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CLOSENESS LAPLACIAN AND CLOSENESS SIGNLESS LAPLACIAN ENERGIES OF NON-COMMUTING GRAPH FOR DIHEDRAL GROUPS*

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Abstract

This research investigates the relationship between algebra and graph theory, specifically how algebra facilitates graph theory. Associating matrices with graphs introduces the concept of graph energies. A new energy formula of non-commuting graphs for dihedral groups

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using closeness Laplacian and closeness signless Laplacian matrices is investigated in this paper. It is found that both energies are always equivalent and are categorized as hyperenergetic.

Keywords: energy of a graph, non-commuting graph, dihedral group, closeness Laplacian matrix, closeness signless Laplacian matrix.
MSC: 05C25, 15A18.

1 Introduction

Algebraic graph theory is currently a prominent topic of research. This study investigates the relationship between algebra and graph theory, focusing on how algebra facilitates graph studies. This field of study is significant, and it contributes to various other areas. Chemical graph theory is a branch of mathematical chemistry that applies graph theory to the mathematical modeling of chemical compounds mentioned by [16]. It also includes a discussion of graph energies presented by [5] by considering a chemical molecule as a graph and estimating the π-electron energy. The eigenvalues of the adjacency matrix denote the energy level of the electron in the molecule.

Neumann [8] introduced the concept of non-commuting graphs. However, [1] conducted extensive research on the features of non-commuting graphs. Let G be a finite group and $Z(G)$ be its center. The non-commuting graph of G, represented by Γ_G, contains G excluding the center of G as its vertex set, with two different vertices v_p and v_q connected by an edge whenever $v_p v_q \neq v_q v_p$.

The vertex set of this research is dihedral groups. For $n \geq 3$, the non-abelian dihedral group of order $2n$ is defined as reflection and rotation motions that return a regular n-gon to its initial state, with composition as the operation, represented by D_{2n}. The n rotations are a^i, and the reflections are $a^i b$, with $1 \leq i \leq n$. Thus, D_{2n} can be expressed as $\langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle$ [3]. For further study, we denote the non-commuting graph for D_{2n} as $\Gamma_{D_{2n}}$. Based on the center of D_{2n}, it is clear that $\Gamma_{D_{2n}}$ has $2n-1$ and $2n-2$ vertices for odd and even n, respectively.

The discussion on $\Gamma_{D_{2n}}$ with several matrices has been done involving closeness matrix [12], Seidel Laplacian and Seidel signless Laplacian matrices [13], Sombor matrix [14], and Wiener Hosoya matrix [15]. In 2022, Zheng and Zhou introduced the definition of closeness Laplacian and closeness signless Laplacian matrices of a graph [17] as the extension of the closeness matrix of a graph [18]. On the other hand, distance-based matrices, especially the distance Laplacian, have gained significant attention for their ability to capture both structural and spectral properties of graphs [11]. Notably, their characteristic polynomials and spectra have been analyzed for algebraically defined graphs, including power graphs [10]. These findings motivated us to construct the matrices for $\Gamma_{D_{2n}}$ and to formulate the spectrum and energy of $\Gamma_{D_{2n}}$.

2 Preliminaries

This section focuses on the basic definitions and theorems from the previous literature. Let d_{pq} be the distance between vertex p and q. We denote $c(p) = \sum_{q \in V(\Gamma_{D_{2n}}) \setminus \{p\}} 2^{-d_{pq}}$, for adjacent vertices p and q. Hence, we have two definitions below.

Definition 1. *[17] The closeness Laplacian matrix of $\Gamma_{D_{2n}}$ is given by $CL(\Gamma_{D_{2n}}) = [l_{pq}]$ whose (p,q)-entry is*

$$l_{pq} = \begin{cases} -2^{-d_{pq}}, & \text{if } p \neq q \\ c(p), & \text{if } p = q. \end{cases}$$

Definition 2. *[17] The closeness signless Laplacian matrix of $\Gamma_{D_{2n}}$ is given by $CSL(\Gamma_{D_{2n}}) = [s_{pq}]$ whose (p,q)-entry is*

$$s_{pq} = \begin{cases} 2^{-d_{pq}}, & \text{if } p \neq q \\ c(p), & \text{if } p = q. \end{cases}$$

Definition 1 and Definition 2 require the distance between any two vertices in a graph. Therefore, we write the following theorem to construct the matrices in the next section.

Theorem 1. *[12] The distance between two distinct vertices p, q in $\Gamma_{D_{2n}}$ is*

1. *for odd n,* $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 1, & \text{otherwise,} \end{cases}$ *, and*

2. *for even n,* $d_{pq} = \begin{cases} 2, & \text{if } v_p, v_q \in G_1 \\ 2, & v_p \in G_2, v_q \in \left\{a^{\frac{n}{2}+i}b\right\} \\ 1, & \text{otherwise,} \end{cases}$

where G_1 is the set of rotations (excluding identity), and G_2 is the set of reflections.

The spectrum of $\Gamma_{D_{2n}}$ is denoted by $Spec_{CL}(\Gamma_{D_{2n}})$ or $Spec_{CSL}(\Gamma_{D_{2n}})$. It is defined as $\left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_m^{k_m}\right\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalues of $CL(\Gamma_{D_{2n}})$ or $CSL(\Gamma_{D_{2n}})$, and $k_1, k_2, \ldots, k_m$ are their respective multiplicities. Moreover, the spectral radius of $\Gamma_{D_{2n}}$ as $max\,\{\lambda|\lambda \in Spec(\Gamma_{D_{2n}})\}$.

Furthermore, the energy of $\Gamma_{D_{2n}}$ is defined as $\sum_{i=1}^{m} |\lambda_i|$ [5]. $\Gamma_{D_{2}n}$ can be categorized as hyperenergetic if the energy is greater than $4(n-1)$ for odd n, or it is greater than $4(n-1)-2$ for even n [7].

3 Main results

3.1 Characteristic polynomial

Let J_n be the matrix of size $n \times n$ with all entries are 1, and I_n be an identity matrix of size $n \times n$. In this subsection, we prove two beneficial theorems to formulate the characteristic polynomial of matrices. In the process of proof, we use row and column operations with the following notation:

1. R_i: the $i-$th row;
2. R_i': the new $i-$th row obtained from a row operation;
3. C_i: the $i-$th column;
4. C_i': the new $i-$th column obtained from a column operation.

Theorem 2. *Suppose $s, t, u,$ and v are real numbers. The characteristic polynomial of a $(2n-1) \times (2n-1)$ matrix,*

$$M = \begin{pmatrix} (s+v)I_{n-1} - vJ_{n-1} & -tJ_{(n-1)\times n} \\ -tJ_{n\times(n-1)} & (u+t)I_n - tJ_n \end{pmatrix}$$

can be simplified into an expression as

$$P_M(\lambda) = (\lambda - s - v)^{n-2}\Big((\lambda - u + t(n-1))(\lambda - s + (n-2)v)$$

$$+(1-n)nt^2\Big)(\lambda - t - u)^{n-1}.$$

Proof. For real numbers s, t, u and v, the characteristic polynomial of M is

$$P_M(\lambda) = \begin{vmatrix} (\lambda - s - v)I_{n-1} + vJ_{n-1} & tJ_{(n-1)\times n} \\ tJ_{n\times(n-1)} & (\lambda - u - t)I_n + tJ_n) \end{vmatrix}.$$

Step 1: Replace R_{1+i} by $R'_{1+i} = R_{1+i} - R_1$, for every $1 \leq i \leq n-2$, and R_{n+i} by $R'_{n+i} = R_{n+i} - R_n$, for every $1 \leq i \leq n-1$ Then we have $P_M(\lambda)$ as follows:

$$P_M(\lambda)$$

$$= \begin{vmatrix} \lambda - s & vJ_{1\times(n-2)} & t & tJ_{1\times(n-1)} \\ (v+s-\lambda)J_{(n-2)\times 1} & (\lambda-s-v)I_{n-2} & 0_{(n-2)\times 1} & 0_{n-1} \\ t & tJ_{1\times(n-2)} & \lambda - u & tJ_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-2)} & (t+u-\lambda)J_{(n-1)\times 1} & (\lambda-u-t)I_{n-1} \end{vmatrix}.$$

Step 2: Replace C_1 with $C'_1 = C_1 + C_2 + \ldots + C_{n-1}$ and replace $C'_n = C_n + C_{n+1} + \ldots + C_{2n-1}$, then we have

$$P_M(\lambda)$$

$$= \begin{vmatrix} \lambda - s + (n-2)v & vJ_{1\times(n-2)} & tn & tJ_{1\times(n-1)} \\ 0_{(n-2)\times 1} & (\lambda-s-v)I_{n-2} & 0_{(n-2)\times 1} & 0_{n-1} \\ t(n-1) & tJ_{1\times(n-2)} & \lambda - u + t(n-1) & tJ_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-2)} & 0_{(n-1)\times 1} & (\lambda-u-t)I_{n-1} \end{vmatrix}.$$

Step 3: Replace R_n by $R'_n = R_n + \frac{t(1-n)}{\lambda-s+(n-2)v}R_1$ and following by

$$R'_n = R_n + \frac{(s-\lambda)+tv}{(\lambda-s+(n-2)v)(\lambda-s-v)}R_2 + \frac{(s-\lambda)+tv}{(\lambda-s+(n-2)v)(\lambda-s-v)}R_3$$

$$+ \ldots + \frac{(s-\lambda)+tv}{(\lambda-s+(n-2)v)(\lambda-s-v)}R_{n-1},$$

then we can write $P_M(\lambda)$ as

$$\begin{vmatrix} \lambda - s + (n-2)v & vJ_{1\times(n-2)} & tn & tJ_{1\times(n-1)} \\ 0_{(n-2)\times 1} & (\lambda-s-v)I_{n-2} & 0_{(n-2)\times 1} & 0_{n-1} \\ 0 & 0_{1\times(n-2)} & \frac{(\lambda-u+t(n-1))(\lambda-s+(n-2)v)+(1-n)nt^2}{\lambda-s+(n-2)v} & \frac{(1-n)t^2+(\lambda-s+(n-2)v)t}{\lambda-s+(n-2)v}J_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times(n-2)} & 0_{(n-1)\times 1} & (\lambda-u-t)I_{n-1} \end{vmatrix}$$

It is a diagonal matrix which implies that

$$P_M(\lambda) = (\lambda - s - v)^{n-2} \Big((\lambda - u + t(n-1))(\lambda - s + (n-2)v)$$

$$+ (1-n)nt^2 \Big) (\lambda - t - u)^{n-1}.$$

□

Theorem 3. *Suppose r, s, t and u are real numbers. The characteristic polynomial of a $(2n-2) \times (2n-2)$ matrix*

$$M = \begin{bmatrix} (r+u)\, I_{n-2} - uJ_{n-2} & -tJ_{(n-2)\times\frac{n}{2}} & -tJ_{(n-2)\times\frac{n}{2}} \\ -tJ_{\frac{n}{2}\times(n-2)} & (s+t)\, I_{\frac{n}{2}} - tJ_{\frac{n}{2}} & (-u+t)I_{\frac{n}{2}} - tJ_{\frac{n}{2}} \\ -tJ_{\frac{n}{2}\times(n-2)} & (-u+t)I_{\frac{n}{2}} - tJ_{\frac{n}{2}} & (s+t)\, I_{\frac{n}{2}} - tJ_{\frac{n}{2}} \end{bmatrix}$$

is

$$\begin{aligned} P_M(\lambda) =& (\lambda - r - u)^{n-3}(\lambda + u - s - 2t)^{\frac{n}{2}-1}(\lambda - s - u)^{\frac{n}{2}} \\ & \Big(\lambda^2 + ((n-2)(u+t) - r - s)\lambda \\ & + (u(n-3) - r)(u - s + (n-2)t) - n(n-2)t^2 \Big). \end{aligned}$$

Proof. The determinant below is the characteristic polynomial of M

$$P_M(\lambda)$$

$$= \begin{vmatrix} (\lambda - r - u)\, I_{n-2} + uJ_{n-2} & tJ_{(n-2)\times\frac{n}{2}} & tJ_{(n-2)\times\frac{n}{2}} \\ tJ_{\frac{n}{2}\times(n-2)} & (\lambda - s - t)\, I_{\frac{n}{2}} + tJ_{\frac{n}{2}} & (u-t)I_{\frac{n}{2}} + tJ_{\frac{n}{2}} \\ tJ_{\frac{n}{2}\times(n-2)} & (u-t)I_{\frac{n}{2}} + tJ_{\frac{n}{2}} & (\lambda - s - t)\, I_{\frac{n}{2}} + tJ_{\frac{n}{2}} \end{vmatrix}.$$

The first stage we operate row operations by replacing $R_{n-2+\frac{n}{2}+i}$ with $R'_{n-2+\frac{n}{2}+i} = R_{n-2+\frac{n}{2}+i} - R_{n-2+i}$, for every $1 \le i \le \frac{n}{2}$, and we have

$$P_M(\lambda)$$

$$= \begin{vmatrix} (\lambda - r - u)\, I_{n-2} + uJ_{n-2} & tJ_{(n-2)\times\frac{n}{2}} & tJ_{(n-2)\times\frac{n}{2}} \\ tJ_{\frac{n}{2}\times(n-2)} & (\lambda - s - t)\, I_{\frac{n}{2}} + tJ_{\frac{n}{2}} & (u-t)I_{\frac{n}{2}} + tJ_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & (-\lambda + s + u)I_{\frac{n}{2}} & (\lambda - s - u)\, I_{\frac{n}{2}} \end{vmatrix}.$$

Then, we consider column operations $C'_{n-2+i} = C_{n-2+i} + C_{n-2+\frac{n}{2}+i}$, for every $1 \le i \le \frac{n}{2} - 1$,

$$P_M(\lambda)$$

$$= \begin{vmatrix} (\lambda - r - u)\, I_{n-2} + uJ_{n-2} & 2tJ_{(n-2)\times\frac{n}{2}} & tJ_{(n-2)\times\frac{n}{2}} \\ tJ_{\frac{n}{2}\times(n-2)} & (\lambda + u - s - 2t)\, I_{\frac{n}{2}} + 2tJ_{\frac{n}{2}} & (u-t)I_{\frac{n}{2}} + tJ_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 0_{\frac{n}{2}} & (\lambda - s - u)\, I_{\frac{n}{2}} \end{vmatrix}.$$

Following by $R'_{n-1+i} = R_{n-1+i} - R_{n-1}$, for every $1 \le i \le \frac{n}{2} - 1$, and replacing C_{n-1} with $C'_{n-1} = C_{n-1} + C_n + C_{n+1} + \ldots + C_{n-2+\frac{n}{2}}$, consequently, $P_M(\lambda)$ can be written as

$$\left| \begin{array}{cccc} \lambda - r & uJ_{1\times(n-3)} & nt & 2tJ_{1\times\left(\frac{n}{2}-1\right)} \\ uJ_{(n-3)\times 1} & (\lambda - r - u)\, I_{n-3} + uJ_{n-3} & ntJ_{(n-3)\times 1} & 2tJ_{(n-3)\times\left(\frac{n}{2}-1\right)} \\ t & tJ_{1\times(n-3)} & \lambda + u - s - t(n-2) & 2tJ_{1\times\left(\frac{n}{2}-1\right)} \\ 0_{\left(\frac{n}{2}-1\right)\times 1} & 0_{\left(\frac{n}{2}-1\right)\times(n-3)} & 0_{\left(\frac{n}{2}-1\right)\times 1} & (\lambda + u - s - 2t)I_{\frac{n}{2}-1} \\ 0 & 0_{1\times(n-3)} & 0 & 0_{1\times\left(\frac{n}{2}-1\right)} \\ 0_{\left(\frac{n}{2}-1\right)\times 1} & 0_{\left(\frac{n}{2}-1\right)\times(n-3)} & 0_{\left(\frac{n}{2}-1\right)\times 1} & 0_{\frac{n}{2}-1} \end{array} \right.$$

$$\left. \begin{array}{cc} t & tJ_{1\times\left(\frac{n}{2}-1\right)} \\ tJ_{(n-3)\times 1} & tJ_{(n-3)\times\left(\frac{n}{2}-1\right)} \\ u & tJ_{1\times\left(\frac{n}{2}-1\right)} \\ (t-u)J_{\left(\frac{n}{2}-1\right)\times 1} & (u-t)I_{\frac{n}{2}-1} \\ \lambda - s - u & 0_{1\times\left(\frac{n}{2}-1\right)} \\ 0_{\left(\frac{n}{2}-1\right)\times 1} & (\lambda - s - u)I_{\frac{n}{2}-1} \end{array} \right|$$

The next step is operating $C'_i = C_i - C_{n-2}$, for $1 \le i \le n-3$ and following by $R'_{n-2} = R_{n-2} + R_1 + R_2 + \ldots + R_{n-3}$. Hence, we derive $P_M(\lambda)$ as follows:

$$\left| \begin{array}{cccc} (\lambda - r - u)I_{n-3} & uJ_{(n-3)\times 1} & ntJ_{(n-3)\times 1} & 2tJ_{(n-3)\times\left(\frac{n}{2}-1\right)} \\ 0_{1\times(n-3)} & \lambda - r + (n-3)u & n(n-2)t & 2(n-2)tJ_{1\times\left(\frac{n}{2}-1\right)} \\ 0_{1\times(n-3)} & t & \lambda + u - s - t(n-2) & 2tJ_{1\times\left(\frac{n}{2}-1\right)} \\ 0_{\frac{n}{2}-1} & 0_{\left(\frac{n}{2}-1\right)\times 1} & 0_{\left(\frac{n}{2}-1\right)\times 1} & (\lambda + u - s - 2t)I_{\frac{n}{2}-1} \\ 0_{1\times(n-3)} & 0 & 0 & 0_{1\times\left(\frac{n}{2}-1\right)} \\ 0_{\frac{n}{2}-1} & 0_{\left(\frac{n}{2}-1\right)\times 1} & 0_{\left(\frac{n}{2}-1\right)\times 1} & 0_{\frac{n}{2}-1} \end{array} \right.$$

$$\left. \begin{array}{cc} tJ_{(n-3)\times\left(\frac{n}{2}-1\right)} & tJ_{(n-3)\times\left(\frac{n}{2}-1\right)} \\ (n-2)t & (n-2)tJ_{1\times\left(\frac{n}{2}-1\right)} \\ u & tJ_{1\times\left(\frac{n}{2}-1\right)} \\ (t-u)J_{\left(\frac{n}{2}-1\right)\times 1} & (u-t)I_{\frac{n}{2}-1} \\ \lambda - s - u & 0_{1\times\left(\frac{n}{2}-1\right)} \\ 0_{\left(\frac{n}{2}-1\right)\times 1} & (\lambda - s - u)I_{\frac{n}{2}-1} \end{array} \right|$$

Then we can simplify $P_M(\lambda)$ as

$$P_M(\lambda) = (\lambda - r - u)^{n-3}(\lambda + u - s - 2t)^{\frac{n}{2}-1}(\lambda - s - u)^{\frac{n}{2}} \Big(\lambda^2 + ((n-2)(u+t) - r - s)\lambda + (u(n-3) - r)(u - s + (n-2)t) - n(n-2)t^2\Big).$$

□

3.2 Closeness Laplacian energy

We begin this subsection by formulating the characteristic polynomial of $\Gamma_{D_{2n}}$ associated with the closeness Laplacian matrix, $CL(\Gamma_{D_{2n}})$.

Theorem 4. *The characteristic polynomial of* $\Gamma_{D_{2n}}$ *associated with* $CL(\Gamma_{D_{2n}})$ *is*

1. *for odd n:* $P_{CL(\Gamma_{D_{2n}})}(\lambda) = \lambda\left(\lambda - \frac{1}{4}(3n-1)\right)^{n-2}\left(\lambda - n + \frac{1}{2}\right)^n,$

2. *for even n:*

$$P_{CL(\Gamma_{D_{2n}})}(\lambda) = \left(\lambda - \frac{3}{4}n + \frac{1}{2}\right)^{n-3}(\lambda - n)^{\frac{n}{2}-1}\left(\lambda - n + \frac{1}{2}\right)^{\frac{n}{2}}\left(\lambda^2 - n\lambda - \frac{1}{2}n\right).$$

Proof. 1. By Theorem 1 for odd n, we have $d_{pq} = 2$, if $v_p, v_q \in G_1$, and zero otherwise. By excluding e from the set of vertex, hence $\Gamma_{D_{2n}}$ has $2n-1$ vertices. Using Definition 1, $CL(\Gamma_{D_{2n}})$ of size $(2n-1)\times(2n-1)$ is as follows:

$$CL(\Gamma_{D_{2n}}) = \begin{bmatrix} \frac{3n-2}{4} & \cdots & -\frac{1}{4} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{4} & \cdots & \frac{3n-2}{4} & -\frac{1}{2} & \cdots & -\frac{1}{2} \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & n-1 & \cdots & -\frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & \cdots & -\frac{1}{2} & -\frac{1}{2} & \cdots & n-1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}J_{n-1} + \frac{3n-1}{4}I_{n-1} & -\frac{1}{2}J_{(n-1)\times n} \\ -\frac{1}{2}J_{(n-1)\times n} & -\frac{1}{2}J_n + \frac{2n-1}{2}I_n \end{bmatrix}.$$

Using Theorem 2, with $s = \frac{3n-2}{4}$, $t = \frac{1}{2}$, $u = n-1$, $v = \frac{1}{4}$, then we obtain the formula of $P_{CL(\Gamma_{D_{2n}})}(\lambda)$,

$$P_{CL(\Gamma_{D_{2n}})}(\lambda) = \lambda\left(\lambda - \frac{1}{4}(3n-1)\right)^{n-2}\left(\lambda - n + \frac{1}{2}\right)^n.$$

2. Now for the even n case, by excluding $e, a^{\frac{n}{2}}$ from the vertex set of $\Gamma_{D_{2n}}$. Therefore, $\Gamma_{D_{2n}}$ consists of $2n-2$ vertices. Based on Theorem 1 and Definition 1, then $CL(\Gamma_{D_{2n}})$ of size $(2n-2)\times(2n-2)$ is as follows:

$$CL(\Gamma_{D_{2n}})$$
$$= \begin{bmatrix} \frac{3}{4}(n-1) & \dots & -\frac{1}{4} & -\frac{1}{2} & \dots & -\frac{1}{2} & -\frac{1}{2} & \dots & -\frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{4} & \dots & \frac{3}{4}(n-1) & -\frac{1}{2} & \dots & -\frac{1}{2} & -\frac{1}{2} & \dots & -\frac{1}{2} \\ -\frac{1}{2} & \dots & -\frac{1}{2} & n-\frac{3}{4} & \dots & -\frac{1}{2} & -\frac{1}{4} & \dots & -\frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & \dots & -\frac{1}{2} & -\frac{1}{2} & \dots & n-\frac{3}{4} & -\frac{1}{2} & \dots & -\frac{1}{4} \\ -\frac{1}{2} & \dots & -\frac{1}{2} & -\frac{1}{4} & \dots & -\frac{1}{2} & n-\frac{3}{4} & \dots & -\frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2} & \dots & -\frac{1}{2} & -\frac{1}{2} & \dots & -\frac{1}{4} & -\frac{1}{2} & \dots & n-\frac{3}{4} \end{bmatrix}.$$

The form of $CL(\Gamma_{D_{2n}})$ can be written as follows:

$$CL(\Gamma_{D_{2n}})$$
$$= \begin{bmatrix} \left(\frac{3}{4}n-\frac{1}{2}\right)I_{n-2}-\frac{1}{4}J_{n-2} & -\frac{1}{2}J_{(n-2)\times\frac{n}{2}} & -\frac{1}{2}J_{(n-2)\times\frac{n}{2}} \\ -\frac{1}{2}J_{\frac{n}{2}\times(n-2)} & \left(n-\frac{1}{4}\right)I_{\frac{n}{2}}-\frac{1}{2}J_{\frac{n}{2}} & \frac{1}{4}I_{\frac{n}{2}}-\frac{1}{2}J_{\frac{n}{2}} \\ -\frac{1}{2}J_{\frac{n}{2}\times(n-2)} & \frac{1}{4}I_{\frac{n}{2}}-\frac{1}{2}J_{\frac{n}{2}} & \left(n-\frac{1}{4}\right)I_{\frac{n}{2}}-\frac{1}{2}J_{\frac{n}{2}} \end{bmatrix}.$$

By Theorem 3 with $r=\frac{3}{4}(n-1)$, $s=n-\frac{3}{4}$, $t=\frac{1}{2}$, and $u=\frac{1}{4}$, we then obtain

$$P_{CL(\Gamma_{D_{2n}})}(\lambda)$$
$$= \left(\lambda-\frac{3}{4}n+\frac{1}{2}\right)^{n-3}(\lambda-n)^{\frac{n}{2}-1}\left(\lambda-n+\frac{1}{2}\right)^{\frac{n}{2}}\left(\lambda^2-n\lambda-\frac{1}{2}n\right).$$

□

We have the following theorem and theorem 6 to determine the $CL-$energy of $\Gamma_{D_{2n}}$.

Theorem 5. *The $CL-$spectral radius for $\Gamma_{D_{2n}}$ is*

1. *for odd n: $\rho_{CL}(\Gamma_{D_{2n}}) = n-\frac{1}{2}$,*
2. *for even n: $\rho_{CL}(\Gamma_{D_{2n}}) = n$.*

Proof. 1. The result follows from Theorem 4 (1), where n is odd, implies there are 3 eigenvalues. First, we have $\lambda_1 = \frac{1}{4}(3n-1)$ of multiplicity $(n-2)$, $\lambda_2 = n - \frac{1}{2}$ of multiplicity n, and a single $\lambda_3 = 0$. Hence, the spectrum of $\Gamma_{D_{2n}}$ as the following:

$$Spec_{CL}(\Gamma_{D_{2n}}) = \left\{ \left(n - \frac{1}{2}\right)^n, \left(\frac{1}{4}(3n-1)\right)^{n-2}, (0)^1 \right\}.$$

The desired result is the spectral radius of $\Gamma_{D_{2n}}$, which we obtain by taking the greatest absolute eigenvalues.

2. Recall from Theorem 4 (2) for even n that $\Gamma_{D_{2n}}$ consists of 5 eigenvalues. It follows that $\lambda_1 = \frac{3}{4}n - \frac{1}{2}$ of multiplicity $n-3$, $\lambda_2 = n$ of multiplicity $\frac{n}{2} - 1$ and $\lambda_3 = n - \frac{1}{2}$ of multiplicity $\frac{n}{2}$ and $\lambda_{4,5} = \frac{1}{2}\left(n \pm \sqrt{n(n-2)}\right)$. Hence, the spectrum of $\Gamma_{D_{2n}}$ as the following:

$$Spec_{CL}(\Gamma_{D_{2n}})$$

$$= \left\{ (n)^{\frac{n}{2}-1}, \left(n - \frac{1}{2}\right)^{\frac{n}{2}}, \left(\frac{3}{4}n - \frac{1}{2}\right)^{n-3}, \left(\frac{1}{2}\left(n + \sqrt{n(n-2)}\right)\right)^1, \left(\frac{1}{2}\left(n - \sqrt{n(n-2)}\right)\right)^1 \right\}.$$

We finish the proof by determining the $CL-$spectral radius of $\Gamma_{D_{2n}}$ as the maximum of $|\lambda_i|$, $i = 1, 2, 3, 4$.

□

Theorem 6. *The $CL-$energy for $\Gamma_{D_{2n}}$ is*

1. *for odd n: $E_{CL}(\Gamma_{D_{2n}}) = \frac{1}{4}(n-1)(7n-2)$*
2. *for even n: $E_{CL}(\Gamma_{D_{2n}}) = \frac{1}{4}\left(7n^2 - 12n + 6\right)$.*

Proof. 1. Let n be odd. By Theorem 5 (1), it follows that the $CL-$energy of $\Gamma_{D_{2n}}$ is

$$E_{CL}(\Gamma_{D_{2n}})$$

$$= (n)\left|n - \frac{1}{2}\right| + (n-2)\left|\frac{1}{4}(3n-1)\right| + (1)\,|0| = \frac{1}{4}(n-1)(7n-2).$$

2. Let n be even. Then according to Theorem 5 (2), the $CL-$energy of $\Gamma_{D_{2n}}$ is

$$E_{CL}(\Gamma_{D_{2n}}) = \left(\frac{n}{2}-1\right)|n| + \left(\frac{n}{2}\right)\left|n-\frac{1}{2}\right| + (n-3)\left|\frac{3}{4}n+\frac{1}{2}\right| + \left|\frac{1}{2}\left(n \pm \sqrt{n(n-2)}\right)\right| = \frac{1}{4}\left(7n^2-12n+6\right).$$

□

3.3 Closeness signless Laplacian

We begin this subsection with the following theorem which describes the characteristic formula of the closeness signless Laplacian matrix of $\Gamma_{D_{2n}}$, $CSL(\Gamma_{D_{2n}})$.

Theorem 7. *The characteristic polynomial of $CSL(\Gamma_{D_{2n}})$ is*

1. *for n is odd:*

$$P_{CSL(\Gamma_{D_{2n}})}(\lambda) = \left(\lambda - \frac{3}{4}(n-1)\right)^{n-2}\left(\lambda - n + \frac{3}{2}\right)^{n-1} \left(\lambda^2 - \frac{5}{2}(n-1)\lambda + \frac{1}{4}(5n-6)(n-1)\right),$$

2. *for n is even:*

$$P_{CSL(\Gamma_{D_{2n}})}(\lambda) = \left(\lambda - \frac{3}{4}n + 1\right)^{n-3}\left(\lambda - n + \frac{3}{2}\right)^{\frac{n}{2}-1}(\lambda - n + 1)^{\frac{n}{2}} \left(\lambda^2 + \left(3 - \frac{5}{2}n\right)\lambda + \frac{1}{4}\left(5n^2 - 13n + 9\right)\right).$$

Proof. We divided the proof into two cases.

1. Suppose n is odd. Based on Theorem 1, by the same argument of the proof of Theorem 4 (1) and Definition 2, we can construct $CSL(\Gamma_{D_{2n}})$

of size $(2n-1)\times(2n-1)$ as follows:

$$CSL(\Gamma_{D_{2n}}) = \begin{bmatrix} \frac{3n-2}{4} & \cdots & \frac{1}{4} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & \cdots & \frac{3n-2}{4} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \cdots & \frac{1}{2} & n-1 & \cdots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \cdots & n-1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{4}J_{n-1}+\frac{3(n-1)}{4}I_{n-1} & \frac{1}{2}J_{(n-1)\times n} \\ \frac{1}{2}J_{(n-1)\times n} & \frac{1}{2}J_n+\frac{2n-3}{2}I_n \end{bmatrix}.$$

Using Theorem 2, with $s=\frac{3n-2}{4}$, $t=-\frac{1}{2}$, $u=n-1$, $v=-\frac{1}{4}$, then we obtain the formula of $P_{CSL(\Gamma_{D_{2n}})}(\lambda)$,

$$P_{CSL(\Gamma_{D_{2n}})}(\lambda) = \left(\lambda-\frac{3}{4}(n-1)\right)^{n-2}\left(\lambda-n+\frac{3}{2}\right)^{n-1} \left(\lambda^2-\frac{5}{2}(n-1)\lambda+\frac{1}{4}(5n-6)(n-1)\right).$$

2. Consider the second case for even n, by Definition 2 we have $CSL(\Gamma_{D_{2n}})$ of size $(2n-2)\times(2n-2)$ as follows:

$$CSL(\Gamma_{D_{2n}})$$
$$= \begin{bmatrix} \frac{3}{4}(n-1) & \cdots & \frac{1}{4} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{4} & \cdots & \frac{3}{4}(n-1) & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \cdots & \frac{1}{2} & n-\frac{3}{4} & \cdots & \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \cdots & n-\frac{3}{4} & \frac{1}{2} & \cdots & \frac{1}{4} \\ \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{4} & \cdots & \frac{1}{2} & n-\frac{3}{4} & \cdots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{4} & \frac{1}{2} & \cdots & n-\frac{3}{4} \end{bmatrix}.$$

Hence, the block matrices of $CSL(\Gamma_{D_{2n}})$ follows:

$$CSL(\Gamma_{D_{2n}})$$

$$= \begin{bmatrix} \left(\frac{3}{4}n-1\right)I_{n-2}+\frac{1}{4}J_{n-2} & \frac{1}{2}J_{(n-2)\times\frac{n}{2}} & \frac{1}{2}J_{(n-2)\times\frac{n}{2}} \\ \frac{1}{2}J_{\frac{n}{2}\times(n-2)} & \left(n-\frac{5}{4}\right)I_{\frac{n}{2}}+\frac{1}{2}J_{\frac{n}{2}} & -\frac{1}{4}I_{\frac{n}{2}}+\frac{1}{2}J_{\frac{n}{2}} \\ \frac{1}{2}J_{\frac{n}{2}\times(n-2)} & -\frac{1}{4}I_{\frac{n}{2}}+\frac{1}{2}J_{\frac{n}{2}} & \left(n-\frac{5}{4}\right)I_{\frac{n}{2}}+\frac{1}{2}J_{\frac{n}{2}} \end{bmatrix}.$$

By Theorem 3 with $r = \frac{3}{4}(n-1)$, $s = n - \frac{3}{4}$, $t = \frac{1}{2}$, and $u = \frac{1}{4}$, we then obtain

$$P_{CSL(\Gamma_{D_{2n}})}(\lambda) = \left(\lambda - \frac{3}{4}n + 1\right)^{n-3}\left(\lambda - n + \frac{3}{2}\right)^{\frac{n}{2}-1}(\lambda - n + 1)^{\frac{n}{2}} \left(\lambda^2 + \left(3 - \frac{5}{2}n\right)\lambda + \frac{1}{4}\left(5n^2 - 13n + 9\right)\right).$$

□

Theorem 8. *The $CSL-$spectral radius for $\Gamma_{D_{2n}}$ is*

1. *for n is odd:* $\rho_{CSL}(\Gamma_{D_{2n}}) = \frac{1}{4}\left(5(n-1) + \sqrt{(n-1)(5n-1)}\right)$,
2. *for n is even:* $\rho_{CSL}(\Gamma_{D_{2n}}) = \frac{1}{4}\left(5n - 6 + \sqrt{n(5n-8)}\right)$.

Proof. 1. The results follows from Theorem 7 (1) for odd n. It can be seen that the eigenvalues of $CSL(\Gamma_{D_{2n}})$ are $\lambda_1 = \frac{3}{4}(n-1)$ of multiplicity $(n-2)$, $\lambda_2 = n - \frac{3}{2}$ of multiplicity $n-1$, and a single $\lambda_{3,4} = \frac{1}{4}\left(5(n-1) \pm \sqrt{(n-1)(5n-1)}\right)$. Thus, the spectrum of $\Gamma_{D_{2n}}$ is as follows:

$$Spec_{CSL}(\Gamma_{D_{2n}})$$
$$= \left\{ \left(\frac{1}{4}\left(5(n-1) + \sqrt{(n-1)(5n-1)}\right)\right)^1, \left(n - \frac{3}{2}\right)^{n-1}, \left(\frac{3}{4}(n-1)\right)^{n-2}, \left(\frac{1}{4}\left(5(n-1) - \sqrt{(n-1)(5n-1)}\right)\right)^1 \right\}.$$

Using the greatest absolute eigenvalues, we can obtain the desired spectral radius of $\Gamma_{D_{2n}}$.

2. It was shown in Theorem 7 (2) for even n that the roots of $P_{CSL(\Gamma_{D_{2n}})}(\lambda) = 0$ deliver five eigenvalues. First, $\lambda_1 = \frac{3}{4}n - 1$ of multiplicity $n-3$, $\lambda_2 = n - \frac{3}{2}$ of multiplicity $\frac{n}{2} - 1$ and $\lambda_3 = n - 1$ of multiplicity $\frac{n}{2}$ and $\lambda_{4,5} = \frac{1}{4}\left(5n - 6 \pm \sqrt{n(5n-8)}\right)$. Hence, the spectrum of $\Gamma_{D_{2n}}$ as the following:

$$Spec_{CSL}(\Gamma_{D_{2n}})$$

$$= \left\{ \left(n - \frac{3}{2}\right)^{\frac{n}{2}-1}, (n-1)^{\frac{n}{2}}, \left(\frac{3}{4}n - 1\right)^{n-3}, \left(\frac{1}{4}\left(5n - 6 + \sqrt{n(5n-8)}\right)\right)^{1}, \left(\frac{1}{4}\left(5n - 6 - \sqrt{n(5n-8)}\right)\right)^{1} \right\}.$$

The $CSL-$spectral radius of $\Gamma_{D_{2n}}$ is the maximum of $|\lambda_i|$, where $i = 1, 2, 3, 4$. This completes the proof.

□

Theorem 9. *The $CSL-$energy for $\Gamma_{D_{2n}}$ is*

1. *for n is odd: $E_{CSL}(\Gamma_{D_{2n}}) = \frac{1}{4}(n-1)(7n-2)$,*
2. *for n is even: $E_{CSL}(\Gamma_{D_{2n}}) = \frac{1}{4}\left(7n^2 - 12n + 6\right)$.*

Proof. 1. We shall consider the first case where n is odd. By Theorem 8 (1), the $CSL-$energy of $\Gamma_{D_{2n}}$ can be calculated as follows:

$$E_{CSL}(\Gamma_{D_{2n}})$$

$$=(n-1)\left|n - \frac{3}{2}\right| + (n-2)\left|\frac{3}{4}(n-1)\right| + \left|\frac{1}{4}\left(5(n-1) \pm \sqrt{(n-1)(5n-1)}\right)\right|$$
$$=\frac{1}{4}(n-1)(7n-2).$$

2. For even n, by Theorem 8 (2), then the $CSL-$energy of $\Gamma_{D_{2n}}$ is

$$E_{CSL}(\Gamma_{D_{2n}}) = \left(\frac{n}{2} - 1\right)\left|n - \frac{3}{2}\right| + \left(\frac{n}{2}\right)|n-1| + (n-3)\left|\frac{3}{4}n - 1\right| + \left|\frac{1}{4}\left(5n - 6 \pm \sqrt{n(5n-8)}\right)\right|$$
$$=\frac{1}{4}\left(7n^2 - 12n + 6\right).$$

□

4 Discussions

Based on Theorem 6 and Theorem 9, in the following, we derive the categorization of the closeness Laplacian and closeness signless Laplacian energies of $\Gamma_{D_{2n}}$.

Corollary 1. $\Gamma_{D_{2n}}$ *is hyperenergetic corresponding with closeness Laplacian and closeness signless Laplacian matrices.*

Furthermore, we can draw the following conclusion:

Corollary 2. *Closeness Laplacian and closeness signless Laplacian energies of* $\Gamma_{D_{2n}}$ *are never an odd integer.*

Corollary 2's statements align with the well-acknowledged facts found in [4] and [9]. Moreover, the following assertion can be used to compare the energy in Theorem 6 and Theorem 9. We conclude this section with the following corollary.

Corollary 3. $E_{CL}(\Gamma_{D_{2n}}) = E_{CSL}(\Gamma_{D_{2n}})$.

It can be seen that the closeness Laplacian energy is always similar to closeness signless Laplacian energy of $\Gamma_{D_{2n}}$.

5 Conclusions

This present research is concentrated on the energy of non-commuting graphs based on the closeness Laplacian and closeness signless Laplacian matrices. Instead of these, many important properties of graphs can be explored. Therefore, further research is recommended to examine the same groups in different types of degree-based matrices.

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