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AN IMPROVED MODEL FOR ATOMIC SHRINKING*

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Abstract

The usual equation for the motion of electrons in the deterministic Rutherford-Bohr atomic model is conservative with a singular potential at the origin. When a dissipation is added, new phenomena appear, mainly a contraction of orbits for large time. A special model is studied in which the dissipation coefficient varies as the inverse of the square of the distance to the nucleus, and it is shown that this equation has contraction properties similar to the case of a linear damping, but the backward equation shows affine growth of the orbits for large time, which is more realistic than the exponential growth in the case of linear damping.

Keywords: gravitation, singular potential, second order differential equation, global solutions, asymptotic bounds.

MSC: 34A34, 34C11, 34D05.

1 Introduction

The usual equation for the motion of electrons in the deterministic Rutherford-Bohr atomic model is conservative with a singular potential at the origin.

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When a dissipation is added to the basic equation

$$mu'' = -\frac{q^2}{4\pi\varepsilon_0}\frac{u}{|u|^3}$$

modeling Coulomb's central force (with q the elementary charge, m the mass of the electron and ε_0 the vacuum permittivity) written as

$$u'' + c_0\frac{u}{|u|^3} = 0, \tag{1}$$

with

$$c_0 := \frac{q^2}{4\pi\varepsilon_0 m},$$

we obtain an equation of the general form

$$u'' + f(t, u, u')u' + c_0\frac{u}{|u|^3} = 0, \tag{2}$$

where f is a suitable non-negative function. In the paper [3] the first author considered the special case where f is a positive constant δ, which leads to the simple equation

$$u'' + \delta u' + c_0\frac{u}{|u|^3} = 0. \tag{3}$$

Such an equation appears when trying to understand the long term dissipation induced by random shocks of wandering cosmic corpuscles with the atoms, and in [3] it was observed that the behavior of the solutions opens the door to new explanations of totally independent phenomena: cosmological redshift (cf. [7, 12]), aging of electrical appliances, gigantism of arthropods and plants in the carboniferous era. Although that equation does not have any solution for which $|u(t)|$ is an exact decreasing exponential $Ce^{-\eta t}$, the type of contraction produced by this kind of strong dissipation is basically uniform and when we try to guess the size of atoms in the distant past, the presumably exponential growth for the backward equation is totally unrealistic.

This observation leads us to study the effect of a somewhat different dissipation mechanism which has already been studied in the literature, in particular by the second author, with a different purpose, cf. eg. [9], in connection with a conjecture of Euler (cf. [2]) on long term collapse of the solar system. Here, we assume that the effect of wandering corpuscles is negligible far from the nucleus and maximal when we approach it. In this case, a reasonable model improving (3) may be

$$u'' + \delta\frac{u'}{a^2 + |u|^2} + c_0\frac{u}{|u|^3} = 0, \tag{4}$$

with a and δ some positive constants.

In the present paper, we report on the asymptotic behavior of the solutions to (2) and more precisely (4). In Section 2 we recall, to make subsequent comparisons easier, the Rutherford-Bohr atomic model with its circular orbits. In Section 3, we collect some general properties of (2). Section 4 is devoted to the special case of (4) and the associate backward problem, with part of the technicalities in the appendix (Section 6). Special properties of real solutions are discussed in Section 5. The physical interpretation of the results is outlined in the concluding Section 7.

2 Recalling the classical Rutherford-Bohr model

The Rutherford model (cf. [10]), which served as a basis for Bohr's model [1] of the hydrogen atom, is based on a corpuscular conception of protons and electrons and the application of Coulomb's law for electrostatic forces. Denoting by q the common absolute value of the charges of proton and electron, the equation of motion, considering the unique proton of the nucleus as the center of coordinates, can be written in the plane of the orbit in complex form. Denoting by u the position of the electron in the complex plane, we have

$$mu'' = -\frac{q^2}{4\pi\varepsilon_0}\frac{u}{|u|^3}. \tag{5}$$

The electron travels in a circular orbit

$$|u(t)| = R \tag{6}$$

so that the solutions take the form

$$u(t) = R\exp(i\omega(t + t_0)) \tag{7}$$

with

$$\omega^2 = \frac{q^2}{4\pi\varepsilon_0 m R^3}. \tag{8}$$

In particular, the constant velocity of the electron is

$$|u'(t)| = v = KR^{-1/2}$$

for some positive constant K. This is consistent with the analogous property given by the third Kepler's law for the planets motion driven by the gravitational field of the sun. In the framework of Bohr's modelization, it is assumed that the radius R can take only the values of a sequence of the form $r_n = r_0 n^2$ with n a positive integer. In particular, the electron cannot "fall" on the nucleus.

Bohr's theory explains neither why all electrons so luckily always choose a circular orbit rather than other elliptic possibilities, nor even how the first atoms of matter could appear, even in the simplest case of the hydrogen atom. If an electron is left without initial velocity in the close surrounding of a proton, an atom will never appear since the electron will collide with the proton in a very short time. The problem of the origin of the initial kinetic energy of the electron is left open. Besides, there is no explanation of why some circular orbits would be stable and not the others. An attempt to circumvent that problem is quantum mechanics leading to Schrödinger's equation (cf. [11]). For the time being, in the same track as [3, 6], we shall avoid quantum considerations, even though such a framework is presently considered necessary to understand the electromagnetic waves produced by hot matter (and therefore to consider spectral properties and the redshift problematics).

3 General results for equation (2)

In this section, we investigate the standard relevant questions for the solutions of the general equation (2).

3.1 Existence and uniqueness

We introduce the total energy

$$E(t) := \frac{1}{2}|u'|^2(t) - \frac{c_0}{|u(t)|},$$

where $u(t)$ is a solution of (2) on some interval $[0, T)$. Assuming for simplicity that $f \in C^1(\mathbb{R}^+ \times \mathbb{C}^2; \mathbb{R}^+)$, it is clear that a unique local solution exists for any initial data $u(0) \in \mathbb{C}^*, u'(0) \in \mathbb{C}$. Moreover the identity

$$E'(t) = -f(t, u, u')|u'|^2 \leq 0$$

implies

$$\forall t \in [0, T),\ E(t) \leq E(0)$$

providing

$$\frac{1}{2}|u'|^2(t) \leq \frac{c_0}{|u(t)|} + E(0).$$

This shows that the only way for the solution to be non-global is vanishing of $u(T)$, in which case the equation does not make sense beyond T.

3.2 Bounded trajectories

The next result generalizes a boundedness result of [3] for any function f.

Theorem 1. *Let $u_0 \neq 0$ and assume the initial smallness condition*

$$|u_0||u_0'|^2 < 2c_0. \tag{9}$$

Then the local solution u of (2) on $[0,T)$ with initial conditions $u(0) = u_0; u'(0) = u_0'$ satisfies the inequality

$$\forall t \in [0,T), \quad |u(t)| \leq \frac{2c_0|u_0|}{2c_0 - |u_0||u_0'|^2}. \tag{10}$$

In particular, if u does not vanish in finite time, u is a global bounded non-vanishing solution.

Proof. The inequality (9) is equivalent to $E(0) < 0$. Then the inequality

$$E(t) := \frac{1}{2}|u'|^2(t) - \frac{c_0}{|u(t)|} \leq E(0) \Longrightarrow \frac{c_0}{|u(t)|} \geq -E(0)$$

gives

$$|u(t)| \leq \frac{c_0}{\frac{c_0}{|u_0|} - \frac{1}{2}|u_0'|^2} = \frac{2c_0|u_0|}{2c_0 - |u_0||u_0'|^2}.$$

□

3.3 Convergence to 0 of non-vanishing bounded solutions

As in the case of the conservative equation (1) which is well known to have hyperbolic and parabolic trajectories, (2) may have some unbounded solutions if the damping is sufficiently weak for large values of u. The next result shows, on the other hand, that in sharp contrast with (1), (2) does not have any periodic trajectory at all under mild conditions on f.

Theorem 2. *Assuming that f is uniformly positive and bounded for u bounded independently of (t,u'), for any solution u of (2) such as $|u(t)|$ is global, positive and bounded on $\mathbb{R}^+$, we have*

$$\lim_{t\to+\infty} |u(t)| = 0. \tag{11}$$

Proof. Since u never vanishes, it is clear that $u \in C^2(\mathbb{R}^+, \mathbb{C})$ and we have

$$E'(t) = -f(t, u(t), u'(t))|u'(t)|^2. \tag{12}$$

In particular, $E(t)$ is non-increasing. Then we have two possibilities

Case 1.

$$\lim_{t\to+\infty} E(t) = -\infty. \tag{13}$$

Then since $\dfrac{c_0}{|u(t)|} \geq -E(t)$ we conclude that

$$\lim_{t\to+\infty} |u(t)| = 0.$$

Case 2.

$$\lim_{t\to+\infty} E(t) = E^* > -\infty. \tag{14}$$

Then $E(t)$ is bounded and by our hypothesis on f we have for some $\nu > 0$

$$\forall t \in \mathbb{R}^+, \quad E(0) - E(t) \geq \nu \int_0^t |u'(s)|^2 ds.$$

In particular, $u' \in L^2(\mathbb{R}^+, \mathbb{C})$. We have assumed that $u(t)$ is bounded, hence precompact in $\mathbb{R}^+$ with values in $\mathbb{C}$. Therefore if (11) is not satisfied we may assume that for some sequence t_n tending to $+\infty$

$$\lim_{n\to+\infty} u(t_n) = w \neq 0. \tag{15}$$

On the other hand, for $u_n(s) = u(t_n + s)$, we have

$$\lim_{n\to+\infty} u_n' = 0 \tag{16}$$

in the strong topology of $L^2(0,1)$ and in particular, by Cauchy-Schwarz-inequality we find

$$\lim_{n\to+\infty} u(t_n + s) = w \tag{17}$$

uniformly on $[0,1]$. Since $w \neq 0$, this implies

$$\lim_{n\to+\infty} \frac{u(t_n + s)}{|u(t_n + s)|^3} = \frac{w}{|w|^3} \tag{18}$$

uniformly on $[0,1]$. But then by the equation

$$\lim_{n\to+\infty} u_n'' = c_0 \frac{w}{|w|^3} \tag{19}$$

in the strong topology of $L^2(0,1)$. This is contradictory with (16). Indeed, it implies for instance

$$\lim_{n\to+\infty}\int_0^1 s(1-s)u''(t_n+s)ds = c_0\int_0^1 s(1-s)ds\frac{w}{|w|^2} = z \neq 0$$

while on the other hand

$$\int_0^1 s(1-s)u''(t_n+s)ds = -\int_0^1 (1-2s)u'(t_n+s)ds \longrightarrow 0.$$

This contradiction concludes the proof. □

As an immediate consequence of the previous theorem and the results of Sections 2 and 3, we obtain:

Corollary 1. *Assume* $|u_0||u_0'|^2 < 2c_0$. *Then assuming that* f *is uniformly positive for* u *bounded independently of* (t, u')*, the local solution of* (2) *with initial conditions* $u(0) = u_0; u'(0) = u_0'$ *either vanishes in finite time, or tends to* 0 *as* t *tends to infinity.*

4 The case of equation (4)

It is not difficult to find some solutions which vanish in finite time, when u_0 and u_0' are collinear with opposite orientation. On the other hand this cannot happen for u_0 and u_0' not collinear. Indeed in such a case $u(t)$ has to remain bounded and as for the simpler model (3), we find that, as a consequence of

$$\frac{d}{dt}(u\wedge u') = -\delta\frac{(u\wedge u')}{a^2+|u|^2},$$

the vector $(u\wedge u')$ remains bounded away from 0 on any finite interval. As a special case of our previous corollary, we have

Corollary 2. *Assume* $|u_0||u_0'|^2 < 2c_0$. *Then the local solution of* (4) *with initial conditions* $u(0) = u_0; u'(0) = u_0'$ *either vanishes in finite time, or tends to* 0 *as* t *tends to infinity.*

Remark 1. *In* [3] *we established that all solutions of equation* (3) *are bounded for* $t \geq 0$*, cf. also* [8, 9]*. For the conservative equation (1), there are many unbounded solutions, for instance real positive solutions with initial positive velocities have a linear growth at infinity as soon as* ${u_0 u_0'}^2 > 2c_0$ *and there are even the unbounded solutions*

$$u(t) = \left(\frac{9c_0}{2}\right)^{\frac{1}{3}}(t+b)^{\frac{2}{3}}$$

(with $b > 0$ arbitrary) corresponding to the limiting case $u_0 {u'_0}^2 = 2c_0$. Since boundedness in (3) is produced by dissipation and the dissipation in (4) becomes very weak for large values of $|u(t)|$, it is natural to wonder whether (4) may have unbounded positive solutions. We show in Section 5 (Proposition 5.1) that it is indeed the case.

Now we state the most important property of (4) which makes the difference with (3) and saves us from one of the big contradictions concerning the size of atoms in the distant past.

Theorem 3. *Let us consider the backward equation of (4)*

$$v'' - \delta \frac{v'}{a^2 + |v|^2} + c_0 \frac{v}{|v|^3} = 0. \tag{20}$$

Then the size of $|v(t)|$ grows up at most linearly for large time. More precisely, for any non-vanishing global solution v of (20) there exists a number $\mu_v \in [0, \infty)$ such that

$$\lim_{t \to +\infty} \frac{|v(t)|}{t} = \mu_v. \tag{21}$$

We shall rely on the identity

$$\frac{d}{dt}\Big\{ \frac{(v, v')}{|v|} - \frac{1}{a}\arctan\frac{|v|}{a} \Big\} = -\frac{c_0}{|v|^2} + \frac{|v \wedge v'|^2}{|v|^3} \tag{22}$$

combined with

$$\frac{d}{dt}(v \wedge v') = \delta \frac{(v \wedge v')}{a^2 + |v|^2}, \tag{23}$$

both proved in the appendix. The invariance under rotations of the equation (20) allows to reduce one dimension in the system. In fact, if we define the new unknowns

$$r = |v|, \;\; \xi = r' - \frac{1}{a} \arctan \frac{r}{a}, \;\; \ell = |r \wedge r'|$$

the above identities lead to the equations

$$r' = \frac{1}{a} \arctan \frac{r}{a} + \xi, \;\; \xi' = -\frac{c_0}{r^2} + \frac{\ell^2}{r^3}, \;\; \ell' = \frac{\delta}{a^2 + r^2} \ell. \tag{24}$$

For the proof of the theorem we shall rely on the following technical lemma

Lemma 1. *Let $W(t) := (r(t), \xi(t), \ell(t))$ be a solution of (24) with $r(0) > 0$ and forward maximal interval $[0, \omega)$ and assume that, for some $m > 0$,*

$$r'(0) \geq m + \frac{c_0}{mr(0)}. \tag{25}$$

Then $\omega = +\infty$ and, for each $t \in [0, \infty)$,

$$r'(t) > m.$$

Proof. We will prove that

$$r'(t) > m, \quad t \in [0, \omega). \tag{26}$$

Then a standard continuation argument implies that $\omega = +\infty$. By contradiction assume that the claim (26) does not hold. We take the first instant $\tau \in (0, \omega)$ such that $r'(\tau) = m$. The definition of τ together with (25) imply that $r'(t) > m$ if $t \in [0, \tau)$. In consequence $r(t) > r(0) + mt$ if $t \in (0, \tau]$. Also, from the second equation of (24),

$$\xi'(t) > -\frac{c_0}{r(t)^2} > -\frac{c_0}{(r(0) + mt)^2}.$$

After integrating this inequality,

$$\xi(\tau) > \xi(0) - \int_0^\tau \frac{c_0}{(r(0) + ms)^2} ds > \xi(0) - \int_0^\infty \frac{c_0}{(r(0) + ms)^2} ds = \xi(0) - \frac{c_0}{mr(0)}.$$

Since $r(t)$ is increasing on $[0, \tau]$ the first equation in (24) implies that

$$r'(\tau) > \frac{1}{a} \arctan \frac{r(0)}{a} + \xi(\tau) > r'(0) - \frac{c_0}{mr(0)}.$$

This is the searched contradiction because $r'(\tau) = m$ is not compatible with the assumption (25). □

We are ready for the proof of the theorem.

Proof. We distinguish two cases.

Case 1: For every $t \in [0, \infty)$, $r'(t) < 2\sqrt{\frac{c_0}{r(t)}}$.

Case 2: There exists some $t_0 \in [0, \infty)$ such that $r'(t_0) \geq 2\sqrt{\frac{c_0}{r(t_0)}}$.

In the first case we solve the differential inequality to deduce that

$$r(t) = \mathcal{O}(t^{2/3}) \ \text{ as } t \to +\infty$$

and $\mu_v = 0$.

Assume now that we are in the second case. We shall apply Lemma 1 to the translate solution $W(t_0 + .)$ with $m = \sqrt{\frac{c_0}{r(t_0)}}$. As a consequence, for each $t \in (t_0, \infty)$, $r'(t) > m$. Along the lines of the proof of the Lemma we observe that the two integrals below are finite:

$$I_1 = \int_0^\infty \frac{ds}{a^2 + r(s)^2}, \ \ I_2 = \int_0^\infty \frac{ds}{r(s)^2}.$$

In particular, the third equation of (24) implies that $\ell(t) \leq \ell(0)e^{\delta I_1}$ and as a consequence

$$I_3 = \int_0^\infty \frac{\ell(s)^2}{r(s)^3} ds < \infty.$$

Now we can use the third and second equations to prove that $\ell(t)$ and $\xi(t)$ have limits at infinity,

$$\ell(\infty) = \ell(0)e^{\delta I_1}, \ \ \xi(\infty) = \xi(0) - c_0 I_2 + I_3.$$

Finally, from the first equation,

$$\lim_{t \to +\infty} r'(t) = \frac{\pi}{2a} + \xi(\infty).$$

After invoking L'Hôpital rule, $\mu_v = r'(\infty)$. □

Remark 2. *The number μ_v can be arbitrarily large. This can be shown by an application of Lemma 1 with $m \to \infty$. The above proof allows to estimate μ_v in terms of the initial conditions if $r'(0) \geq 2\sqrt{\frac{c_0}{r(0)}}$. Note that $I_1 \leq \frac{1}{a}\sqrt{\frac{r(0)}{c_0}}(\frac{\pi}{2} - \arctan(\frac{r(0)}{a}))$, $\ell(\infty) \leq \ell(0)e^{\delta I_1}$, $I_3 \leq \frac{\ell(\infty)^2}{2c_0^{1/2} r(0)^{3/2}}$.*

5 Some special results for real solutions of equation (4)

5.1 The equation (4) has unbounded solutions

Proposition 1. *Let u_0, u_0' be both positive and such that*

$$u_0' > \max\{\frac{64\pi\delta}{7a}, 2\frac{c_0^{1/2}}{{u_0}^{1/2}}\}. \tag{27}$$

Then u is unbounded and more precisely

$$\forall t \geq 0, \quad u(t) \geq u_0 + \frac{u_0'}{4} t. \tag{28}$$

Proof. We establish by contradiction that

$$\forall t \geq 0, \quad u'(t) \geq \frac{u_0'}{4}. \tag{29}$$

Assuming the contrary, let

$$T := \inf\{t > 0, u'(t) < \frac{u_0'}{4}\}. \tag{30}$$

Then by definition

$$\forall t \in (0, T), \quad u'(t) \geq \frac{u_0'}{4} \tag{31}$$

and $u'(T) = \frac{u_0'}{4}$, hence

$$E(T) = \frac{1}{32}{u_0'}^2 - \frac{c_0}{u(T)} = E(0) - \delta \int_0^T \frac{u'^2(s)}{a^2 + u^2(s)} ds$$

implying

$$E(T) > \frac{1}{4}{u_0'}^2 - \delta \int_0^T \frac{u'^2(s)}{a^2 + u^2(s)} ds \tag{32}$$

since $u_0' > 2\frac{c_0^{1/2}}{{u_0}^{1/2}}$ implies $E(0) > \frac{1}{4}{u_0'}^2$. In addition, u is increasing on $(0, T)$ and the inequality

$$E(t) = \frac{1}{2}u'^2(t) - \frac{c_0}{u(t)} \leq \frac{1}{2}{u_0'}^2 - \frac{c_0}{u_0}$$

shows that

$$\forall t \in (0, T), \quad u'(t) \leq u_0'. \tag{33}$$

Using this inequality in (32) we obtain

$$\frac{1}{32}{u_0'}^2 > \frac{1}{4}{u_0'}^2 - \delta {u_0'}^2 \int_0^T \frac{1}{a^2 + u^2(s)} ds.$$

Multiplying by 32 and simplifying by ${u_0'}^2$, we obtain

$$7 < 32\delta \int_0^T \frac{1}{a^2 + u^2(s)} ds.$$

As a consequence of (31), we have $u(s) > \frac{u_0'}{4}s$ and in particular this now implies

$$7 < 32\delta \int_0^T \frac{1}{a^2 + \frac{{u_0'}^2}{16}s^2} ds < 32\delta \int_0^\infty \frac{1}{a^2 + \frac{{u_0'}^2}{16}s^2} ds. \tag{34}$$

A standard change of variable $\sigma = \dfrac{u_0'}{4a} s$ provides the value

$$\int_0^\infty \frac{1}{a^2 + \frac{{u_0'}^2}{16} s^2} ds = \frac{2\pi}{a u_0'},$$

and we end up with the inequality

$$7 < \frac{64\pi\delta}{a u_0'} \Rightarrow u_0' < \frac{64\pi\delta}{7a}$$

contradicting our assumption on u_0'. □

5.2 Real solutions of the backward equation

For real solutions of the backward equation (20), we have the more precise estimate given by

Proposition 2. *For any real solution v and more generally if v and v' are collinear, we have*

$$\forall t \geq 0, \quad |v(t)| \leq |v_0| + t(\frac{\pi}{2a} + |v_0'|). \tag{35}$$

Proof. Since v and v' are collinear, our basic identity reduces to

$$\frac{d}{dt}\left\{ \frac{(v, v')}{|v|} - \frac{1}{a}\arctan\frac{|v|}{a} \right\} = -\frac{c_0}{|v|^2} \leq 0 \tag{36}$$

which implies in particular

$$\frac{d}{dt}|v(t)| = \frac{(v, v')}{|v|} \leq +\frac{\pi}{2a} + \frac{(v_0, v_0')}{|v_0|} - \frac{1}{a}\arctan\frac{|v_0|}{a}$$

and finally

$$|v(t)| \leq |v_0| + t\left(\frac{\pi}{2a} + |v_0'|\right).$$

□

Remark 3. *The sublinear rate of growth of solutions of* (20) *is likely to be optimal. Actually, large solutions tend to behave as solutions of the simpler equation*

$$v'' - \delta\frac{v'}{|v|^2} + c_0\frac{v}{|v|^3} = 0 \tag{37}$$

which has the explicit real positive solutions

$$v(t) = \frac{c_0}{\delta}(t + C).$$

However, the optimality of inequality (35) *itself is not clear, since as a tends to* 0*, the right hand side blows-up.*

6 Appendix

6.1 Proof of (23)

Since $v' \wedge v' = 0$ we have

$$\frac{d}{dt}(v \wedge v') = v \wedge v'' = \delta \frac{v \wedge v'}{a^2 + |v|^2},$$

because $v \wedge v = 0$.

6.2 The derivative of $|v(t)|$

Since $|v(t)| = \sqrt{|v(t)|^2}$, we have

$$\frac{d}{dt}|v(t)| = \frac{2(v(t), v'(t))}{2\sqrt{|v(t)|^2}} = \frac{(v(t), v'(t))}{|v(t)|}.$$

6.3 Proof of (22)

We first compute

$$\frac{d}{dt}\left(\arctan \frac{|v(t)|}{a}\right) = \frac{1}{1 + \frac{|v(t)|^2}{a^2}} \frac{(v(t), v'(t))}{a|v(t)|} = a \frac{(v(t), v'(t))}{|v(t)|(a^2 + |v(t)|^2)}. \quad (38)$$

Then, dropping for convenience the variable t in all subsequent calculations, we evaluate

$$\frac{d}{dt}\frac{(v(t), v'(t))}{|v(t)|} = \left(\frac{v}{|v|}, v''\right) + \left(\left(\frac{v}{|v|}\right)', v'\right) = \left(\frac{v}{|v|}, v''\right) + \frac{|v'|^2}{|v|} + \left(v', \left(\frac{1}{|v|}\right)' v\right).$$

Since

$$\left(\frac{1}{|v|}\right)' = -\frac{(|v|)'}{|v|^2} = -\frac{(v, v')}{|v|^3}$$

we end up with

$$\frac{d}{dt}\frac{(v(t), v'(t))}{|v(t)|} = \left(\frac{v}{|v|}, v''\right) + \frac{|v'|^2}{|v|} - \frac{(v, v')^2}{|v|^3} = \left(\frac{v}{|v|}, v''\right) + \frac{|v \wedge v'|^2}{|v|^3}. \quad (39)$$

By replacing in (39) the vector v'' by its expression in (20), we find, after combining with (38), the equality

$$\frac{d}{dt}\left\{\frac{(v, v')}{|v|} - \frac{1}{a}\arctan\frac{|v|}{a}\right\} = -\frac{c_0}{|v|^2} + \frac{|v \wedge v'|^2}{|v|^3},$$

whence (22).

7 Conclusion

Under minimal assumptions on the function f, we have obtained that all bounded solutions of the general equation (2) converge to 0 for t large. This applies in particular to the model (4) and at least for all sufficiently small initial data. Ultimately, the rate of decay should be similar to (3). On the other hand, the linear expansion for the backward equation (20) makes the model more realistic than (3). However, the exponential rate of decay for positive time, as examined in [3] is not very realistic either, and if we want to avoid an unrealistic apocalypse prediction in 20 000 years or so, it is probably necessary to take account of some other phenomena precluding that electrons approach the nucleus too closely. It is in fact quite possible that the electro-static attraction becomes repulsive at short distances, cf [5] for some mathematical models in this direction.

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