Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 18, No. 2/2026

A SIMPLE CONSTRUCTION OF KIRKMAN TRIPLE SYSTEMS OF ORDER 3^h*

Antonio Causa[†] Leonardo Fragapane[‡] Mario Gionfriddo[§]
Elena Guardo[¶]

Communicated by G. Failla

DOI 10.56082/annalsarscimath.2026.2.225

Abstract

A Steiner triple system (STS) of order v is a 3-uniform hypergraph with v vertices in which every 2-subset of vertices has degree 1. A Kirkman triple system (KTS) is a resolvable Steiner triple system, that is, a partition of the blocks of the triple system into classes which are themselves partitions of the set of vertices into disjoint blocks. In this paper we give a construction of KTS of order $v = 3^h$ much simpler and less technical than previously known constructions.

Keywords: Kirkman triple systems, resolvable designs.
MSC: 05B07, 05B10.

1 Introduction

A Steiner system $S(h, k, v)$ is a pair $\Sigma = (X, \mathcal{B})$, where X is a v-set and $\mathcal{B}$ is a family of k-subsets of X such that every h-subset of X is contained in

---

[*] Accepted for publication on March 03, 2026

[†] `causa@dmi.unict.it`, Dipartimento di Matematica e Informatica, University of Catania, Viale A. Doria, 6 - 95100 - Catania, Italy

[‡] `leonardo.fragapane@phd.unict.it`, Dipartimento di Matematica e Informatica, University of Catania, Viale A. Doria, 6 - 95100 - Catania, Italy

[§] `gionfriddo@dmi.unict.it`, Dipartimento di Matematica e Informatica, University of Catania, Viale A. Doria, 6 - 95100 - Catania, Italy

[¶] `guardo@dmi.unict.it`, Dipartimento di Matematica e Informatica, University of Catania, Viale A. Doria, 6 - 95100 - Catania, Italy

exactly one member of $\mathcal{B}$ [4,6,10,11]. Using Hypergraph Theory terminology, a Steiner system is a hypergraph $\Sigma = (X, \mathcal{B})$ of order v, uniform of rank k, such that every h-subset $Y \subseteq X$ has degree $d(Y) = 1$ [1,14]. Steiner systems $S(h, k, v)$ were defined for the first time by Woolhouse in 1844 [15], who asked for which positive integers h, k, v an $S(h, k, v)$ exists. This problem remains unsolved in general until today, even if many partial results have been given. A Steiner Triple System (STS) is a system $S(2, 3, v)$, in 1847 T. Kirkman [9] and J. Steiner [13], independently, showed that *an* $STS(v)$ *exists if and only if* $v \equiv 1 \text{ or } 3 \pmod 6$.

Actually, there is an additional parameter to be considered, the so-called *index*. A Steiner system $S_\lambda(h, k, v)$ is a pair $\Sigma = (X, \mathcal{B})$, where X is a v-set and $\mathcal{B}$ is a family of k-subset of X such that every h-subset of X is contained in exactly λ members of $\mathcal{B}$. The first definition of Steiner system was given considering $\lambda = 1$

Other results have been determined by H. Hanani about the spectrum of $S(3, 4, v)$ and $S(2, 4, v)$, respectively in 1960 [7] and in 1962 [8]. After the solution on the spectrum of STS, the famous *fifteen schoolgirl problem* was posed. It is a problem proposed by Thomas Penyngton Kirkman in 1850 as Query VI in *The Lady's and Gentleman's Diary* (pag.48). The problem states:

"*Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.*"

A solution to this problem is an example of a Kirkman triple system (KTS) [10] which is a Steiner triple system having a parallelism, that is, a partition of the blocks of the triple system into parallel classes which are themselves partitions of the points into disjoint blocks. Such Steiner systems that have a parallelism are also called resolvable and the partition is called a resolution of the systems.

Soon after the problem about the fifteen schoolgirls, in 1851 T. Kirkman determined the spectrum of all the KTS proving that: a KTS exists if and only if $v \equiv 3 \pmod 6$.

Many constructions of Kirkman triple systems of all admissible orders are already known in the literature, and this research still attracts a lot of interest (see [2,3,5,12,16] just to cite some of them). In this paper we give a construction of KTS having order $v = 3^h$, using the so-called method of differences, which allows us to obtain a construction that is much simpler and less technical than those previously available in the literature.

2 Partition of differences

Let $v = 3^h$. Consider $\mathbb{Z}_v$ and define

$$D_v = \left\{1, 2, \ldots, \frac{v-1}{2}\right\}$$

which is called *set of differences in* $\mathbb{Z}_v$.

Let $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r\}$ be the partition of D_v such that:

$$\forall\ i = 1, \ldots, r \qquad a \in \mathcal{F}_i \iff a \text{ is multiple of } 3^{i-1} \text{ and } a \text{ not multiple of } 3^i$$

Theorem 1. *If* $\mathcal{F}_i = \{a_{i,1}, \ldots, a_{i,s_i}\}$*, for* $i = 1, \ldots, r$*, and* $a_{i,1} = \min_j a_{i,j}$*, then:*

1. $a_{i,1} = 3^{i-1}$*;*
2. *we define* $a_{i,j+1}$ *as follows,*

$$a_{i,j+1} = \begin{cases} 2a_{i,j} & \text{if } 2a_{i,j} \le \frac{v-1}{2} \\ v - 2a_{i,j} & \text{if } 2a_{i,j} > \frac{v-1}{2} \end{cases};$$

3. $|\mathcal{F}_i| = 3^{h-i}$*;*
4. $|\mathcal{F}| = h$*.*

Proof. 1. By definition, 3^{i-1} is the minimum of $\mathcal{F}_i$, then $a_{i,1} = 3^{i-1}$.

2. The statement follows considering that $v = 3^h$ and $a_{i,j+1}$ is a multiple of 3^{i-1} if and only if both $2a_{i,j}$ and $v - 2a_{i,j}$ are multiple of 3^{i-1} (same for the other condition).

3. By definition, $\mathcal{F}_h = \{3^{h-1}\}$ and $|\mathcal{F}_h| = 1 = 3^0$.

   - Consider $\mathcal{F}_{h-1}$.
     There are nine multiples of 3^{h-2} in $\mathbb{Z}_v$, namely $3^{h-2}, 2{\cdot}3^{h-2}, \ldots, 9 \cdot 3^{h-2}$:
     - one of them is v;
     - among the remaining eight, four belong to D_v while the other four belong to $\mathbb{Z}_v \setminus D_v$;
     - among the four elements in D_v, there is $3^{h-1} \in \mathcal{F}_h$.

     Therefore, $|\mathcal{F}_{h-1}| = 3 = 3^1$

- $\mathcal{F}_i$, for $1 \leq i \leq h-2$.
  There are 3^{h-i+1} multiples of 3^{i-1} in $\mathbb{Z}_v$:
  - one of them is v;
  - among the remaining $3^{h-i+1}-1$, $\frac{3^{h-i+1}-1}{2}$ belong to D_v while the other $\frac{3^{h-i+1}-1}{2}$ belong to $\mathbb{Z}_v \setminus D_v$;
  - among the $\frac{3^{h-i+1}-1}{2}$ elements in D_v, there are also the ones which already belong to the sets $\mathcal{F}_k$, for $k > i$.

  Therefore,

$$\begin{aligned} |\mathcal{F}_i| &= \frac{3^{h-i+1}-1}{2} - \sum_{j=i+1}^{h} |\mathcal{F}_j| = \\ &= \frac{(3-1)(3^{h-i}+ ... +1)}{2} - \sum_{j=i+1}^{h} 3^{h-j} = \\ &= (3^{h-i}+ ... +1) - (3^{h-i-1}+ ... +1) = 3^{h-i}. \end{aligned}$$

4. By definition, $\mathcal{F}_i \neq \emptyset$ for $i = 1, \ldots, h$ while $\mathcal{F}_{h+1} = \emptyset$. □

Observe that $a_{h,2} = v - 2a_{h,1} = 3^h - 2 \cdot 3^{h-1} = 3^{h-1} = a_{h,1}$.

3 Base blocks and a system $S_3(2, 3, v)$

From now on, consider $\mathbb{Z}_v = \{0, 1, \ldots, v-1\}$. For every $a \in D_v$, define the *base block of* a, $\mathcal{B}_a$, as:

$$\mathcal{B}_a = \{v - a, 0, a\}.$$

$\mathcal{B}_a$ is either associated with the triple of differences $\{a, a, 2a\}$ if $2a \leq \frac{v-1}{2}$ or to $\{a, a, v-2a\}$ if $2a > \frac{v-1}{2}$.
If $a = a_{h,1}$, the differences for the base block are $\{a, a, a\}$, making it the one and only base block having $a_{h,1}$ as a difference (all the others appear in two of them).
For every $a \in D_v$, let Γ_a be the *set of traslates of* $\mathcal{B}_a$,

$$\Gamma_a = \{\mathcal{B}_{a,k} = \{v - a + k, k, a + k\} \subseteq \mathbb{Z}_v : \ a \in D_v, \ k = 0, \ldots, v-1\}.$$

Theorem 2. *If* $X = \mathbb{Z}_v$ *and* $\mathcal{B} = \bigcup_{a \in D_v} \Gamma_a$, *then* $\Sigma = (X, \mathcal{B})$ *is an* $S_3(2, 3, v)$, *for* $v = 3^h$.

Proof. It is immediate that Σ is a uniform hypergraph of rank 3, with order equal to $v = 3^h$. It remains to prove that every pair of distinct elements of X, let it be $\{x, y\}$ with $x < y$, has degree equal to 3, $d(x, y) = 3$.
Let a be the difference between x and y, $a = y-x$ or $a = v-(y-x)$ based on which one belongs to D_v; consider the base block $\mathcal{B}_a = \mathcal{B}_{a,0} = \{v - a, 0, a\}$:

- if $a = y - x$, $\mathcal{B}_a = \{v + x - y, 0, y - x\} \subseteq \mathbb{Z}_v$;
- if $a = v - (y - x)$, $\mathcal{B}_a = \{v - v + y - x, 0, v + x - y\} = \{y - x, 0, v + x - y\} \subseteq \mathbb{Z}_v$.

Independently from the expression of a, the base block is $\{v+x-y, 0, y-x\}$: assume that $a = y - x \in D_v$.
At first, let $a \neq \frac{v}{3}$: in Γ_a there are

$$\mathcal{B}_{a,x} = \{v + 2x - y, x, y\} \quad \mathcal{B}_{a,y} = \{x, y, 2y - x\}.$$

In this case, $v + 2x - y \neq 2y - x$, hence $\mathcal{B}_{a,x} \neq \mathcal{B}_{a,y}$.

Consider now another base block (depending from the value of a):

- if a is even, $\mathcal{B}_{\frac{a}{2}} = \mathcal{B}_{\frac{a}{2},0} = \{v - \frac{y-x}{2}, 0, \frac{y-x}{2}\} \subseteq \mathbb{Z}_v$;
- if a is odd, $\mathcal{B}_{\frac{v-a}{2}} = \mathcal{B}_{\frac{v-a}{2},0} = \{\frac{v+y-x}{2}, 0, \frac{v-y+x}{2}\} \subseteq \mathbb{Z}_v$.

It is important to specify that all the operations are done in $\mathbb{Z}_v$.
For each case, we are interested in a specific translate:

- if a is even, $\mathcal{B}_{\frac{a}{2},\frac{x+y}{2}} = \{v + x, \frac{x+y}{2}, y\} = \{x, \frac{x+y}{2}, y\}$;
- if a is odd, $\mathcal{B}_{\frac{v-a}{2},\frac{v+y+x}{2}} = \{v + y, \frac{v+y+x}{2}, v + x\} = \{y, \frac{v+y+x}{2}, x\}$.

This proves that $d(x, y) \geq 3$.

The last case is $a = \frac{v}{3} = 3^{h-1}$, that is the only element of $\mathcal{F}_h$: the base block is $\mathcal{B}_{\frac{v}{3}} = \mathcal{B}_{\frac{v}{3},1} = \{2 \cdot \frac{v}{3}, 0, \frac{v}{3}\}$.
Consider now these three translates:

$$\mathcal{B}_{\frac{v}{3},x} = \{2 \cdot \frac{v}{3} + x, x, y\};$$
$$\mathcal{B}_{\frac{v}{3},y} = \{x, y, \frac{v}{3} + y\};$$
$$\mathcal{B}_{\frac{v}{3},y+\frac{v}{3}} = \{y, y + \frac{v}{3}, x\}.$$

This proves that $d(x,y) \geq 3$ even in this case.
In order to prove that $d(x,y) = 3$, it would be sufficient to consider that the difference a does not appear in any other base block.
However, we will confirm this result calculating the cardinality of $\mathcal{B} = \bigcup_{a \in D_v} \Gamma_a$:

$$|\mathcal{B}| = \sum_{a \in D_v} |\Gamma_a| = \frac{v-1}{2} \cdot v = 3 \cdot \frac{v(v-1)}{6},$$

exactly the number of blocks of an $S_3(2,3,v)$. □

4 Construction of $KTS(3^h)$

In the sequel, we use the following notation. Given a base block

$$\mathcal{B}_a = \{v-a, 0, a\},$$

written in this order, we will say that the element 0 is the *central* vertex of $\mathcal{B}_a$, the element a is its *right* vertex and the element $v-a$ is its *left* vertex.
For every $i = 1, \dots, h$, associate with $\mathcal{F}_i$ the set $\mathcal{U}_i \subseteq \mathbb{Z}_v$ defined as follows:

$$\begin{array}{rcl} \mathcal{F}_h & \rightarrow & \mathcal{U}_h = \{0, 1, \dots, 3^{h-1}-1\} \\ & = & \{u : u \equiv 0, 1, \dots, 3^{h-1}-1 \pmod{3^h}\} \\ \mathcal{F}_{h-1} & \rightarrow & \mathcal{U}_{h-1} = \{u : u \equiv 0, 1, \dots, 3^{h-2}-1 \pmod{3^{h-1}}\} \\ & \vdots & \\ \mathcal{F}_i & \rightarrow & \mathcal{U}_i = \{u : u \equiv 0, 1, \dots, 3^{i-1}-1 \pmod{3^i}\} \\ & \vdots & \\ \mathcal{F}_3 & \rightarrow & \mathcal{U}_3 = \{u : u \equiv 0, 1, \dots, 8 \pmod{27}\} \\ \mathcal{F}_2 & \rightarrow & \mathcal{U}_2 = \{u : u \equiv 0, 1, 2 \pmod{9}\} \\ \mathcal{F}_1 & \rightarrow & \mathcal{U}_1 = \{u : u \equiv 0 \pmod{3}\} \end{array}$$

Each difference $a \in D_v$ belongs to a single set $\mathcal{F}_i$ and detects a base block $\mathcal{B}_a = \{v-a, 0, a\}$.

Associate with each $i = 1, \dots, h$ and $a \in \mathcal{F}_i$ the following set:

$$\mathcal{C}_{a,i} = \{\mathcal{B}_{a,u} = \{v-a+u, u, a+u\} : u \in \mathcal{U}_i\}.$$

Observe that the total amount of these sets is

$$\sum_{i=1}^{h} |\mathcal{F}_i| = \sum_{i=1}^{h} 3^{h-i} = \frac{v-1}{2}.$$

Theorem 3. *Let* $\mathcal{C}_i = \bigcup_{a \in \mathcal{F}_i} \mathcal{C}_{a,i}$*. If* $X = \mathbb{Z}_v$ *and* $\mathcal{B} = \bigcup_{i=1,2,\ldots,h} \mathcal{C}_i$*, then* $\Sigma = (X, \mathcal{B})$ *is a* STS *of order* $v = 3^h$*.*

Proof. It is immediate that Σ is a uniform hypergraph of rank 3, with order equals to $v = 3^h$. It remains to prove that every pair of distinct elements of $\mathbb{Z}_v$, let it be $\{x, y\}$ with $x < y$, appears in exactly one block.
Let a be the difference between x and y, $a = y - x$ or $a = v - (y - x)$. Based on which one belongs to D_v, consider the base block $\mathcal{B}_a = \mathcal{B}_{a,0} = \{v - y + x, 0, y - x\}$.
There exists a single i such that $a \in \mathcal{F}_i$. Consider

$$\mathcal{U}_i = \{u : u \equiv 0, 1, \ldots, 3^{i-1} - 1 \pmod{3)^i}\}.$$

- If $x \in \mathcal{U}_i$, the pair $\{x, y\}$ is contained in the block $\mathcal{B}_{a,x} = \{v - y + 2x, x, y\}$.
- If $y \in \mathcal{U}_i$, the pair $\{x, y\}$ is contained in the block $\mathcal{B}_{a,y} = \{x, y, 2y - x\}$.
- If both $x, y \notin \mathcal{U}_i$, then they are not central elements of the block containing the pair. This means that we have to look at the *previous* difference of a, which is $a/2$.
  If a is even, the pair $\{x, y\}$ is contained in the block $\mathcal{B}_{\frac{a}{2}, x+\frac{a}{2}} = \{x, x + \frac{a}{2}, y\}$.
  If a is odd, then it exists at least an $\alpha \in \mathcal{F}_i$ such that $a = v - 2\alpha$. It follows that the pair $\{x, y\}$ is contained in the block $\mathcal{B}_{\alpha, y+\alpha} = \{y, y+\alpha, x\}$.

Therefore, it always exists at least a block containing any given pair of vertices.
In order to prove the uniqueness of the block, we proceed by counting the blocks of the entire system:

$$|\mathcal{B}| = \sum_{i=1}^{h} |\mathcal{F}_i| \cdot \frac{v}{3} = (3^{h-1} + \ldots + 3 + 1) \cdot (3 - 1) \cdot \frac{v}{6} = (3^h - 1) \cdot \frac{v}{6} = \frac{v \cdot (v-1)}{6},$$

which is exactly the number of blocks of an $STS(v)$. $\square$

Now, it remains to prove that the family

$$\Pi = \{\mathcal{C}_{a,i} :, i = 1, \ldots, h;\ a \in \mathcal{F}_i\}$$

is a *resolution* for the system $\Sigma = (\mathbb{Z}_v, \mathcal{B})$. This means that every class $\mathcal{C}_{a,i}$ of Π is a parallel class: in other words a class of pairwise disjoint blocks. At last, it will be proved that Σ is a $KTS(3^h)$.

Theorem 4. *The family* $\Pi = \{\mathcal{C}_{a,i} : i = 1, \ldots, h; \quad a \in \mathcal{F}_i\}$ *is a* resolution *for the system* $\Sigma = (\mathbb{Z}_v, \mathcal{B})$*. Hence,* Σ *is a* $KTS(3^h)$*.*

Proof. Consider any class $\mathcal{C} \in \Pi$: there exist an $i = 1, \ldots, h$ and an $a \in \mathcal{F}_i$ such that:

$$\mathcal{B}_{a,u} = \{v - a + u, u, a + u\} \in \mathcal{C}_{a,i} :- \mathcal{C}, \quad u \in \mathcal{U}_i.$$

Therefore, $\mathcal{C}$ contains all of the blocks of type:

$$\{x - a, x, x + a\},\ \forall x \in \mathcal{U}_i : a \in \mathcal{F}_i.$$

Now, we prove that $\mathcal{C}$ is a parallel class for the system $\Sigma = (\mathbb{Z}_v, \mathcal{B})$. First of all, consider that all the *central* vertices x of these blocks of $\mathcal{C}$ are all different among them and this implies that also all the *right* vertices are different amomg them and all the *left* vertices are different among them. We prove that also vertices of different type are different among them.
Let $x, y \in \mathcal{U}_i, x < y$, $a = y - x \in \mathcal{F}_i$. Then $x + a$ is a *right* vertex and $y - a$ is a *left* vertex.

We prove that:

1. *Every right vertex is different from every central vertex.*
   Indeed, if $x + a = y$, then in the the same class associated with the difference a they should be the blocks $\{x-a, x, x+a = y\}$ and $\{y-a = x, y, y+a\}$, both containing the pair $\{x, y\}$. It follows that: $x+a \neq y$.
2. *Every left vertex is different from every central vertex.*
   Indeed, if $y = x - a$, then in the the same class associated with the difference a they should be the blocks $\{x - a = y, x, x + a\}$ and $\{y - a, y = x - a, y + a = x\}$, both containing the pair $\{x, y\}$. It follows that: $x - a \neq y$.
3. *Every left vertex is different from every right vertex.*
   Indeed, if $y - a = x + a$ [or $x - a = y + a$], then $y = x + 2a$, $x = y - 2a$ and in the same class associated with the difference $2a \in \mathcal{F}_i$, there should be the blocks

   $$\{x - 2a, x, x + 2a = y\} \text{ and } \{y - 2a = x, y, y + 2a\},$$

   both containing the pair $\{x, y\}$. Observe that if the *successor* of a in $\mathcal{F}_i$ were $v - 2a$ instead of $2a$, the same situation would occur.

Therefore, in every class $\mathcal{C} \in \Pi$ there are $v/3$ pairwise disjoint blocks and Π is so a *resolution* of $\mathcal{B}$.
This proves that the system is a KTS of order $v = 3^h$. □

5 Construction of $\boldsymbol{KTS(3^h)}$ of small order

1) $\boldsymbol{KTS}$ **of order** $\boldsymbol{v = 9 = 3^2}$
Let $v = 9 = 3^2, h = 2$.
Let $X = \mathbb{Z}_9$, the set of vertices, and $D_9 = \{1, 2, 3, 4\}$.

Differences and central vertices

$\boldsymbol{i = 1}$

$\mathcal{F}_1 = \{1, 2, 4\} \longrightarrow \mathcal{U}_1 = \{(0) - (3) - (6)]\}$

Difference Triples $\mathcal{T}_1 :$ $(1, 1, 2), (2, 2, 4), (4, 4, 1)$

$\boldsymbol{i = 2}$

$\mathcal{F}_2 = \{3\} \longrightarrow \mathcal{U}_2 = \{(0, 1, 2)\}$

Difference Triples $\mathcal{T}_2 :$ $(3, 3, 3)$

Base blocks and derived blocks

Blocks from the differences in $(\mathcal{F}_1, \mathcal{U}_1)$:

$$\begin{array}{ccc} \mathbf{8 \cdot 0 \cdot 1} & \mathbf{7 \cdot 0 \cdot 2} & \mathbf{5 \cdot 0 \cdot 4} \\ 2 \cdot 3 \cdot 4 & 1 \cdot 3 \cdot 5 & 8 \cdot 3 \cdot 7 \\ 5 \cdot 6 \cdot 7 & 4 \cdot 6 \cdot 8 & 2 \cdot 6 \cdot 1 \end{array}$$

Blocks from the differences in $(\mathcal{F}_2, \mathcal{U}_2)$:

$$\begin{array}{c} \mathbf{6 \cdot 0 \cdot 3} \\ 7 \cdot 1 \cdot 4 \\ 8 \cdot 2 \cdot 5 \end{array}$$

2) $\boldsymbol{KTS}$ **of order** $\boldsymbol{v = 27 = 3^3}$
Let $v = 27 = 3^3, h = 3$. Let $X = \mathbb{Z}_{27}$ be the set of vertices, and $D_{27} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$

Differences and central vertices

$\boldsymbol{i = 1}$

$\mathcal{F}_1 = \{1, 2, 4, 8, 11, 5, 10, 7, 13\} \longrightarrow \mathcal{U}_1 = \{(0) - (3) - (6) - (9) - (12) -$

$(15) - (18) - (21) - (24)\}$

Difference Triples $\mathcal{T}_1$: $(1,1,2), (2,2,4), (4,4,8), (8,8,11), (11,11,5), (5,5,10), (10,10,7), (7,7,13), (13,13,1)$

***i* = 2**

$$\mathcal{F}_2 = \{3,6,12\} \longrightarrow \mathcal{U}_2 = \{(0-1-2)-(9-10-11)-(18-19-20)\}$$

Difference Triples $\mathcal{T}_2$: $(3,3,6), (6,6,12), (12,12,3)$

***i* = 3**

$$\mathcal{F}_3 = \{9\} \longrightarrow \mathcal{U}_3 = \{(0-1-2-3-4-5-6-7-8)\}$$

Difference Triples $\mathcal{T}_3$: $(9,9,9)$

Base blocks and derived blocks

Blocks from the differences in $(\mathcal{F}_1,\mathcal{U}_1)$:

26 · 0 · 1	**25 · 0 · 2**	**23 · 0 · 4**	**19 · 0 · 8**	**16 · 0 · 11**
2 · 3 · 4	1 · 3 · 5	26 · 3 · 7	22 · 3 · 11	19 · 3 · 14
5 · 6 · 7	4 · 6 · 8	2 · 6 · 10	25 · 6 · 14	22 · 6 · 17
8 · 9 · 10	7 · 9 · 11	5 · 9 · 13	1 · 9 · 17	25 · 9 · 20
11 · 12 · 13	10 · 12 · 14	8 · 12 · 16	4 · 12 · 20	1 · 12 · 23
14 · 15 · 16	13 · 15 · 17	11 · 15 · 19	7 · 15 · 23	4 · 15 · 26
17 · 18 · 19	16 · 18 · 20	14 · 18 · 22	10 · 18 · 26	7 · 18 · 2
20 · 21 · 22	19 · 21 · 23	17 · 21 · 25	13 · 21 · 2	10 · 21 · 5
23 · 24 · 25	22 · 24 · 26	20 · 24 · 1	20 · 24 · 5	13 · 24 · 8
22 · 0 · 5	**17 · 0 · 10**	**20 · 0 · 7**	**14 · 0 · 13**	
25 · 3 · 8	20 · 3 · 13	23 · 3 · 10	17 · 3 · 16	
1 · 6 · 11	23 · 6 · 16	26 · 6 · 13	20 · 6 · 19	
4 · 9 · 14	26 · 9 · 19	2 · 9 · 16	23 · 9 · 22	
7 · 12 · 17	2 · 12 · 22	5 · 12 · 19	26 · 12 · 25	
10 · 15 · 20	5 · 15 · 25	8 · 15 · 22	2 · 15 · 1	
13 · 18 · 23	8 · 18 · 1	11 · 18 · 25	5 · 18 · 4	
16 · 21 · 26	11 · 21 · 4	14 · 21 · 1	8 · 21 · 7	
19 · 24 · 2	14 · 24 · 7	17 · 24 · 4	11 · 24 · 10	

Blocks from the differences in $(\mathcal{F}_2, \mathcal{U}_2)$:

$\mathbf{24 \cdot 0 \cdot 3}$	$\mathbf{21 \cdot 0 \cdot 6}$	$\mathbf{15 \cdot 0 \cdot 12}$
$25 \cdot 1 \cdot 4$	$22 \cdot 1 \cdot 7$	$16 \cdot 1 \cdot 13$
$26 \cdot 2 \cdot 5$	$23 \cdot 2 \cdot 8$	$17 \cdot 2 \cdot 14$
$6 \cdot 9 \cdot 12$	$3 \cdot 9 \cdot 15$	$24 \cdot 9 \cdot 21$
$7 \cdot 10 \cdot 13$	$4 \cdot 10 \cdot 16$	$25 \cdot 10 \cdot 22$
$8 \cdot 11 \cdot 14$	$5 \cdot 11 \cdot 17$	$26 \cdot 11 \cdot 23$
$15 \cdot 18 \cdot 21$	$12 \cdot 18 \cdot 24$	$6 \cdot 18 \cdot 3$
$16 \cdot 19 \cdot 22$	$13 \cdot 19 \cdot 25$	$7 \cdot 19 \cdot 4$
$17 \cdot 20 \cdot 23$	$14 \cdot 20 \cdot 26$	$8 \cdot 20 \cdot 5$

Blocks from the differences in $(\mathcal{F}_3, \mathcal{U}_3)$:

$\mathbf{18 \cdot 0 \cdot 9}$
$19 \cdot 1 \cdot 10$
$20 \cdot 2 \cdot 11$
$21 \cdot 3 \cdot 12$
$22 \cdot 4 \cdot 13$
$23 \cdot 5 \cdot 14$
$24 \cdot 6 \cdot 15$
$25 \cdot 7 \cdot 16$
$26 \cdot 8 \cdot 17$

3) **KTS of order $v = 81 = 3^4$**
Let $v = 81 = 3^4, h = 4$.
Let $X = \mathbb{Z}_{81}$ be the set of vertices, and $D_{81} = \{1, 2, 3, 4, \ldots, 38, 39, 40\}$

Difference triples and central vertices

$i = 1$
$\mathcal{F}_1 \longrightarrow \quad \mathcal{U}_1$ with

$\mathcal{F}_1 = \{1, 2, 4, 8, 16, 32, 17, 34, 13, 26, 29, 23, 35, 11, 22, 37, 7, 14, 28, 25, 31, 19, 38, 5, 10, 20, 40\}$
and
$\mathcal{U}_1 = \{(0) - (3) - (6) - (9) - (12) - (15) - (18) - (21) - (24) - (27) - (30) - (33) - (36) - (39) - (42) - (45) - (48) - (51) - (54) - (57) - (60) - (63) - (66) - (69) - (72) - (75) - (78)\}$

Differences Triples $\mathcal{T}_1$:
$(1, 1, 2), (2, 2, 4), (4, 4, 8), (8, 8, 16), (16, 16, 32), (32, 32, 17), (17, 17, 34), (34, 34, 13),$

$(13, 13, 26), (26, 26, 29), (29, 29, 23), (23, 23, 35), (35, 35, 11), (11, 11, 22), (22, 22, 37),$
$(37, 37, 7), (7, 7, 14), (14, 14, 28), (28, 28, 25), (25, 25, 31), (31, 31, 19), (19, 19, 38),$
$(38, 38, 5), (5, 5, 10), (10, 10, 20), (20, 20, 40), (40, 40, 1)$

$\boldsymbol{i = 2}$
$\mathcal{F}_2 \longrightarrow \quad \mathcal{U}_2$ with

$\mathcal{F}_2 = \{3, 6, 12, 15, 21, 24, 30, 33, 39\}$
and

$\mathcal{U}_2 = \{(0-1-2)-(9-10-11)-(18-19-20)-(27-28-29)-$
$(36-37-38)-(45-46-47), (54-55-56)-(63-64-65)-(72-73-74)\}$

Differences Triples $\mathcal{T}_2$:
$(3, 3, 6), (6, 6, 12), (12, 12, 24), (24, 24, 33), (33, 33, 15),$
$(15, 15, 30), (30, 30, 21), (21, 21, 39), (39, 39, 3)$

$\boldsymbol{i = 3}$
$\mathcal{F}_3 \longrightarrow \quad \mathcal{U}_3$ with

$\mathcal{F}_3 = \{9, 18, 36\}$
and
$\mathcal{U}_3 = \{(0-1-2-3-4-5-6-7-8)-(27-28-29-30-$
$31-32-33-34-35), (54-55-56-57-58-59-60-61-62)\}$

Differences Triples $\mathcal{T}_3$:
$(9, 9, 18), (18, 18, 36), (36, 36, 9)$

$\boldsymbol{i = 4}$
$\mathcal{F}_4 \longrightarrow \quad \mathcal{U}_4$ with

$\mathcal{F}_4 = \{27\}$
and
$\mathcal{U}_4 = \{(0-1-2-3-4-5-6-7-8-9-10-11-12-13-14-15-$

$16-17-18-19-20-21-22-23-24-25-26)\}$

Differences Triples $\mathcal{T}_4$:
$(27, 27, 27)$

Base blocks and derived blocks

Blocks from the differences in $(\mathcal{F}_1, \mathcal{U}_1)$:

$$\begin{array}{ccccc}
\mathbf{80 \cdot 0 \cdot 1} & \mathbf{79 \cdot 0 \cdot 2} & \mathbf{77 \cdot 0 \cdot 4} & \cdots & \mathbf{41 \cdot 0 \cdot 40} \\
2 \cdot 3 \cdot 4 & 1 \cdot 3 \cdot 5 & 80 \cdot 3 \cdot 7 & \cdots & 44 \cdot 3 \cdot 43 \\
5 \cdot 6 \cdot 7 & 4 \cdot 6 \cdot 8 & 2 \cdot 6 \cdot 10 & \cdots & 47 \cdot 6 \cdot 46 \\
8 \cdot 9 \cdot 10 & 7 \cdot 9 \cdot 11 & 5 \cdot 9 \cdot 13 & \cdots & 50 \cdot 9 \cdot 49 \\
11 \cdot 12 \cdot 13 & 10 \cdot 12 \cdot 14 & 8 \cdot 12 \cdot 16 & \cdots & 53 \cdot 12 \cdot 52 \\
14 \cdot 15 \cdot 16 & 13 \cdot 15 \cdot 17 & 11 \cdot 15 \cdot 19 & \cdots & 56 \cdot 15 \cdot 55 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
71 \cdot 72 \cdot 73 & 70 \cdot 72 \cdot 74 & 68 \cdot 72 \cdot 76 & \cdots & 32 \cdot 72 \cdot 31 \\
74 \cdot 75 \cdot 76 & 73 \cdot 75 \cdot 77 & 71 \cdot 75 \cdot 79 & \cdots & 35 \cdot 75 \cdot 34 \\
77 \cdot 78 \cdot 79 & 76 \cdot 78 \cdot 80 & 74 \cdot 78 \cdot 1 & \cdots & 38 \cdot 78 \cdot 37
\end{array}$$

Blocks from the differences in $(\mathcal{F}_2, \mathcal{U}_2)$:

$$\begin{array}{ccccc}
\mathbf{78 \cdot 0 \cdot 3} & \mathbf{75 \cdot 0 \cdot 6} & \mathbf{69 \cdot 0 \cdot 12} & \cdots & \mathbf{42 \cdot 0 \cdot 39} \\
79 \cdot 1 \cdot 4 & 76 \cdot 1 \cdot 7 & 70 \cdot 1 \cdot 13 & \cdots & 43 \cdot 1 \cdot 40 \\
80 \cdot 2 \cdot 5 & 77 \cdot 2 \cdot 8 & 71 \cdot 2 \cdot 14 & \cdots & 44 \cdot 2 \cdot 41 \\
6 \cdot 9 \cdot 12 & 3 \cdot 9 \cdot 15 & 78 \cdot 9 \cdot 21 & \cdots & 51 \cdot 9 \cdot 48 \\
7 \cdot 10 \cdot 13 & 4 \cdot 10 \cdot 16 & 79 \cdot 10 \cdot 22 & \cdots & 52 \cdot 10 \cdot 49 \\
8 \cdot 11 \cdot 14 & 5 \cdot 11 \cdot 17 & 80 \cdot 11 \cdot 23 & \cdots & 53 \cdot 11 \cdot 50 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
69 \cdot 72 \cdot 75 & 66 \cdot 72 \cdot 78 & 60 \cdot 72 \cdot 3 & \cdots & 33 \cdot 72 \cdot 30 \\
70 \cdot 73 \cdot 76 & 67 \cdot 73 \cdot 79 & 61 \cdot 73 \cdot 4 & \cdots & 34 \cdot 73 \cdot 31 \\
71 \cdot 74 \cdot 77 & 68 \cdot 74 \cdot 80 & 62 \cdot 74 \cdot 5 & \cdots & 35 \cdot 74 \cdot 32
\end{array}$$

Blocks from the differences in $(\mathcal{F}_3, \mathcal{U}_3)$:

72 · 0 · 9	**63 · 0 · 18**	**45 · 0 · 36**
73 · 1 · 10	64 · 1 · 19	46 · 1 · 37
⋮	⋮	⋮
80 · 8 · 17	71 · 8 · 26	53 · 8 · 44
18 · 27 · 36	9 · 27 · 45	72 · 27 · 63
19 · 28 · 37	10 · 28 · 46	73 · 28 · 64
20 · 29 · 38	11 · 29 · 47	74 · 29 · 65
⋮	⋮	⋮
26 · 35 · 44	17 · 35 · 53	80 · 35 · 71
45 · 54 · 63	36 · 54 · 72	18 · 54 · 9
46 · 55 · 64	37 · 55 · 73	19 · 55 · 10
⋮	⋮	⋮
53 · 62 · 71	44 · 62 · 80	26 · 62 · 17

Blocks from the differences in $(\mathcal{F}_4, \mathcal{U}_4)$:

54 · 0 · 27	**63 · 9 · 36**	**72 · 18 · 45**
55 · 1 · 28	64 · 10 · 37	73 · 19 · 46
56 · 2 · 29	65 · 11 · 38	74 · 20 · 47
57 · 3 · 30	66 · 12 · 39	75 · 21 · 48
58 · 4 · 31	67 · 13 · 40	76 · 22 · 49
59 · 5 · 32	68 · 14 · 41	77 · 23 · 50
60 · 6 · 33	69 · 15 · 42	78 · 24 · 51
61 · 7 · 34	70 · 16 · 43	79 · 25 · 52
62 · 8 · 35	71 · 17 · 44	80 · 26 · 53

Acknowledgments. Causa, Gionfriddo and Guardo were partially supported by the Universita degli Studi di Catania,"PIACERI 2024/26" and by GNSAGA Indam; Guardo has been supported by the project PRIN 2022, "0-dimensional Schemes, Tensor Theory and Applications", funded by the European Union Next Generation EU, Mission 4, Component 2 – CUP: E53D23005670006.

References

[1] C. Berge, *Graphs and Hypergraphs*, North Holland, New York (USA), 1997.

[2] J.C. Bermond, An application of the solution of Kirkman's schoolgirl problem: the decomposition of the simmetric oriented complete graph into 3-circuit, *Discrete Math.* 8 (1974), 301-304.

[3] S. Bonvicini, M. Buratti, M. Garonzi, G. Rinaldi and T. Traetta, The first families of highly symmetric Kirkman Triple Systems whose orders fill a congruence class, *Des. Codes Cryptogr.* 89 (2021), 2725-2757.

[4] C. Colbourn and A. Rosa, *Triple Systems*, Oxford University Press, New York, 1999.

[5] G. Falcone and M. Pavone, Kirkman's tetrahedron and the fifteen schoolgirl problem, *Am. Math. Mon.* 118 (2011), 887-900.

[6] M. Gionfriddo, L. Milazzo and V. Voloshin, *Hypergraphs and Designs*, Nova Science Publisher, New York, 2015.

[7] H. Hanani, On quadruple systems, *Can. J. Math.* 12 (1960), 145-157.

[8] H. Hanani, On some tactical configurations, *Can. J. Math.* 15 (1963), 702-722.

[9] T.P. Kirkman, On a problem in combinations, *Cambridge and Dublin Math. J.* 2 (1847), 191-204.

[10] C.C. Lindner and C.A. Rodger, *Design Theory*, CRC Press, Boca Raton, 2012.

[11] C.C. Lindner and A. Rosa, Steiner quadruple systems - A survey, *Discrete Math.* 21 (1978), 147-181.

[12] D.K. Ray-Chaudhuri and R.M. Wilson, Solution of Kirkman's schoolgirl problem, *Proc. Symposia Pure Math.*, Combinatorics, 19 (1971), 187-203.

[13] J. Steiner, Combinatorische Aufgabe, *J. Reine Angew. Math.* 45 (1853), 181-182.

[14] V. Voloshin, *Introduction to Graph and Hypergraph Theory*, Nova Science Publisher, New York, 2009.

[15] W.S.B. Woolhouse, *Prize Question 1733*, Lady's and Gentleman's Diary, 1844.

[16] L.Y. Xiao, D.X. Zhao and X.C. Wan, A new method of constructing Kirkman triple system, *Proceedings of the 2011 Chinese Control and Decision Conference*, DOI: 10.1109/CCDC.2011.5968970.