BC}\mathcal{F}_m^{(k)}\end{bmatrix}$$

$$\begin{bmatrix}\bar{\beta}_{m+n+1}^{(k)}e_1 + \beta_{m+n+1}^{(k)}e_2 & \bar{\beta}_{m+n}^{(k)}e_1 + \beta_{m+n}^{(k)}e_2\\ \bar{\beta}_{m+n}^{(k)}e_1 + \beta_{m+n}^{(k)}e_2 & \bar{\beta}_{m+n-1}^{(k)}e_1 + \beta_{m+n-1}^{(k)}e_2\end{bmatrix} = \begin{bmatrix}F_n & F_{n-1}\\ F_{n-1} & F_{n-2}\end{bmatrix}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{m+2}^{(k)} & \mathbb{BC}\mathcal{F}_{m+1}^{(k)}\\ \mathbb{BC}\mathcal{F}_{m+1}^{(k)} & \mathbb{BC}\mathcal{F}_m^{(k)}\end{bmatrix}.$$

The equation (25) is obtained by comparing the corresponding entries of the two matrix equations. Likewise, Substituting $n \longrightarrow m+n-2$ into equations (18) and (19) we have

$$P(HQ^{m+n-3})P = (PHQ^{m-2})(PQ^{n-1}),$$
$$L(Q^{m+n-2})L = (LQ^{m-1})(LQ^{n-1}).$$

If we equate the corresponding entries on both sides of the matrix equations, we obtain Equations (26) and (27) respectively. □

Corollary 1. *For a positive integer n, the following equality holds:*

$$\begin{aligned}\left(\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\right)^2 + \left(\mathbb{BC}\mathcal{F}_n^{(k)}\right)^2 &= (\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2)(\bar{\beta}_{2n+1}^{(k)}e_1 + \beta_{2n+1}^{(k)}e_2)\\ &+ (\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2)(\bar{\beta}_{2n}^{(k)}e_1 + \beta_{2n}^{(k)}e_2).\end{aligned}$$

Proof. Substituting $m \longrightarrow n$ into the equation (27) and using equation (8) we get

$$\begin{aligned}\left(\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\right)^2 + \left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2 =& \mathbb{BC}\mathcal{F}_{2}^{(k)}\mathbb{BC}\mathcal{F}_{2n}^{(k)} + \mathbb{BC}\mathcal{F}_{1}^{(k)}\mathbb{BC}\mathcal{F}_{2n-1}^{(k)} \\ =& \left(\mathbb{BC}\mathcal{F}_{1}^{(k)} + \mathbb{BC}\mathcal{F}_{0}^{(k)}\right)\mathbb{BC}\mathcal{F}_{2n}^{(k)} + \mathbb{BC}\mathcal{F}_{1}^{(k)}\mathbb{BC}\mathcal{F}_{2n-1}^{(k)} \\ =& \mathbb{BC}\mathcal{F}_{1}^{(k)}\left(\mathbb{BC}\mathcal{F}_{2n}^{(k)} + \mathbb{BC}\mathcal{F}_{2n-1}^{(k)}\right) + \mathbb{BC}\mathcal{F}_{0}^{(k)}\mathbb{BC}\mathcal{F}_{2n}^{(k)} \\ =& \mathbb{BC}\mathcal{F}_{1}^{(k)}\mathbb{BC}\mathcal{F}_{2n+1}^{(k)} + \mathbb{BC}\mathcal{F}_{0}^{(k)}\mathbb{BC}\mathcal{F}_{2n}^{(k)} \\ =& (\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2)(\bar{\beta}_{2n+1}^{(k)}e_1 + \beta_{2n+1}^{(k)}e_2) \\ & + (\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2)(\bar{\beta}_{2n}^{(k)}e_1 + \beta_{2n}^{(k)}e_2)\end{aligned}$$

□

Corollary 2. *For positive integer n, the following equality holds:*

$$\begin{aligned}\left(\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\right)^2 - \left(\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\right)^2 =& (\bar{\beta}_1^{(k)}e_1 + \beta_1^{(k)}e_2)(\bar{\beta}_{2n}^{(k)}e_1 + \beta_{2n}^{(k)}e_2) \\ & + (\bar{\beta}_0^{(k)}e_1 + \beta_0^{(k)}e_2)(\bar{\beta}_{2n-1}^{(k)}e_1 + \beta_{2n-1}^{(k)}e_2).\end{aligned}$$

Proof. Firstly, we get

$$\begin{aligned}\left(\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\right)^2 - \left(\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\right)^2 =& \left(\left(\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\right)^2 + \left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2\right) \\ & - \left(\left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2 + \left(\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\right)^2\right).\end{aligned} \tag{28}$$

Next, we make the following computations.

$$\begin{aligned}\left(\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\right)^2 + \left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2 &= \begin{bmatrix}\mathbb{BC}\mathcal{F}_{n+1}^{(k)} & \mathbb{BC}\mathcal{F}_{n}^{(k)}\end{bmatrix}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{n+1}^{(k)} \\ \mathbb{BC}\mathcal{F}_{n}^{(k)}\end{bmatrix} \\ &= \begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} & \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix} Q^n Q^n \begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} \\ \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix} \\ &= \begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} & \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix} Q^{2n} \begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} \\ \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix}.\end{aligned} \tag{29}$$

Similarly, we also have

$$
\begin{aligned}
\left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2+\left(\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\right)^2 &=\begin{bmatrix}\mathbb{BC}\mathcal{F}_{n}^{(k)} & \mathbb{BC}\mathcal{F}_{n-1}^{(k)}\end{bmatrix}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{n}^{(k)}\\ \mathbb{BC}\mathcal{F}_{n-1}^{(k)}\end{bmatrix}\\
&=\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} & \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix} Q^{n-1}Q^{n-1}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)}\\ \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix}\\
&=\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} & \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix} Q^{2n-2}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)}\\ \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix}. \qquad (30)
\end{aligned}
$$

By using equations(28)-(30), we have

$$
\begin{aligned}
\left(\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\right)^2-\left(\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\right)^2 =&\left(\left(\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\right)^2+\left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2\right)\\
&-\left(\left(\mathbb{BC}\mathcal{F}_{n}^{(k)}\right)^2+\left(\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\right)^2\right)\\
=&\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} & \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix} Q^{2n}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)}\\ \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix}\\
&-\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} & \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix} Q^{2n-2}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)}\\ \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix}\\
=&\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} & \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix} \left(Q^{2n}-Q^{2n-2}\right)\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)}\\ \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix}\\
=&\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)} & \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix} Q^{2n-2}\left(Q^{2}-I\right)\begin{bmatrix}\mathbb{BC}\mathcal{F}_{1}^{(k)}\\ \mathbb{BC}\mathcal{F}_{0}^{(k)}\end{bmatrix}.
\end{aligned}
$$

By Cayley-Hamilton theorem it follows that,

$$Q^2-Q-I=[0]_{2\times 2}.$$

Also we get

$$
\begin{aligned}
\left(\mathbb{BC}\mathcal{F}_{n+1}^{(k)}\right)^2-\left(\mathbb{BC}\mathcal{F}_{n-1}^{(k)}\right)^2 &= \begin{bmatrix}\mathbb{BC}\mathcal{F}_1^{(k)} & \mathbb{BC}\mathcal{F}_0^{(k)}\end{bmatrix} Q^{2n-2}Q \begin{bmatrix}\mathbb{BC}\mathcal{F}_1^{(k)}\\ \mathbb{BC}\mathcal{F}_0^{(k)}\end{bmatrix}\\
&= \begin{bmatrix}\mathbb{BC}\mathcal{F}_1^{(k)} & \mathbb{BC}\mathcal{F}_0^{(k)}\end{bmatrix} Q^{2n-1} \begin{bmatrix}\mathbb{BC}\mathcal{F}_1^{(k)}\\ \mathbb{BC}\mathcal{F}_0^{(k)}\end{bmatrix}\\
&= \begin{bmatrix}\mathbb{BC}\mathcal{F}_1^{(k)} & \mathbb{BC}\mathcal{F}_0^{(k)}\end{bmatrix} \begin{bmatrix}F_{2n} & F_{2n-1}\\ F_{2n-1} & F_{2n}\end{bmatrix} \begin{bmatrix}\mathbb{BC}\mathcal{F}_1^{(k)}\\ \mathbb{BC}\mathcal{F}_0^{(k)}\end{bmatrix}\\
&= \begin{bmatrix}\mathbb{BC}\mathcal{F}_1^{(k)} & \mathbb{BC}\mathcal{F}_0^{(k)}\end{bmatrix} \begin{bmatrix}\mathbb{BC}\mathcal{F}_{2n}^{(k)}\\ \mathbb{BC}\mathcal{F}_{2n-1}^{(k)}\end{bmatrix}\\
&=\mathbb{BC}\mathcal{F}_1^{(k)}\mathbb{BC}\mathcal{F}_{2n}^{(k)}+\mathbb{BC}\mathcal{F}_0^{(k)}\mathbb{BC}\mathcal{F}_{2n-1}^{(k)}\\
&=(\bar{\beta}_1^{(k)}e_1+\beta_1^{(k)}e_2)(\bar{\beta}_{2n}^{(k)}e_1+\beta_{2n}^{(k)}e_2)+(\bar{\beta}_0^{(k)}e_1+\beta_0^{(k)}e_2)\\
&\quad(\bar{\beta}_{2n-1}^{(k)}e_1+\beta_{2n-1}^{(k)}e_2).
\end{aligned}
$$

□

Theorem 9. *For the bicomplex Fibonacci finite operator, we have*

$$
\begin{aligned}
\mathbb{BC}\mathcal{F}_{n+r}^{(k)}\mathbb{BC}\mathcal{F}_{n+s}^{(k)}-\mathbb{BC}\mathcal{F}_{n}^{(k)}\mathbb{BC}\mathcal{F}_{n+r+s}^{(k)} =&(-1)^nF_r\left((\mathcal{F}_1^{(k)})^2-\mathcal{F}_0^{(k)}\mathcal{F}_2^{(k)}\right)\big((\bar{\beta}_1e_1+\beta_1e_2)\\
&(\bar{\beta}_se_1+\beta_se_2)-(\bar{\beta}_0e_1+\beta_0e_2)(\bar{\beta}_{s+1}e_1+\beta_{s+1}e_2)\big).
\end{aligned}
$$

Proof. With the use of equations (24) and (25), we can compute :

$$
\begin{aligned}
\begin{bmatrix}\mathbb{BC}\mathcal{F}_{n+r}^{(k)} & \mathbb{BC}\mathcal{F}_{n}^{(k)}\end{bmatrix} &= \begin{bmatrix}\mathbb{BC}\mathcal{F}_{n+1}^{(k)} & \mathbb{BC}\mathcal{F}_{n}^{(k)}\end{bmatrix}\begin{bmatrix}F_r & 0\\ F_{r-1} & 1\end{bmatrix}\\
&= \begin{bmatrix}\mathbb{BC}F_1 & \mathbb{BC}F_0\end{bmatrix}\begin{bmatrix}\mathcal{F}_{n+1}^{(k)} & \mathcal{F}_{n}^{(k)}\\ \mathcal{F}_{n}^{(k)} & \mathcal{F}_{n-1}^{(k)}\end{bmatrix}\begin{bmatrix}F_r & 0\\ F_{r-1} & 1\end{bmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{bmatrix}\mathbb{BC}\mathcal{F}_{n+s}^{(k)}\\ -\mathbb{BC}\mathcal{F}_{n+r+s}^{(k)}\end{bmatrix} &= \begin{bmatrix}1 & 0\\ -F_{r-1} & F_r\end{bmatrix}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{n+s}^{(k)}\\ -\mathbb{BC}\mathcal{F}_{n+s+1}\end{bmatrix}\\
&= \begin{bmatrix}1 & 0\\ -F_{r-1} & F_r\end{bmatrix}\begin{bmatrix}\mathcal{F}_{n-1}^{(k)} & -\mathcal{F}_{n}^{(k)}\\ -\mathcal{F}_{n}^{(k)} & \mathcal{F}_{n+1}^{(k)}\end{bmatrix}\begin{bmatrix}\mathbb{BC}F_s\\ -\mathbb{BC}F_{s+1}\end{bmatrix}.
\end{aligned}
$$

Also, we have the following computation:

$$\begin{aligned}
&\begin{bmatrix}\mathcal{F}_{n+1}^{(k)} & \mathcal{F}_{n}^{(k)}\\ \mathcal{F}_{n}^{(k)} & \mathcal{F}_{n-1}^{(k)}\end{bmatrix}\begin{bmatrix}F_r & 0\\ F_{r-1} & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -F_{r-1} & F_r\end{bmatrix}\begin{bmatrix}\mathcal{F}_{n-1}^{(k)} & -\mathcal{F}_{n}^{(k)}\\ -\mathcal{F}_{n}^{(k)} & \mathcal{F}_{n+1}^{(k)}\end{bmatrix}\\
&= \begin{bmatrix}\mathcal{F}_{n+1}^{(k)} & \mathcal{F}_{n}^{(k)}\\ \mathcal{F}_{n}^{(k)} & \mathcal{F}_{n-1}^{(k)}\end{bmatrix}(F_r I)\begin{bmatrix}\mathcal{F}_{n-1}^{(k)} & -\mathcal{F}_{n}^{(k)}\\ -\mathcal{F}_{n}^{(k)} & \mathcal{F}_{n+1}^{(k)}\end{bmatrix}\\
&= F_r\begin{bmatrix}\mathcal{F}_{n+1}^{(k)} & \mathcal{F}_{n}^{(k)}\\ \mathcal{F}_{n}^{(k)} & \mathcal{F}_{n-1}^{(k)}\end{bmatrix}\begin{bmatrix}\mathcal{F}_{n-1}^{(k)} & -\mathcal{F}_{n}^{(k)}\\ -\mathcal{F}_{n}^{(k)} & \mathcal{F}_{n+1}^{(k)}\end{bmatrix}\\
&= F_r\begin{bmatrix}\mathcal{F}_{n+1}^{(k)}\mathcal{F}_{n-1}^{(k)} - (\mathcal{F}_{n}^{(k)})^2 & 0\\ 0 & -(\mathcal{F}_{n}^{(k)})^2 + \mathcal{F}_{n-1}^{(k)}\mathcal{F}_{n+1}^{(k)}\end{bmatrix}\\
&= -F_r\big((\mathcal{F}_{n}^{(k)})^2 - \mathcal{F}_{n-1}^{(k)}\mathcal{F}_{n+1}^{(k)}\big)I\\
&= (-1)^n F_r\big((\mathcal{F}_{1}^{(k)})^2 - \mathcal{F}_{0}^{(k)}\mathcal{F}_{2}^{(k)}\big)I.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\mathbb{BC}\mathcal{F}_{n+r}^{(k)}\mathbb{BC}\mathcal{F}_{n+s}^{(k)} - \mathbb{BC}\mathcal{F}_{n}^{(k)}\mathbb{BC}\mathcal{F}_{n+r+s}^{(k)}\\
&= \begin{bmatrix}\mathbb{BC}\mathcal{F}_{n+r}^{(k)} & \mathbb{BC}\mathcal{F}_{n}^{(k)}\end{bmatrix}\begin{bmatrix}\mathbb{BC}\mathcal{F}_{n+s}^{(k)}\\ -\mathbb{BC}\mathcal{F}_{n+r+s}^{(k)}\end{bmatrix}\\
&= (-1)^n F_r\big((\mathcal{F}_{1}^{(k)})^2 - \mathcal{F}_{0}^{(k)}\mathcal{F}_{2}^{(k)}\big)\begin{bmatrix}\mathbb{BC}F_1 & \mathbb{BC}F_0\end{bmatrix}\begin{bmatrix}\mathbb{BC}F_s\\ -\mathbb{BC}F_{s+1}\end{bmatrix}\\
&= (-1)^n F_r\big((\mathcal{F}_{1}^{(k)})^2 - \mathcal{F}_{0}^{(k)}\mathcal{F}_{2}^{(k)}\big)(\mathbb{BC}F_1\mathbb{BC}F_s - \mathbb{BC}F_0\mathbb{BC}F_{s+1})\\
&= (-1)^n F_r\big((\mathcal{F}_{1}^{(k)})^2 - \mathcal{F}_{0}^{(k)}\mathcal{F}_{2}^{(k)}\big)\big((\bar{\beta}_1 e_1 + \beta_1 e_2)(\bar{\beta}_s e_1 + \beta_s e_2)\\
&\quad - (\bar{\beta}_0 e_1 + \beta_0 e_2)(\bar{\beta}_{s+1} e_1 + \beta_{s+1} e_2)\big).
\end{aligned}$$

□

Corollary 3. *Equations* (22) *and* (23) *are obtained by substituting* n *by* $n-1$ *and* $r = s = 1$ *in the aforementioned theorem.*

3 Conclusions

In this paper, we have presented a systematic study of bicomplex Fibonacci finite operator sequences through the idempotent representation of bicomplex numbers. Unlike previous works on hypercomplex Fibonacci-type structures that mainly relied on classical component-wise formulations, our approach is based explicitly on the orthogonal idempotent decomposition intrinsic to the bicomplex algebra. The main novelty of this work lies in deriving recurrence relations, polynomial representations, and matrix representations directly in terms of the idempotent basis. This framework simplifies the structural analysis by decomposing the sequences into independent complex components, leading to clearer proofs and new structural identities that are not immediately visible in the standard representation. It is worth noting that, unlike quaternion algebra introduced by William Rowan Hamilton —which is noncommutative and does not admit a non-trivial idempotent representation— the bicomplex algebra is commutative and possesses orthogonal idempotent elements. This distinctive feature makes the idempotent method particularly effective for studying operator sequences. The results obtained here provide a refined structural perspective and open new directions for further investigations on recursive operator sequences within the bicomplex framework and related commutative hypercomplex systems.

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