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EUCLIDEAN-LAGRANGE AND CANTOR-LAGRANGE QUARTIC POLYNOMIALS AND ASSOCIATED CUBIC CURVES*

Mircea Crasmareanu†

Dedicated to Professor Gheorghe Moroşanu on the occasion of his 75th anniversary

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Abstract

The purpose of this paper is to introduce and examine two classes of quartic real polynomials P having the same Euclidean norm as their Lagrange resolvent, respectively, having the square of the Euclidean norm equal to the height of the Lagrange resolvent. The reduced form of the polynomial P is provided, which eliminates the cubic term. Finding such polynomials with integer coefficients is always of interest. The Lagrange resolvent associates a cubic curve with each of these polynomials as a cubic polynomial. It highlights the situations in which these cubic curves are elliptic curves.

Keywords: quartic real polynomial, Lagrange resolvent, Euclidean-Lagrange polynomial, cubic curve, Cantor-Lagrange polynomial.

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†mcrasm@uaic.ro, Faculty of Mathematics, *Alexandru Ioan Cuza* University, Iasi, 700506, Romania

1 Introduction

Five techniques to solve a quartic equation are identified by the names Descartes-Euler-Cardano, Ferrari-Lagrange, Neumark, Christianson-Brown, and Yacoub-Fraidenraich-Brown. In this study, we concentrate on the latter, which relates a specific monic (and reduced) quartic polynomial P to its Lagrange resolvent $LR(P)$ represented as a monic cubic polynomial

This note begins by treating both polynomials P and $LR(P)$ as vectors in $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively, and consequently connecting their Euclidean norms $\|\cdot\|$. It pertains to a specific category of P, specifically those for which $\|P_d\| = \|LR(P_d)\|$, and this collection is the focus of our investigation referred to as *Euclidean-Lagrange polynomials*. Here P_d represents the depressed version of the original quartic polynomial P.

The key condition for the isometry of the Lagrange map $P \to LR(P)$ is formulated as a quadratic equation regarding the coefficients (p, q, r) of P. Since this equation resembles a Pythagorean theorem, we solve it entirely using two real parameters $\alpha \neq 0$, $\beta \neq 0$. The specific case $\alpha = \beta$ is fully addressed. Another Lagrange-Euclidean polynomial is obtained as fixed point of the Lagrange map above expressed only in terms of coefficients.

We also link a cubic curve $LC(P) : y^2 = LR(P)(x)$ and concentrate on situations where this curve is an elliptic curve. Indeed, we found all of them on the renowned database https://www.lmfdb.org/ where many specifics can be accessed.

In the final section, we present a new category of polynomials P, specifically those satisfying $\|P_d\|^2 = h(LR(P_d))$, where h denotes the Cantor height of the cubic polynomial $LR(P_d)$. Three instances of these polynomials, referred to as *Cantor-Lagrange*, are examined.

2 Euclidean-Lagrange quartic polynomials

Fix a natural number $n \in \mathbb{N}^*$. The setting of this work is provided by the n-dimensional real linear space of monic polynomials of grade n:

$$\mathbb{R}_n^{monic}[x] := \{P(x) = x^n + a_1 x^{n-1} + ... + a_n; a_1, ..., a_n \in \mathbb{R}\}.$$

It is an Euclidean space with respect to the usual scalar product of $\mathbb{R}^n$ and hence the square of the induced norm is the sum of the square of the coefficients:

$$\|P\|^2 := a_1^2 + ... + a_n^2. \tag{1}$$

Remark 1. *It is well-known that in order to prove that the set of real algebraic numbers is countable Cantor defines the height of above P as the positive (real) number:*

$$h(P) := n + |a_0 = 1| + |a_1| + ... + |a_n|.$$

It follows the inequality:

$$(h(P) - n - 1)^2 \leq n\|P\|^2$$

with equality only for the polynomial $\Phi_n(x) = \frac{x^{n+1}-1}{x-1} = x^n + x^{n-1} + ... + x + 1.$

Our study focuses on a fixed quartic polynomial $P \in \mathbb{R}_4^{monic}[x]$:

$$P(x) := x^4 + Ax^3 + Bx^2 + Cx + D$$

which with a Cardano-type transformation $x = y - \frac{A}{4}$ have the *reduced* form (here the subscript d means *depressed*):

$$P_d(y) := y^4 + py^2 + qy + r. \tag{2}$$

The *Lagrange resolvent* of P is a cubic polynomial $LR(P)$ constructed with the coefficients of P_d; for more details see [1, p. 322] while for an universal method to solve the quartic equation see [5]:

$$LR(P)(u) := u^3 - pu^2 - 4ru + (4pr - q^2) = 0. \tag{3}$$

Inspired by [2] we introduce:

Definition 1. *i) P (or equivalently P_d) is an Euclidean-Lagrange polynomial if it preserves the Euclidean norm with respect to Lagrange transformation DR:*

$$\|P_d\| = \|LR(P)\|. \tag{4}$$

Hence, the restriction of LR to the set of Euclidean-Lagrange polynomials is an isometry.
ii) The Lagrange-cubic curve associated to (arbitrary) P_d is:

$$LC(P_d) : y^2 = LR(P)(x). \tag{5}$$

We have immediately:

Proposition 1. *i) The only Euclidean-Lagrange polynomials with* $r = 0$ *are* $P_d^0(y) = y^4$ *and the two 1-parameter families* $P_d^{\pm}(y) = y^4 + py^2 \pm y$, $p \in \mathbb{R}$.
ii) Let P *with* $r \neq 0$. *Then* P *is an Euclidean-Lagrange polynomial if and only if there exists the non-zero real numbers* $\alpha \leq \beta$ *such that:*

$$q = \alpha^2 + \beta^2 > 0, \quad r = \frac{2\alpha\beta}{\sqrt{15}}, \quad p = \frac{\sqrt{15}}{8\alpha\beta}[(\alpha^2 + \beta^2)^2 + \beta^2 - \alpha^2]. \tag{6}$$

Proof. The equality (4) reads as a Pythagorean relation:

$$q^2 = 15r^2 + (4pr - q^2)^2 \tag{7}$$

hence, we use the well-known parametrization of Pythagorean triples ([4]); there exists $\alpha \leq \beta$ such that:

$$\sqrt{15}r = 2\alpha\beta, \quad 4pr - q^2 = \beta^2 - \alpha^2, \quad q = \alpha^2 + \beta^2 \tag{8}$$

and the claimed relations follow immediately. □

Example 1. *The Lagrange-cubic curves associated to the polynomials from i) are:*

$$LC(P^0) : y^2 = x^3, \quad LC(P^{\pm}) : y^2 = x^3 - 1 = (x - 1)(x^2 + x + 1) \tag{9}$$

which are the semicubical parabola and the elliptic curve given by https://www.lmfdb.org/EllipticCurve/Q/144/a/3. This elliptic curve has complex multiplication.

Example 2. *We choose* $\alpha = \beta \neq 0$ *in (8) obtaining:*

$$P_d^{\alpha}(y) = y^4 + \alpha^2\left(\frac{\sqrt{15}}{2}y^2 + 2y + \frac{2}{\sqrt{15}}\right) = y^4 + \alpha^2\left(\sqrt{\frac{\sqrt{15}}{2}}y + \sqrt{\frac{2}{\sqrt{15}}}\right)^2, \tag{10}$$

$$LC(P^{\alpha}) : y^2 = x^3 - \alpha^2\left(\frac{\sqrt{15}}{2}x^2 - \frac{8}{\sqrt{15}}x\right), \|P_d^{\alpha}\|^2 = \|LR(P^{\alpha})\|^2 = \frac{481}{60}\alpha^2. \tag{11}$$

In order to obtain a cubic curve with integral coefficients in the right-hand-side we choose $\alpha^2 = 2\sqrt{15} = \sqrt{\frac{5!}{2}}$ *and therefore:*

$$LC\left(P^{\pm\sqrt{2\sqrt{15}}}\right) : y^2 = x^3 - 15x^2 - 16x = x(x + 1)(x - 6). \tag{12}$$

This elliptic curve is https://www.lmfdb.org/EllipticCurve/Q/27864/c/1 and its Weierstrass expression is:

$$LC\left(P^{\pm\sqrt{2\sqrt{15}}}\right): y^2 = X^3 - 75X - 266. \tag{13}$$

The cubic polynomial $LC\left(P^{\pm\sqrt{2\sqrt{15}}}\right)$ *having real distinct roots is a strictly hyperbolic polynomial according to a well-known terminology, see [2] or [3].*

3 The fixed points of the Lagrange map

A second way to find Euclidean-Lagrange polynomials is to find the fixed points of the *Lagrange map*:

$$LR: (p, q, r) \in \mathbb{R}^3 \to (-p, -4r, 4pr - q^2) \in \mathbb{R}^3. \tag{14}$$

Also, the Lagrange map is a surjective map on its image $\{(A, B, C) \in \mathbb{R}^3; AB - C \geq 0\}$ with:

$$LR^{-1}(A, B, C) = \left(-A, \pm\sqrt{AB - C}, -\frac{B}{4}\right). \tag{15}$$

A direct computation yields:

Proposition 2. *The Lagrange map* (1) *has only two fixed points:* $(0, 0, 0)$ *and* $\left(0, \frac{1}{4}, -\frac{1}{16}\right)$.

This result gives a new Euclidean-Lagrange polynomial:

$$P_d^{fixed}(y) = y^4 + \frac{y}{4} - \frac{1}{16}, \quad \|P_d^{fixed}\| = \frac{\sqrt{17}}{16}, \quad LR(P^{fixed})(u) = u^3 + \frac{u}{4} - \frac{1}{16} \tag{16}$$

and the only real (hence strictly positive) root of $LR(P^{fixed})$ is:

$$u = \frac{\sqrt[3]{9 + \sqrt{129}}}{2 \times 6^{2/3}} - \frac{1}{\sqrt[3]{6(9 + \sqrt{129})}} \simeq 0.21193. \tag{17}$$

The Euclidean-Lagrange polynomials P_d^{fixed} has only two real roots, one strictly negative and one strictly positive and the associated cubic curve is:

$$LC(P^{fixed}): y^2 = x^3 + \frac{x}{4} - \frac{1}{16}. \tag{18}$$

Recall that for a real number α the α-it homothetical transformation of the Weierstrass cubical curve $\Gamma : y^2 = x^3 + Px + Q$ is the cubical curve:

$$H_\alpha(\Gamma) : y^2 = x^3 + (\alpha^2 P)x + (\alpha^3 Q). \tag{19}$$

Hence, choosing $\alpha = 4$ for (5) we obtain the elliptical curve:

$$H_4(LC(P^{fixed})) : y^2 = x^3 + 4x - 4 \tag{20}$$

which is exactly the curve https://www.lmfdb.org/EllipticCurve/Q/688/a/1.

It is worth to point out that, excepting $P_d(y) = y^4$, all previous obtained polynomials have distinct roots. Indeed $P_d(y) = y^4$ is the only Euclidean-Lagrangian polynomial with multiple roots since the equality $P_d(y) = (y^2 + ay + b)^2$ means: $a = 0 = q$, $p = 2b$, $r = b^2$. From the characterization (7) we have also $p = r = 0$.

4 Cantor-Lagrange quartic polynomials

We introduce now a new class of quartic polynomial in the same setting above but now considering the Cantor height.

Definition 2. *i) P (or equivalently P_d) is a Cantor-Lagrange polynomial if:*

$$\|P_d\|^2 = h(LR(P)) \tag{21}$$

which means:

$$p^2 + q^2 + r^2 = 4 + |p| + 4|r| + |4pr - q^2|. \tag{22}$$

We analyze some interesting cases according to the pair (p, r) of coefficients.

Case I) Suppose that $r = 0$; it follows the characterization $p^2 = 4 + |p|$ and we have two solutions $p_\pm = \frac{\sqrt{17}\pm 1}{2}$. Hence we have two 1-parameters families $P_d^{(I,\pm,q)}(y) = y^4 + p_\pm y^2 + qy$.

Case II) $p = 0$ implies the characterization $r^2 = 4+4|r|$ with the solutions $r_\pm = \pm 2(\sqrt{2} + 1)$. In conclusion, we have again the 1-parameter families of Cantor-Lagrange polynomials $P_d^{(II,\pm,q)}(y) = y^4 + qy + r_\pm$.

Case III) $4pr < q^2$, $p > 0$ and $r > 0$ give the following quadratic equation, representing a hyperbola in the plane (p, r):

$$H : p^2 + 4pr + r^2 - p - 4r - 4 = 0. \tag{23}$$

The eccentricity of H is $e = 2$ and H contains an infinite sequence of lattice points e.g. $(p, r) = (1, 2)$. So, an example of this case is:

$$\begin{cases} P_d(y) = y^4 + y^2 + 3y + 2, \quad LR(P)(u) = u^3 - u^2 - 8u - 1, \\ \|P_d\|^2 = h(LR(P)) = 14. \end{cases} \tag{24}$$

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