

AN ERGODIC THEOREM FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT(0) SPACES*

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Abstract

In the framework of CAT(0) spaces, we prove that the mean of iterations of an asymptotically nonexpansive mapping with nonempty fixed point set converges weakly to a fixed point. The result extends the similar previous result for nonexpansive mappings in CAT(0) spaces [13] and generalizes the results of [11] from Hilbert spaces to Hadamard spaces.

Keywords: asymptotically nonexpansive, ergodic theorem, weak convergence, fixed point, CAT(0) space, Hadamard space.

MSC: 47H25, 47J25, 40A05.

1 Introduction

Let X be a normed linear space with norm $\|\cdot\|$ and let $C \subset X$ be a nonempty, closed, and convex set. A mapping $T : C \rightarrow C$ is called:

1. nonexpansive if for each $x, y \in C$, $\|Tx - Ty\| \leq \|x - y\|$

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2. asymptotically nonexpansive if for each $x, y \in C$

$$\|T^n x - T^n y\| \leq (1 + \alpha_n)\|x - y\|, \quad (1)$$

where α_n is a positive real sequence that converges to 0 as $n \rightarrow \infty$.

For a mapping $T : C \rightarrow C$, $x \in C$ is called a fixed point of T if $Tx = x$. The set of all fixed points of T is denoted by $\text{Fix}(T)$. The asymptotic behavior of iterations of nonexpansive mappings is a significant issue in nonlinear analysis. A simple example in \mathbb{R} illustrates that iterations of a nonexpansive mapping is not generally convergent. However, Baillon [4] demonstrated that if the mapping has a fixed point, or equivalently, if the sequence of its iterations is bounded, then the average of the iterations converges weakly to a fixed point of the mapping in a Hilbert space. This result is known as Baillon's nonlinear mean ergodic theorem, which serves as a nonlinear extension of the ergodic theorem of von Neumann [23] for linear mappings. Since then, several extensions of Baillon's theorem have been proposed, including extensions to Banach spaces by Beauzamy et al., Bruck, Reich and Hirano in [5–7, 12, 21, 22]. Another approach to extending this theorem involves replacing the nonexpansive mapping with a more general nonlinear mapping, such as asymptotically nonexpansive mappings, which were first defined by Goebel and Kirk in [10] to extend the Kirk fixed point theorem. Baillon's ergodic theorem has been further extended for asymptotically nonexpansive mappings by Hirano and Takahashi [11] in Hilbert spaces and by Oka [19] in Banach spaces.

Recently, numerous results across various branches of nonlinear analysis have been extended to nonlinear contexts such as Hadamard manifolds and CAT(0) spaces. These include fixed point theory for nonexpansive mappings, convex optimization, proximal algorithms, and nonlinear semigroup theory. The ergodic theory of nonexpansive mappings has been investigated in [1, 2, 13–15, 17] for CAT(0) spaces. In this paper, we establish an ergodic theorem for asymptotically nonexpansive mappings in CAT(0) spaces.

Let (\mathcal{H}, d) be a metric space. A geodesic between two points x_0 and x_1 in \mathcal{H} is a mapping $\gamma : [0, d(x_0, x_1)] \rightarrow \mathcal{H}$ such that $\gamma(0) = x_0$, $\gamma(d(x_0, x_1)) = x_1$, and $d(\gamma(t), \gamma(t')) = |t - t'|$, $\forall t, t' \in [0, d(x_0, x_1)]$. A metric space in which any two points are connected by a geodesic is called a geodesic metric space. The image of a geodesic connecting two points x_0 and x_1 is denoted by $[x_0, x_1]$, which is called a segment between x_0 and x_1 . Each point $x_t \in [x_0, x_1]$, satisfying $d(x_t, x_0) = td(x_0, x_1)$ and $d(x_t, x_1) = (1 - t)d(x_0, x_1)$ for some $t \in [0, 1]$, is represented as $(1 - t)x_0 \oplus tx_1$. A subset C of \mathcal{H} is called convex if for every $x, y \in C$, the segment $[x, y] \subset C$. For a subset C of \mathcal{H} , the

closed convex hull of C is denoted by $\overline{\text{co}}C$, which is the intersection of all closed and convex sets containing C . A geodesic metric space whose metric d satisfies the following condition for each metric segment $[x_0, x_1]$ and any arbitrary point $y \in \mathcal{H}$ is known as a CAT(0) space.

$$(d(x_t, y))^2 \leq (1-t)(d(x_0, y))^2 + t(d(x_1, y))^2 - t(1-t)(d(x_0, x_1))^2. \quad (2)$$

From this point until the end of the paper, we will denote $(d(x, y))^2$ by $d^2(x, y)$. A complete CAT(0) space is referred to as a Hadamard space. A Hadamard space \mathcal{H} has the $\overline{Q_4}$ property if, for any arbitrary points $x, y, p, q \in \mathcal{H}$ and any point m in the segment $[x, y]$, the following inequality holds:

$$d(p, x) \leq d(q, x) \ \& \ d(p, y) \leq d(q, y) \ \Rightarrow \ d(p, m) \leq d(q, m).$$

Hilbert spaces, \mathbb{R} -trees, and CAT(0) spaces of constant curvature all possess the $\overline{Q_4}$ property (see [2, 16]). It is easily seen that the $\overline{Q_4}$ property implies that the set $F(x, y) := \{z \in \mathcal{H} : d(x, z) \leq d(y, z)\}$ is convex for all $x, y \in \mathcal{H}$.

Let (\mathcal{H}, d) be a Hadamard space, and let $\{x_n\}$ be a sequence in \mathcal{H} . We define

$$r(x, \{x_n\}) = \limsup_{n \rightarrow +\infty} d(x, x_n), \quad x \in \mathcal{H}.$$

The asymptotic radius of the sequence $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf_{x \in \mathcal{H}} r(x, \{x_n\}),$$

and the asymptotic center of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in \mathcal{H} : r(\{x_n\}) = r(x, \{x_n\})\}.$$

According to [8, Proposition 7], $A(\{x_n\})$ is a singleton in Hadamard spaces. A sequence $\{x_n\}$ in a Hadamard space \mathcal{H} is said to converge weakly to $x \in \mathcal{H}$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ [8, 9, 16]. In this case, we denote this weak convergence as $x_n \rightharpoonup x$.

Let C be a closed convex subset of a Hadamard space \mathcal{H} . For an arbitrary mapping $T : C \rightarrow C$ and $x \in C$, the mean of the iterates $T^k x, \dots, T^{n+k-1} x$ is denoted by $\sigma_n^k x$ and is defined as follows:

$$\sigma_n^k x := \text{Argmin}_{y \in \mathcal{H}} \sum_{i=k}^{n+k-1} d^2(T^i x, y), \quad \sigma_n x := \sigma_n^0 x. \quad (3)$$

Lemma 1. [2, 13] For $\sigma_n^k x$ defined above and an arbitrary $y \in \mathcal{H}$, we have:

1. $d^2(\sigma_n^k x, y) \leq \frac{1}{n} \sum_{i=0}^{n-1} d^2(T^{k+i} x, y) - \frac{1}{n} \sum_{i=0}^{n-1} d^2(T^{k+i} x, \sigma_n^k x)$
2. $d(\sigma_n^k x, y) \leq \frac{1}{n} \sum_{i=0}^{n-1} d(T^{k+i} x, y).$

Nonexpansive and asymptotically nonexpansive mappings are defined in metric spaces similarly to how they are defined in normed spaces, by replacing the norm with the corresponding metric.

2 Main result

The following theorem is the main result of the paper.

Theorem 1. *Let \mathcal{H} be a Hadamard space with the $\overline{Q_4}$ property, C be a closed convex subset of \mathcal{H} , and $T : C \rightarrow C$ be an asymptotically nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. Then, for each $x \in C$, the sequence $\sigma_n x$ (defined in (3)) converges weakly to a fixed point of T , which is the asymptotic center of the sequence $\{T^k x\}$.*

To prove Theorem 1, we first need to establish some lemmas.

Lemma 2. *Assume that the conditions of Theorem 1 on C and T hold. Then, for each $x \in C$ and $y \in \text{Fix}(T)$, we have the following:*

1. $\lim_{n \rightarrow +\infty} d(T^n x, y)$ exists.
2. The sequence $\{\sigma_n^k\}_n$ is bounded for $k = 0, 1, \dots$.
3. $d(\sigma_n x, \sigma_n^k x) \rightarrow 0$ as $n \rightarrow +\infty$ for $k = 1, 2, \dots$.
4. $\sigma_n^k x \in \overline{\text{co}}\{T^m x\}_{m \geq k}$ for all $n \geq 1$.

Proof. **Part 1.** For all $y \in \text{Fix}(T)$, we have

$$d(T^{m+n} x, y) = d(T^m T^n x, y) \leq (1 + \alpha_m) d(T^n x, y).$$

Taking the limsup as $m \rightarrow +\infty$ and the liminf as $n \rightarrow +\infty$, we conclude that $\lim d(T^n x, y)$ exists.

Part 2. This follows directly from Part 1 of this lemma and Part 2 of Lemma 1.

Parts 3,4. The proofs for these parts are identical to those in Parts (iii) and (iv) of [13, Lemma 2.3]. \square

Lemma 3. *Let C and T satisfy the assumptions of Theorem 1. Then we have:*

$$\limsup_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} d(\sigma_n x, T^m \sigma_n x) = 0 \quad (4)$$

Proof. In Part 1 of Lemma 1 with $k = 0$, let $y = T^m \sigma_n x$, where $m \leq n$. Then, we have:

$$\begin{aligned}
d^2(\sigma_n, T^m \sigma_n x) &\leq \frac{1}{n} \sum_{i=0}^{m-1} d^2(T^i x, T^m \sigma_n x) + \frac{1}{n} \sum_{i=m}^{n-1} d^2(T^i x, T^m \sigma_n x) \\
&\quad - \frac{1}{n} \sum_{i=0}^{n-1} d^2(T^i x, \sigma_n x) \\
&\leq \frac{1}{n} \sum_{i=0}^{m-1} d^2(T^i x, T^m \sigma_n x) + (1 + \alpha_m)^2 \frac{1}{n} \sum_{i=0}^{n-m-1} d^2(T^i x, \sigma_n x) \\
&\quad - \frac{1}{n} \sum_{i=0}^{n-1} d^2(T^i x, \sigma_n x) \\
&\leq \frac{1}{n} \sum_{i=0}^{m-1} d^2(T^i x, T^m \sigma_n x) + (\alpha_m^2 + 2\alpha_m) \frac{1}{n} \sum_{i=0}^{n-m-1} d^2(T^i x, \sigma_n x)
\end{aligned}$$

Taking lim sup as $n \rightarrow +\infty$, we obtain

$$\limsup_{n \rightarrow +\infty} d^2(\sigma_n x, T^m \sigma_n x) \leq (\alpha_m^2 + 2\alpha_m) D,$$

where $D = \sup_{n, i \geq 0} d^2(T^i x, \sigma_n x) < +\infty$ (by Lemma 2, the sequences $\{T^k x\}$ and $\{\sigma_n x\}$ are bounded). Now, taking the lim sup as $m \rightarrow \infty$, we get (3). \square

Lemma 4. *Let the assumptions of Theorem 1 regarding C and T be satisfied. If the subsequence $\sigma_{n_j} x$ of $\sigma_n x$ converges weakly to a point y , then $y \in \text{Fix}(T)$.*

Proof. To prove this, it is sufficient to show that $T^k y \rightarrow y$. By (2), the triangle inequality, and asymptotic nonexpansiveness of T , we have:

$$\begin{aligned}
&d^2(\sigma_n x, \frac{1}{2} y \oplus \frac{1}{2} T^k y) \\
&\leq \frac{1}{2} d^2(\sigma_n x, y) + \frac{1}{2} d^2(\sigma_n x, T^k y) - \frac{1}{4} d^2(y, T^k y) \\
&\leq \frac{1}{2} d^2(\sigma_n x, y) + \frac{1}{2} (d(\sigma_n x, T^k \sigma_n x) + d(T^k \sigma_n x, T^k y))^2 - \frac{1}{4} d^2(y, T^k y) \\
&\leq \frac{1}{2} d^2(\sigma_n x, y) + \frac{1}{2} (d(\sigma_n x, T^k \sigma_n x) + (1 + \alpha_k) d(\sigma_n x, y))^2 - \frac{1}{4} d^2(y, T^k y).
\end{aligned}$$

Substituting n by n_j in the above expression and taking the limit superior as $j \rightarrow +\infty$, we obtain:

$$\begin{aligned}
& \limsup_{j \rightarrow +\infty} d^2(\sigma_{n_j}x, y) + \frac{1}{4}d^2(y, T^k y) \\
& \leq \limsup_{j \rightarrow +\infty} d^2(\sigma_{n_j}x, \frac{1}{2}y \oplus \frac{1}{2}T^k y) + \frac{1}{4}d^2(y, T^k y) \\
& \leq \left(\frac{1}{2} + \frac{1}{2}(1 + \alpha_k)^2\right) \limsup_{j \rightarrow +\infty} d^2(\sigma_{n_j}x, y) \\
& \quad + \frac{1}{2} \limsup_{n \rightarrow +\infty} d^2(\sigma_n x, T^k \sigma_n x) + M(1 + \alpha_k) \limsup_{n \rightarrow +\infty} d(\sigma_n x, T^k \sigma_n x),
\end{aligned}$$

where $M := \sup_{n \geq 0} d(\sigma_n x, y)$. Taking the limit superior as $k \rightarrow +\infty$, and applying Lemma 3, we arrive at the desired result. \square

Proof of Theorem 1. Since, by Part 2 of Lemma 2, the sequence $\{\sigma_n x\}$ is bounded, there exists a subsequence $\sigma_{n_i} x$ of $\sigma_n x$ that converges weakly to a point $y \in \text{Fix}(T)$ (as stated in Lemma 4). We will prove that y is the asymptotic center of the sequence $\{T^k x\}$. Let x_0 be the asymptotic center of $\{T^k x\}$. If $y \neq x_0$, then, by Part 1 of Lemma 2, we have:

$$\limsup_k d(T^k x, x_0) < \limsup_k d(T^k x, y) = \lim_k d(T^k x, y).$$

Therefore, there exists $k_0 > 0$ such that for each $k \geq k_0$, we have $T^k x \in F(x_0, y) = \{z \in C : d(z, x_0) \leq d(z, y)\}$. The closedness and convexity of $F(x_0, y)$ imply that $\overline{\text{co}}\{T^k x : k \geq k_0\} \subset F(x_0, y)$. Since y does not belong to $F(x_0, y)$, it follows that $y \notin \overline{\text{co}}\{T^k x : k \geq k_0\}$. However, by Lemma 4 and by Parts 3 and 4 of Lemma 2, we know that $y \in \text{Fix}(T) \cap \overline{\text{co}}\{T^k x : k \geq k_0\}$. This leads to a contradiction, demonstrating that every weak clusters point of $\{\sigma_n x\}$ is the asymptotic center of $\{T^k x\}$. Therefore, we conclude that $\sigma_n x \rightharpoonup y \in \text{Fix}(T)$ as $n \rightarrow \infty$. \square

Remark 1. *If the conditions of Theorem 1 on T and C are satisfied, and if the mapping T is asymptotically regular, meaning that $d(T^{n+1}x, T^n x) \rightarrow 0$ as $n \rightarrow +\infty$ for each $x \in C$, then $T^n x$ converges weakly to an element of $\text{Fix}(T)$.*

To see why this is true, let y be a weak convergence point of a subsequence $T^{n_j} x$ of $T^n x$. By the asymptotic regularity of T , we have $d(TT^{n_j} x, T^{n_j} x) \rightarrow 0$ as $j \rightarrow +\infty$. The demiclosed principle for asymptotically nonexpansive mappings in $\text{CAT}(0)$ spaces [18] implies that $y \in \text{Fix}(T)$. Furthermore, by Lemma 2, $\lim_{n \rightarrow +\infty} d(T^n x, y)$ exists for each $y \in \text{Fix}(T)$. Opial's lemma in $\text{CAT}(0)$ spaces [20, Lemma 2.1] then shows that $T^n x \rightharpoonup y \in \text{Fix}(T)$ as $n \rightarrow +\infty$.

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