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EXTENDED FOUR PARAMETER CHEBYSHEV-HALLEY-TYPE METHODS OF ORDER SIX*

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Abstract

A study of the local and the semilocal convergence is carried out for the Chebyshev-Halley-type iterative methods under ω-type conditions. The conditions are imposed only on the first-order derivatives. In both cases, the convergence region and the region of uniqueness of the solution is established. The new technique is a usefull alternative to expensive Taylor series used to study the convergence of iterative methods requiring high order derivatives not on the methods. The results of a numerical experiment are presented to check the convergence conditions.

Keywords: complete normed space, Chebyshev-Halley-type methods, local and semi-local convergence, order six.

MSC: 65J15, 65H10, 65G99, 47H30.

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1 Introduction

Mathematical models of the complex physical or technological processes described often by nonlinear problems, in particular by systems of nonlinear algebraic or transcendental equations, nonlinear integral equations, nonlinear boundary value problems and other. It is extremely rare to find an exact solution to such problems. Therefore, research and development of methods for numerically solving nonlinear problems is an important task. In general, these problems are written in the form of an operator equation [1,8,11,12,14]

$$F(x) = 0. \tag{1}$$

Here the operator $F : \Omega \subset B_1 \to B_2$ is Frechét-differentiable, Ω is an open and convex set, whereas B_1 and B_2 are Banach spaces.

There is a large number of iterative methods for the numerical solving of the equation (1), in particular Newton method or methods with divided differences. One of the characteristics of iterative methods is the amount of calculations of the values of F and its derivatives (or divided differences). In most cases, at each step of single-step methods, it is necessary to calculate one value of F and find one inverse operator for F' or its approximation. To increase the convergence order, it is necessary to increase the number of calculations of F and F' which leads to increasing computational costs and a decreasing of the efficiency of the method. Such methods had been investigated in [2–7, 9, 10, 13].

In this paper we consider the three step method with four real parameters b, c, p and q $(b \neq 0)$, which is defined for $x_0 \in \Omega$ and each $n = 0, 1, 2, \ldots$ by [9]

$$\begin{aligned}
y_n &= x_n - \frac{2}{3}F'(x_n)^{-1}F(x_n),\\
A_n &= I - F'(x_n)^{-1}F'(y_n),\ M_n = I - \frac{c}{b}A_n,\\
T_n &= I + a_1A_n + a_2A_n^2,\ a_1 = \frac{1}{4}\frac{3b-2}{b},\ a_2 = \frac{1}{8}\frac{9b^2-3b-4c+2}{b^2},\\
z_n &= x_n - \left(I + \frac{1}{2b}M_n^{-1}A_n\right)T_nF'(x_n)^{-1}F(x_n),\\
x_{n+1} &= z_n - (pI + qA_n)F'(x_n)^{-1}F(z_n).
\end{aligned} \tag{2}$$

The local convergence order six is shown in [9] for $B_1 = B_2 = \mathbb{R}^m$, $p = 1$, $q = \dfrac{3}{2}$ using the Taylor series technique. There are constraints with this technique.

Motivation for writing this paper.

(E_1) The local analysis (see [9]) is restricted only on $\mathbb{R}^m$ and the assumption is made that $F^{(5)}$ (the fifth derivative exists and is bounded on Ω. But this high order derivative is not on the method (2). Hence, the results in [9] are applicable to solve equation (1)) provided that the derivatives reaching up to order five exist, which may not be true. As an example, let $m = 1$ and say $\Omega = (-1.3, 1.3)$. Define the function $F : \Omega \to \mathbb{R}$ by

$$F(t) = \begin{cases} d_1 t^5 \log t^2 + d_2 t^6 + d_3 t^7, & t \neq 0 \\ 0, & t = 0, \end{cases}$$

where $d_1 \neq 0$ and $d_2 + d_3 = 0$. It follows by this definition that $t^* = 1 \in \Omega$ solves the equation $F(t) = 0$. But the function $F^{(5)}$ is unbounded at $t = 0 \in \Omega$. Thus, the results in [9] cannot assure the convergence of the method (2) to t^*. But the method (2) converges to $t^* = 1$ if say $x_0 = 0.95$, $b = c = p = 1$ and $q = \dfrac{3}{2}$. This observation indicates that if an alternative to the Taylor series technique is used the sufficient convergence conditions in [9] may be weakened.

(E_2) No radius of convergence is given in [9]. So, the initial points x_0 are given in not known.

(E_3) There are no previous knowledge of the integer N such that $\|x^* - x_n\| < \varepsilon \; \forall n \geq N$ and some $\varepsilon \geq 0$.

(E_4) There are no isolation of the solution results.

(E_5) The more difficult and important semi-local analysis of method (2) is not previously studied.

The technique of this paper positively addresses all issues (E_1)-(E_5) and series as an alternative to Taylor series for studying the convergence of method (2) as well as other similar methods analogously [4–7, 10, 13]. In particular, we achieve:

$(E_1)'$ The local analysis of method (2) is shown by using only the operators on it, i.e. F and F'.

$(E_2)'$ A computable radius of convergence is provided. So, the initial points ate selected from a certain ball about x^* or x_0.

$(E_3)'$ The number of iterations to be carried out, i.e. N is known a priori.

$(E_4)'$ Domains are determined which contain only one solution.

$(E_5)'$ Majorizing scalar sequences [1] are employed to determine the convergence of $\{x_n\}$ generated by the method (2). The analysis is presented in the more general setting of a Banach space. Moreover, the derivative F' is controlled by generalized continuity conditions used to control it and also sharpen the error distances $\|x^* - x_n\|$ [1–3].

The new local convergence analysis is shown using only the operators on the method, i.e. F and F'. Moreover, the semi-local analysis not previously studied is presented using majorizing sequences. Both analyses provided in the more general setting of a Banach space and also depend on generalized continuity which controls the derivative F'. The same technique can be used to extend the applicability of other methods along the same lines [4–7,10,13].

2 Local convergence

First we introduce the following notation and conditions used in the study of local convergence of the method (2).

Denote by $U(x, r)$ and $U[x, r]$ open and closed balls, respectively, with center at the point x and of radius r. Let us set $S = [0, \infty)$.

Suppose

(LC_1) There exists a function $\omega_0 : S \to S$, which is continuous and strictly increasing on S such that equation $\omega_0(t) - 1 = 0$ has at least one positive root. We denote by ϱ_0 the smallest such root and set $S_0 = [0, \varrho_0)$.

(LC_2) There exists a function $\omega : S_0 \to S$, which is continuous and strictly increasing on S_0 such that for function $g_1 : S_0 \to S$ given by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta + \frac{1}{3}\left(1 + \int_0^1 w_0(\theta t)d\theta\right)}{1 - w_0(t)}, \tag{3}$$

equation $g_1(t) - 1 = 0$ has at least one root in the interval $(0, \varrho_0)$. We denote by r_1 the smallest such root.

(LC_3) The equation $\mu(t) - 1 = 0$ has at least one root in the interval $(0, \varrho_0)$. We denote by ϱ the smallest such root. Set $S_1 = [0, \varrho)$.

Define the function $g_2 : S_1 \to S$ by

$$g_2(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1 - w_0(t)} + \frac{\lambda(t)\left(1 + \int_0^1 w_0(\theta t)d\theta\right)}{1 - w_0(t)}$$

$$+\frac{(1+\lambda(t))\overline{w}(t)}{2|b|(1-w_0(t))^2(1-\mu(t))}\left(1+\int_0^1 w_0(\theta t)d\theta\right),$$

where

$$\overline{w}(t)=\begin{cases} w((1+g_1(t))t) \\ or \\ w_0(t)+w_0(g_1(t)t), \end{cases} \qquad \mu(t)=\left|\frac{c}{b}\right|\frac{\overline{w}(t)}{1-w_0(t)},$$

$$\lambda(t)=\frac{\overline{w}(t)}{1-w_0(t)}\left(|a_1|+|a_2|\frac{\overline{w}(t)}{1-w_0(t)}\right).$$

The equation

$$g_2(t)-1=0$$

has at least one root in the interval $(0,r_1)$. We denote by r_2 the smallest such root.

Define the function $g_3 : S_1 \to S$ by

$$g_3(t)=\left[\frac{\int_0^1 w((1-\theta)g_2(t)t)d\theta}{1-w_0(g_2(t)t)}+\frac{\overline{\overline{w}}(t)\left(1+\int_0^1 w_0(\theta g_2(t)t)d\theta\right)}{(1-w_0(t))(1-w_0(g_2(t)t))}\right.$$

$$\left.+\left(|p-1|+\frac{|q|\overline{w}(t)}{1-w_0(t)}\right)\frac{\left(1+\int_0^1 w_0(\theta t)d\theta\right)}{1-w_0(t)}\right]g_2(t),$$

where

$$\overline{\overline{w}}(t)=\begin{cases} w((1+g_2(t))t) \\ or \\ w_0(t)+w_0(g_2(t)t), \end{cases}$$

The equation

$$g_3(t)-1=0$$

has at least one root in the interval $(0,r_2)$. We denote by r_3 the smallest such root.

Define the parameter

$$r^*=\min\{r_j\}, \quad j=1,2,3. \tag{4}$$

This parameter is shown to be a radius of convergence in Theorem 1 for the method (2). Set $S_2=[0,r^*)$.

It follows by these definitions that

$$0 \le \omega_0(t) < 1 \quad \forall t \in S_2 \quad \textbf{and} \quad 0 \le \omega_0(g_2(t)t) < 1 \quad \forall t \in S_2 \tag{5}$$

and $\forall t \in S_2$

$$0 \le g_i(t) < 1, \; i = 1, 2, 3. \tag{6}$$

(LC_4) There exists a solution $x^* \in \Omega$ of the equation $F(x) = 0$, $L \in \mathcal{L}(B_1, B_2)$ such that $L^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\|L^{-1}(F'(x) - L)\| \le \omega_0(\|x - x^*\|) \quad \forall x \in \Omega.$$

Set $\Omega_0 = \Omega \cap U(x^*, \varrho_0)$.

(LC_5) $\|L^{-1}(F'(x) - F'(y))\| \le \omega(\|x - y\|) \quad \forall x, y \in \Omega_0$.

(LC_6) $U(x^*, r^*) \subset \Omega$.

Theorem 1. *Suppose that conditions (LC_1)-(LC_6) hold and $x_0 \in U(x^*, r^*)$. Then, the sequence $\{x_n\}$ generated by method (2) is well defined in $U(x^*, r^*)$ for each $n = 0, 1, \ldots$ and converges to the solution $x^* \in \Omega$ of the equation (1). Moreover, the following error estimates hold for each $n = 0, 1, \ldots$*

$$\|y_n - x^*\| \le g_1(r^*)\|x_n - x^*\| \le \|x_n - x^*\| < r^*, \tag{7}$$

$$\|z_n - x^*\| \le g_2(r^*)\|x_n - x^*\| \le \|x_n - x^*\| \tag{8}$$

and

$$\|x_{n+1} - x^*\| \le g_3(r^*)\|x_n - x^*\| \le \|x_n - x^*\|. \tag{9}$$

Proof. The proof is carried out by mathematical induction. Using conditions (LC_4) and (5), we obtain in turn that

$$\|L^{-1}(F'(x_n) - L)\| \le \omega_0(\|x_n - x^*\|) \le \omega_0(r^*) < 1. \tag{10}$$

The Banach Lemma on invertible linear operators [1,11] and (10) imply that $F'(x_n)^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\|F'(x_n)^{-1}L\| \le \frac{1}{1 - \omega_0(\|x_n - x^*\|)}. \tag{11}$$

Then, by the first substep of (2), we can write in turn

$$\begin{aligned} y_n - x^* &= x_n - x^* - F'(x_n)^{-1}F(x_n) + \frac{1}{3}F'(x_n)^{-1}F(x_n) \\ &= F'(x_n)^{-1}\left(F'(x_n) - \int_0^1 F'(x^* + \theta(x_n - x^*))d\theta\right)(x_n - x^*) \\ &\quad + \frac{1}{3}F'(x_n)^{-1}\left(\int_0^1 F'(x^* + \theta(x_n - x^*))d\theta - F'(x^*) + F'(x^*)\right) \\ &\quad \times (x_n - x^*). \end{aligned}$$

Taking into account conditions (LC_4), (LC_5), (6), (11) and the last equality, we have

$$\begin{aligned} \|y_n - x^*\| &\leq \frac{\int_0^1 w((1-\theta)\|x_n - x^*\|)d\theta + \frac{1}{3}\left(1 + \int_0^1 w_0(\theta\|x_n - x^*\|)d\theta\right)}{1 - w_0(\|x_n - x^*\|)} \\ &\quad \times \|x_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r^*. \end{aligned}$$

Using conditions (LC_4), (LC_5) and (11), we have

$$\begin{aligned} \|A_n\| &= \|I - F'(x_n)^{-1}F'(y_n)\| = \|F'(x_n)^{-1}(F'(x_n) - F'(x^*) \\ &\quad + F'(x^*) - F'(y_n))\| \leq \|F'(x_n)^{-1}L\|\Big(\|L^{-1}(F'(x_n) - F'(x^*))\| \\ &\quad + \|L^{-1}(F'(x^*) - F'(y_n))\|\Big) \leq \frac{\omega_0(\|x_n - x^*\|) + \omega_0(\|y_n - x^*\|)}{1 - \omega_0(\|x_n - x^*\|)} \\ &= \frac{\omega_0(\|x_n - x^*\|) + \omega_0(g_1(\|x_n - x^*\|)\|x_n - x^*\|)}{1 - \omega_0(\|x_n - x^*\|)} \end{aligned}$$

or

$$\begin{aligned} \|A_n\| &= \|I - F'(x_n)^{-1}F'(y_n)\| = \|F'(x_n)^{-1}(F'(x_n) - F'(y_n))\| \\ &\leq \|F'(x_n)^{-1}L\|\left(\|L^{-1}(F'(x_n) - F'(y_n))\|\right) \\ &\leq \frac{\omega(\|x_n - y_n\|)}{1 - \omega_0(\|x_n - x^*\|)} \leq \frac{\omega((1 + g_1(\|x_n - x^*\|))\|x_n - x^*\|)}{1 - \omega_0(\|x_n - x^*\|)} \end{aligned}$$

and

$$\|A_n\| \leq \frac{\overline{\omega}(\|x_n - x^*\|)}{1 - \omega_0(\|x_n - x^*\|)}. \tag{12}$$

In view of the following equality holds

$$I - T_n = -a_1 A_n - a_2 A_n^2,$$

we obtain using the estimates (10) and (12)

$$\begin{aligned}
\|I-T_n\| &\leq |a_1|\frac{\overline{w}(\|x_n-x^*\|)}{1-w_0(\|x_n-x^*\|)}+|a_2|\left(\frac{\overline{w}(\|x_n-x^*\|)}{1-w_0(\|x_n-x^*\|)}\right)^2\\
&= \lambda(\|x_n-x^*\|)=\lambda_n, && (13)\\
\|T_n\| &\leq 1+\lambda_n, && (14)
\end{aligned}$$

$$\left\|\frac{c}{b}A_n\right\| \leq \left|\frac{c}{b}\right|\frac{\overline{w}(\|x_n-x^*\|)}{1-w_0(\|x_n-x^*\|)}=\mu(\|x_n-x^*\|)=\mu_n<1$$

and

$$\|M_n^{-1}\| = \|(I-(I-M_n))^{-1}\|=\left\|\left(I-\frac{c}{b}A_n\right)^{-1}\right\|\leq\frac{1}{1-\mu_n}. \quad (15)$$

Then, by the second substep of (2), we can write in turn

$$\begin{aligned}
z_n-x^* &= x_n-x^*-F'(x_n)^{-1}F(x_n)+(I-T_n)\,F'(x_n)^{-1}F(x_n)\\
&\quad -\frac{1}{2b}M_n^{-1}A_nT_nF'(x_n)^{-1}F(x_n)\\
&= F'(x_n)^{-1}\left(F'(x_n)-\int_0^1 F'(x^*+\theta(x_n-x^*))d\theta\right)(x_n-x^*)\\
&\quad +(I-T_n)\,F'(x_n)^{-1}\left(\int_0^1 F'(x^*+\theta(x_n-x^*))d\theta\right.\\
&\quad \left.-F'(x^*)+F'(x^*)\right)(x_n-x^*)\\
&\quad -\frac{1}{2b}M_n^{-1}A_nT_nF'(x_n)^{-1}\left(\int_0^1 F'(x^*+\theta(x_n-x^*))d\theta\right.\\
&\quad \left.-F'(x^*)+F'(x^*)\right)(x_n-x^*).
\end{aligned}$$

By using (LC_4), (LC_5), estimates (6), (11), (12)–(15), we get

$$\begin{aligned}
\|z_n-x^*\| &\leq \left[\frac{\int_0^1 w((1-\theta)\|x_n-x^*\|)d\theta}{1-w_0(\|x_n-x^*\|)}\right.\\
&\quad +\frac{\lambda_n\left(1+\int_0^1 w_0(\theta\|x_n-x^*\|)d\theta\right)}{1-w_0(\|x_n-x^*\|)}\\
&\quad +\frac{(1+\lambda_n)\overline{w}(\|x_n-x^*\|)}{2|b|(1-w_0(\|x_n-x^*\|))^2(1-\mu_n)}\\
&\quad \left.\times\left(1+\int_0^1 w_0(\theta\|x_n-x^*\|)d\theta\right)\right]\|x_n-x^*\|\\
&= g_2(\|x_n-x^*\|)\|x_n-x^*\|\leq\|x_n-x^*\|.
\end{aligned}$$

Then, by the third substep of (2), we can write in turn

$$
\begin{aligned}
x_{n+1} - x^* &= z_n - x^* - F'(z_n)^{-1}F(z_n) + \left(F'(z_n)^{-1} - F'(x_n)^{-1}\right) F(z_n) \\
&\quad -(p-1)F'(x_n)^{-1}F(z_n) - qA_nF'(x_n)^{-1}F(z_n), \\
&= F'(z_n)^{-1}\left(F'(z_n) - \int_0^1 F'(x^* + \theta(z_n - x^*))d\theta\right)(z_n - x^*) \\
&\quad + \left(F'(z_n)^{-1} - F'(x_n)^{-1}\right)\left(\int_0^1 F'(x^* + \theta(z_n - x^*))d\theta\right. \\
&\quad \left. -F'(x^*) + F'(x^*)\right)(z_n - x^*) - \left((p-1)I + qA_n\right)F'(x_n)^{-1} \\
&\quad \times\left(\int_0^1 F'(x^* + \theta(z_n - x^*))d\theta - F'(x^*) + F'(x^*)\right) \\
&\quad \times(z_n - x^*).
\end{aligned}
$$

Using condition (LC_3), (5) and (8), we obtain in turn that

$$
\|L^{-1}(F'(z_n) - L)\| \le \omega_0(\|z_n - x^*\|) \le \omega_0(g_2(r^*)r^*) < 1. \tag{16}
$$

The Banach Lemma on invertible linear operators [1,11] and (16) imply that $F'(z_n)^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$
\|F'(z_n)^{-1}L\| \le \frac{1}{1 - \omega_0(\|z_n - x^*\|)}. \tag{17}
$$

By the following equality

$$
F'(z_n)^{-1} - F'(x_n)^{-1} = F'(x_n)^{-1}\left(F'(x_n) - F'(z_n)\right)F'(x_n)^{-1},
$$

and taking into account the conditions (LC_4), (LC_5), estimates (6), (8), (11), (12) and (17), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\le \left[\frac{\overline{\overline{w}}(\|x_n - x^*\|)\left(1 + \int_0^1 w_0(\theta\|z_n - x^*\|)d\theta\right)}{(1 - w_0(\|x_n - x^*\|))(1 - w_0(\|z_n - x^*\|))}\right. \\
&\quad + \frac{\int_0^1 w((1-\theta)\|z_n - x^*\|)d\theta}{1 - w_0(\|z_n - x^*\|)} \\
&\quad + \left(|p-1| + \frac{|q|\overline{w}(\|x_n - x^*\|)}{1 - w_0(\|x_n - x^*\|)}\right) \\
&\quad \left.\times\frac{\left(1 + \int_0^1 w_0(\theta\|z_n - x^*\|)d\theta\right)}{1 - w_0(\|x_n - x^*\|)}\right]\|z_n - x^*\| \\
&\le g_3(\|x_n - x^*\|)\|x_n - x^*\| \le \|x_n - x^*\|.
\end{aligned}
$$

Moreover, by (9) there exists $\alpha \in [0,1)$ such that

$$\|x_{n+1} - x^*\| \leq \alpha \|x_n - x^*\| \leq \alpha^{n+1} \|x_0 - x^*\| < r^*. \tag{18}$$

Therefore, it follows from (18) that the iterate $x_{n+1} \in U(x^*, r^*)$ and $\lim_{n \to \infty} x_n = x^*$. □

Next, we present a result for uniqueness of the solution of the equation (1).

Proposition 1. *Suppose:*
(a) The condition (LC_4) holds in the ball $U(x^*, \varrho_1)$ for some $\varrho_1 > 0$.
(b) There exists $\varrho_2 \geq \varrho_1$ such that

$$\int_0^1 \omega_0\Big(\theta \varrho_2\Big) d\theta < 1. \tag{19}$$

Set $\Omega_1 = U[x^*, \varrho_2] \cap \Omega$.
Then, the equation $F(x) = 0$ has the unique solution x^* in the region Ω_1.

Proof. Suppose that there exists $u^* \in \Omega_1$, $u^* \neq x^*$ and $F(u^*) = 0$. Let $T = \int_0^1 F'(x^* + \theta(u^* - x^*)) d\theta$. It follows by (a) and (b) that

$$\begin{aligned} \|L^{-1}(T - L)\| &\leq \int_0^1 \omega_0(\|x^* + \theta(u^* - x^*) - x^*\|) d\theta \leq \\ &\leq \int_0^1 \omega_0\Big(\theta \|u^* - x^*\|\Big) d\theta \int_0^1 \omega_0\Big(\theta \varrho_2\Big) d\theta < 1. \end{aligned}$$

Hence, the operator T is invertible. Then, by the identity

$$u^* - x^* = T^{-1}(F(u^*) - F(x^*)) = T^{-1}(0 - 0) = 0,$$

we conclude that $u^* = x^*$. □

3 Semi-local convergence

Majorizing sequences are used to provide the semi-local analysis.

Suppose:

(SLC_1) There exists a function $v_0 : S \to S$ which is continuous and nondecreasing such that the equation

$$v_0(t) - 1 = 0$$

has a smallest solution ϱ_3. Set $S_3 = [0, \varrho_3)$.

(SLC_2) There exists a function $v : S_3 \to S$ which is continuous and nondecreasing.

The majorant sequence $\{\alpha_n\}$ is defined for $\alpha_0 = 0$,

$$\beta_0 \geq \frac{2}{3}\|F'(x_0)^{-1}F(x_0)\|$$

and $\forall n = 0, 1, 2, \ldots$ by

$$\overline{v}_n = \begin{cases} v(\beta_n - \alpha_n), \\ or \\ v_0(\alpha_n) + v_0(\beta_n), \end{cases}$$

$$\overline{\lambda}_n = \left(|a_1| + |a_2|\frac{\overline{v}_n}{1 - v_0(\alpha_n)}\right)\frac{\overline{v}_n}{1 - v_0(\alpha_n)}, \quad \mu_n = \left|\frac{c}{b}\right|\frac{\overline{v}_n}{1 - v_0(\alpha_n)},$$

$$\gamma_n = \beta_n + \frac{3}{2}\left(\frac{1}{3} + \overline{\lambda}_n + \frac{(1 + \lambda_n)\overline{v}_n}{2|b|(1 - v_0(\alpha_n))(1 - \mu_n)}\right)(\beta_n - \alpha_n),$$

$$\begin{aligned} \xi_n &= \int_0^1 v((1 - \theta)(\gamma_n - \alpha_n))d\theta(\gamma_n - \alpha_n) + (1 + v_0(\alpha_n))(\gamma_n - \beta_n) \\ &\quad + \frac{1}{2}(1 + v_0(\alpha_n))(\beta_n - \alpha_n), \end{aligned}$$

$$\alpha_{n+1} = \gamma_n + \left(|p| + |q|\frac{\overline{v}_n}{1 - v_0(\alpha_n)}\right)\frac{\xi_n}{1 - v_0(\alpha_n)}, \tag{20}$$

$$\begin{aligned} \delta_{n+1} &= \int_0^1 v((1 - \theta)(\alpha_{n+1} - \alpha_n))d\theta(\alpha_{n+1} - \alpha_n) \\ &\quad + (1 + v_0(\alpha_n))(\alpha_{n+1} - \beta_n) + \frac{1}{2}(1 + v_0(\alpha_n))(\beta_n - \alpha_n) \end{aligned}$$

and

$$\beta_{n+1} = \alpha_{n+1} + \frac{2}{3}\frac{\delta_{n+1}}{1 - v_0(\alpha_{n+1})}.$$

(SLC_3)

Lemma 1. *Suppose that* $\forall n = 0, 1, 2, \ldots$

$$0 \leq v_0(\alpha_n) < 1, \quad \mu_n < 1, \quad 0 \leq \alpha_n < \overline{\alpha} \; for \; some \; \overline{\alpha} > 0. \tag{21}$$

Then, the sequence $\{\alpha_n\}$ *given by the formula (20) is nondecreasing and convergent to its unique least upper bound* $\alpha^* \in [0, 1]$.

Proof. The sequence $\{\alpha_n\}$ is nondecreasing and bounded from above by $\overline{\alpha}$ and as such it is convergent to α^*. □

The additional conditions shall be used in the semi-local convergence analysis of method (2).

(SLC_4) There exist points $x_0 \in \Omega$ and parameter β_0 such that $F'(x_0)^{-1}$, $L_0^{-1} \in \pounds(Y, X)$ and

$$\frac{2}{3}\|F'(x_0)^{-1}F(x_0)\| \leq \beta_0.$$

(SLC_5) $\|L^{-1}(F'(x) - F'(x_0))\| \leq v_0(\|x - x_0\|) \quad \forall\, x \in \Omega.$
Set $\Omega_2 = U(x_0, \varrho_3) \cap \Omega$.

(SLC_6) $\|L^{-1}(F'(x) - F'(y))\| \leq v(\|x - y\|) \quad \forall\, x, y \in \Omega_2.$

(SLC_7) $U[x_0, \alpha^*] \subset \Omega.$

Next, the semi-local convergence of the method (2) is presented based on the conditions (SLC_1)-(SLC_7) and the preceding terminology.

Theorem 2. *Suppose that the conditions (SLC_1)-(SLC_7) hold. Then, the sequence $\{x_n\}$ generated by the method (2) is well defined in $U(x_0, \alpha^*)$, remains in $U(x_0, \alpha^*)$ for all $n = 0, 1, 2, ...$ and converges to a solution $x^* \in U[x_0, \alpha^*]$ of the equation $F(x) = 0$. Moreover, the following error estimates hold*

$$\|x^* - x_n\| \leq \alpha^* - \alpha_n. \tag{22}$$

Proof. By the condition (SLC_4) the iterate y_0 is well defined, since from the first substep of the method for $n = 0$ we have

$$\|y_0 - x_0\| \leq \frac{2}{3}\|F'(x_0)^{-1}F(x_0)\| \leq \beta_0 - \alpha_0 = \beta_0 < \alpha^*,$$

and the iterate $y_0 \in U(x_0, \alpha^*)$.

Suppose that $y_i, z_i, x_{i+1} \in U(x_0, \alpha^*)$ for $i = 0, \ldots, n-1$ and

$$\begin{aligned}
\|y_i - x_i\| &\leq \beta_i - \alpha_i, \\
\|z_i - y_i\| &\leq \gamma_i - \beta_i, \\
\|x_{i+1} - y_i\| &\leq \alpha_{i+1} - \beta_i, \\
\|x_{i+1} - z_i\| &\leq \alpha_{i+1} - \gamma_i.
\end{aligned}$$

Then for $i = n$, we get following estimates.

Using conditions (SLC_3) and (SLC_5), we obtain in turn that

$$\|L^{-1}(F'(x_n)-L)\| \leq v_0(\|x_n-x_0\|) \leq v_0(\alpha_n) < 1. \tag{23}$$

By Banach Lemma [1] and (23) the operator $F'(x_n)^{-1} \in \mathcal{L}(B_2, B_1)$ and

$$\|F'(x_n)^{-1}L\| \leq \frac{1}{1-v_0(\|x_n-x_0\|)}. \tag{24}$$

By the identity

$$\begin{aligned}
F(x_n) &= F(x_n)-F(x_{n-1})-\frac{3}{2}F'(x_{n-1})(y_{n-1}-x_{n-1}) \\
&= F(x_n)-F(x_{n-1})-F'(x_{n-1})(x_n-x_{n-1})+F'(x_{n-1})(x_n-y_{n-1}) \\
&\quad -\frac{1}{2}F'(x_{n-1})(y_{n-1}-x_{n-1}) \\
&= \int_0^1 \Big(F'(x_{n-1}+\theta(x_n-x_{n-1}))-F'(x_{n-1})\Big)d\theta(x_n-x_{n-1}) \\
&\quad +F'(x_{n-1})(x_n-y_{n-1})-\frac{1}{2}F'(x_{n-1})(y_{n-1}-x_{n-1}),
\end{aligned}$$

we get

$$\begin{aligned}
\|L^{-1}F(x_n)\| &\leq \int_0^1 v\Big((1-\theta)(\alpha_n-\alpha_{n-1})\Big)d\theta(\alpha_n-\alpha_{n-1}) \\
&\quad +(1+v_0(\alpha_{n-1}))(\alpha_n-\beta_{n-1}) \\
&\quad +\frac{1}{2}(1+v_0(\alpha_{n-1}))(\beta_{n-1}-\alpha_{n-1}) = \delta_n.
\end{aligned} \tag{25}$$

We have by the first substep of the method

$$\|y_n-x_n\| \leq \frac{2}{3}\|F'(x_n)^{-1}L\|\|L^{-1}F(x_n)\| \leq \frac{2}{3}\frac{\delta_n}{1-v_0(\alpha_n)} = \beta_n-\alpha_n$$

and

$$\|y_n-x_0\| \leq \|y_n-x_n\|+\|x_n-x_0\| \leq \beta_n-\alpha_n+\alpha_n = \beta_n < \alpha^*.$$

Subtracting the third and second substeps of the method gives

$$\begin{aligned}
z_n-y_n &= -\frac{1}{3}F'(x_n)^{-1}F(x_n)+(I-T_n)\,F'(x_n)^{-1}F(x_n) \\
&\quad -\frac{1}{2b}M_n^{-1}A_nT_nF'(x_n)^{-1}F(x_n).
\end{aligned} \tag{26}$$

Using (SLC_5), (SLC_6) and (23), we have

$$\begin{aligned}\|A_n\| &= \|I - F'(x_n)^{-1}F'(y_n)\| \\ &= \|F'(x_n)^{-1}(F'(x_n) - F'(x_0) + F'(x_0) - F'(y_n))\| \\ &\le \|F'(x_n)^{-1}L\|\Big(\|L^{-1}(F'(x_n) - F'(x_0))\| \\ &\quad +\|L^{-1}(F'(x_0) - F'(y_n))\|\Big) \\ &\le \frac{v_0(\|x_n - x_0\|) + v_0(\|y_n - x_0\|)}{1 - v_0(\|x_n - x_0\|)} \le \frac{v_0(\alpha_n) + v_0(\beta_n)}{1 - v_0(\alpha_0)}\end{aligned}$$

or

$$\begin{aligned}\|A_n\| &= \|I - F'(x_n)^{-1}F'(y_n)\| = \|F'(x_n)^{-1}(F'(x_n) - F'(y_n))\| \\ &\le \|F'(x_n)^{-1}L\|\|L^{-1}(F'(x_n) - F'(y_n))\| \le \frac{v(\|x_n - y_n\|)}{1 - v_0(\|x_n - x_0\|)} \\ &\le \frac{v_0(\alpha_n - \beta_n)}{1 - v_0(\alpha_0)}\end{aligned}$$

and

$$\|A_n\| \le \frac{\overline{v}(\|x_n - y_n\|)}{1 - v_0(\|x_n - x_0\|)} \le \frac{\bar{v}_n}{1 - v_0(\alpha)}. \tag{27}$$

Here $\overline{v}(\|x_n - y_n\|) = v_0(\|x_n - x_0\|) + v_0(\|y_n - x_0\|)$ or
$\overline{v}(\|x_n - y_n\|) = v(\|x_n - y_n\|)$.

From the following equality

$$I - T_n = -a_1 A_n - a_2 A_n^2,$$

and using the estimate (27), we obtain that

$$\|I - T_n\| \le |a_1|\frac{\overline{v}(\|x_n - x_0\|)}{1 - v_0(\|x_n - x_0\|)} + |a_2|\left(\frac{\overline{v}(\|x_n - x_0\|)}{1 - v_0(\|x_n - x_0\|)}\right)^2 \tag{28}$$

$$\le \overline{\lambda}_n \tag{29}$$

and

$$\|T_n\| \le 1 + \overline{\lambda}_n. \tag{30}$$

Moreover,

$$\Big\|\frac{c}{b}A_n\Big\| \le \Big|\frac{c}{b}\Big|\frac{\overline{v}(\|x_n - x_0\|)}{1 - v_0(\|x_n - x_0\|)} = \mu_n < 1,$$

$$\|M_n^{-1}\| = \|(I - (I - M_n))^{-1}\| = \left\|\left(I - \frac{c}{b}A_n\right)^{-1}\right\| \le \frac{1}{1 - \mu_n}. \tag{31}$$

Taking into account the equality (26), and estimates (23), (27), (28), (30), (31), we get in turn

$$\|z_n - y_n\| \leq \left(\frac{1}{3} + \overline{\lambda}_n + \frac{(1+\overline{\lambda}_n)\overline{v}_n}{2|b|(1-v_0(\alpha_n))(1-\mu_n)}\right)\frac{3}{2}(\beta_n - \alpha_n) = \gamma_n - \beta_n$$

and

$$\|z_n - x_0\| \leq \|z_n - y_n\| + \|y_n - x_0\| \leq \gamma_n - \beta_n + \beta_n = \gamma_n < \alpha^*.$$

Since,

$$\begin{aligned} F(z_n) &= F(z_n) - F(x_n) - \frac{3}{2}F'(x_n)(y_n - x_n) \\ &= F(z_n) - F(x_n) - F'(x_n)(z_n - x_n) + F'(x_n)(z_n - y_n) \\ &\quad -\frac{1}{2}F'(x_n)(y_n - x_n) \\ &= \left(\int_0^1 (F'(x_n + \theta(z_n - x_n)) - F'(x_n))d\theta\right)(z_n - x_n) \\ &\quad +F'(x_n)(z_n - y_n) - \frac{1}{2}F'(x_n)(y_n - x_n). \end{aligned}$$

Then, using (SLC_5) and (SLC_6), we have

$$\begin{aligned} \|L^{-1}F(z_n)\| &\leq \int_0^1 v((1-\theta)(\gamma_n - \alpha_n))d\theta(\gamma_n - \alpha_n) + (1 + v_0(\alpha_n)) \\ &\quad \times(\gamma_n - \beta_n) + \frac{1}{2}(1 + v_0(\alpha_n))(\beta_n - \alpha_n) = \xi_n. \qquad (32) \end{aligned}$$

We get by the last substep of the method, (23), (27) and (32)

$$\|x_{n+1} - z_n\| \leq \left(|p| + |q|\frac{\overline{v}_n}{1 - v_0(\alpha_n)}\right)\frac{\xi_n}{1 - v_0(\alpha_n)} = \alpha_{n+1} - \gamma_n$$

and

$$\|x_{n+1} - x_0\| \leq \|x_{n+1} - z_n\| + \|z_n - x_0\| \leq \alpha_{n+1} - \gamma_n + \gamma_n = \alpha_{n+1} < \alpha^*.$$

Thus, the iterates $y_n, z_n, x_{n+1} \in U(x_0, \alpha^*)$.

It follows from the obtained estimates that the sequence $\{x_n\}$ is complete in a Banach space B_1. Hence, it converges to some point $x^* \in U[x_0, \alpha^*]$. Furthermore, by letting $n \to \infty$ in (25) and using the continuity of F we conclude that $F(x^*) = 0$. Finally, by the estimate

$$\begin{aligned} \|x_{n+i} - x_n\| &\leq \|x_{n+i} - x_{n+i-1}\| + ... + \|x_{n+1} - x_n\| \\ &\leq \alpha_{n+i} - \alpha_{n+i-1} + ... + \alpha_{n+1} - \alpha_n = \alpha_{n+i} - \alpha_n, \end{aligned}$$

we show (22) if $i \to \infty$. □

Proposition 2. *Suppose:*
(1) There exists a solution $x_* \in U(x_0, \varrho_4)$ *of the equation (1) for some* $\varrho_4 > 0$.
(2) Condition (SLC_5) *holds on* $U(x_0, \varrho_4)$.
(3) There exist $\varrho_5 > \varrho_4$ *such that*

$$\int_0^1 v_0\Big((1-\theta)\varrho_5 + \theta\varrho_4\Big)d\theta < 1. \tag{33}$$

Set $\Omega_3 = U[x_0, \varrho_5] \cap \Omega$.
Then, the equation (1) has unique solution x^* *in the region* Ω_3.

Proof. Suppose that there exists $y^* \in \Omega_3$, $y^* \neq x^*$ and $F(y^*) = 0$. Define the linear operator G by $G = \int\limits_0^1 F'(y^* + \theta(x^* - y^*))d\theta$. Using (SLC_2) and (33), we obtain

$$\begin{aligned} \|L^{-1}(G - F'(x_0))\| &\leq \int_0^1 v_0(\|y^* + \theta(x^* - y^*) - x_0\|)d\theta \\ &\leq \int_0^1 v_0\Big((1-\theta)\|y^* - x_0\| + \theta\|x^* - x_0\|\Big)d\theta \\ &\leq \int_0^1 v_0\Big((1-\theta)\varrho_5 + \theta\varrho_4\Big)d\theta < 1. \end{aligned}$$

So, the linear operator G is invertible. Therefore, from the identity

$$y^* - x^* = G^{-1}(F(y^*) - F(x^*)) = G^{-1}(0 - 0) = 0,$$

we conclude that $y^* = x^*$. □

4 Numerical examples

In this section, we give the results of verifying the convergence conditions of the Theorems 1 and 2 for the considered method (2). The experiments were conducted in GNU Octave 7.3.0 software. To stop the iterative process the condition $\|x_{n+1} - x_n\| \leq \varepsilon$ was used. The calculations were performed with $\varepsilon = 10^{-8}$ and the norms $\|\cdot\|_\infty$ is used.

Example 1. *Consider the nonlinear integral equations [1]*

$$F(x)(s) = x(s) - 5s\int_0^1 tx(t)^3dt, \ \ s, t \in [0, 1].$$

Here, $B_1 = B_2 = C[0, 1]$, $\Omega \subseteq C[0, 1]$ *and the exact solution* $x^*(s) = 0$.

The derivative of operator F is defined by the following formula

$$F'(x)h(s) = h(s) - 15s\int_0^1 tx(t)^2h(t)dt,\ h \in C[0,1]$$

and we have that $F'(x^*) = I$ (I is an identity operator),

$$F'(x)h(s) - F'(y)h(s) = 15s\int_0^1 t(y(t)+x(t))(y(t)-x(t))h(t)dt.$$

Let's choose $L = F'(x^*)$ and $\Omega = U(x^*,1)$ for a local case. Then, we have $\omega_0(\|x-x^*\|) = 7.5\|x-x^*\|$, $\rho_0 = \dfrac{2}{15}$, $\Omega_0 = \left(x^*, \dfrac{2}{15}\right)$ and a function $\omega(\|x-y\|) = 2\|x-y\|$, $x,y \in \Omega_0$. The radii obtained for different values of parameters are given in Table 1.

Table 1: Radii for Example 1.

Parameters	Radius r^*
$b=3,\ c=p=q=1$	5.3551e-02
$b=3,\ c=p=1,\ q=3/2$	5.1918e-02
$b=1,\ c=p=1,\ q=3/2$	5.1268e-02
$b=-3,\ c=p=1,\ q=3/2$	4.8902e-02

Example 2. *Consider the system of m equations*

$$\sum_{j=1}^{m} x_j + e^{x_i} - 1 = 0, \quad i = 1,\ldots,m.$$

Here, $B_1 = B_2 = \mathbb{R}^m$, $\Omega \subseteq \mathbb{R}^m$ and the exact solution $x^* = (0,\ldots,0)^T$.

For this problem elements of the Jacobian matrix are calculated by formulas

$$F'(x)_{ij} = \begin{cases} e^{x_i}+1, & i=j, \\ 1, & i \neq j, \end{cases} \quad \text{and} \quad F'(x^*)_{ij} = \begin{cases} 2, & i=j, \\ 1, & i \neq j. \end{cases}$$

Let's choose $L = F'(x^*)$ and $\Omega = U(x^*,1)$ for a local case. Then, we have

$$L^{-1}(F'(x)-L) = L^{-1}diag\left\{e^{x_1}-1,\ldots,e^{x_m}-1\right\},$$
$$L^{-1}(F'(x)-F'(y)) = L^{-1}diag\left\{e^{x_1}-e^{y_1},\ldots,e^{x_m}-e^{y_m}\right\}.$$

Therefore, functions ω_0 and ω have the following form

$$\omega_0(\|x - x^*\|) = \sigma(e - 1)\|x - x^*\|$$

and

$$\omega(\|x - y\|) = \sigma e^{\min\{1,\varrho_0\}}\|x - y\|,$$

where $\sigma = (F'(x^*))^{-1}_{11}$.

Local case. Let $m = 5$, $b = 3$, $c = p = q = 1$. Then, $\Omega_0 \approx (x^*, 0.6984)$,

$$r^* \approx \min\{0.2658,\ 0.1393,\ 0.1267\} \approx 0.1267$$

and $U[x^*, r^*] \approx [-0.1267,\ 0.1267]^m \subset \Omega_0$. The method (2) converges to the exact solution at two iterations for starting approximation

$$x_0 = (0.12, \dots, 0.12)^T$$

and error estimates (7)-(9) hold for each $n \geq 0$ (see Table 2).

Table 2: Error estimates at iteration for Example 2.

n	$\|x_n - x^*\|$	g_3^*	$\|y_n - x^*\|$	g_1^*	$\|z_n - x^*\|$	g_2^*
0	1.2000e-01	–	4.0849e-02	6.8860e-02	2.6447e-07	9.7156e-02
1	1.8766e-09	1.2000e-01	6.2554e-10	1.0769e-09	7.3635e-17	1.5194e-09
2	7.0097e-17	1.8766e-09				

Similar results are obtained for $b = 3$, $c = p = 1$ and $q = \dfrac{3}{2}$: $\Omega_0 \approx (x^*, 0.6984)$, $r^* \approx \min\{0.2658,\ 0.1393,\ 0.1211\} \approx 0.1211$ and $U[x^*, r^*] \approx [-0.1211,\ 0.1211]^m \subset \Omega_0$. Error estimates at iteration are given in Table 3.

Table 3: Error estimates at iteration for Example 2.

n	$\|x_n - x^*\|$	g_3^*	$\|y_n - x^*\|$	g_1^*	$\|z_n - x^*\|$	g_2^*
0	1.2000e-01	–	4.0849e-02	6.7307e-02	2.6447e-07	1.2775e-01
1	6.3512e-11	1.2000e-01	2.1171e-11	3.5623e-11	7.6134e-17	4.6702e-11
2	7.0513e-17	6.3512e-11	0	0	0	0

In Tables 2 and 3 we use following notations: $g_1^* = g_1(r^*)\|x_n - x^*\|$, $g_2^* = g_2(r^*)\|x_n - x^*\|$ and $g_3^* = g_3(r^*)\|x_{n-1} - x^*\|$.

5 Conclusions

The paper studies the convergence of a three-step iterative method containing inverse linear operators under the weak conditions. Moreover, these conditions contain only operators that are in the method. A local and a semilocal convergence analysis of this method under generalized Lipschitz conditions for only the first-order derivatives is presented. The regions of convergence and uniqueness of the solution are established. The results of a numerical experiment are also presented. The new technique is an alternative to expensive Taylor series usually employed to study the convergence of iterative methods. The same technique is applicable to extend other methods [4–14].

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