

LOCAL ERGODIC PROPERTIES FOR POWERS OF OPERATORS IN BANACH SPACES*

Laurian Suciu[†]

*Dedicated to the memory of Professor Mihail Megan for his scientific
contributions and enduring enthusiasm for mathematics*

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Abstract

In this paper we deal with the local convergence of a sequence $\{T_n\}$ in the convex hull of the powers of an operator T on a Banach space $\mathcal{X}$. We refer to the case when the sequence $\{T_n(T - I)\}$ is bounded, and to the subspaces of $\mathcal{X}$ on which these two sequences converge to zero, respectively. Also, we study the behavior of $f(T_n x)$ or $f(T_n(T - I)x)$ for $x \in \mathcal{X}$ and f in the dual space $\mathcal{X}^*$ of $\mathcal{X}$, in particular when $\mathcal{X}$ is a Banach algebra and f a character of $\mathcal{X}$. Some consequences concerning the Cesàro means of higher order are also derived.

Keywords: Cesàro mean, ergodicity, ascent of an operator.

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1 Introduction and preliminaries

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ be the Banach space of all bounded linear operators $T : \mathcal{X} \rightarrow \mathcal{Y}$. As usual denote $\mathcal{B}(\mathcal{X}) = \mathcal{B}(\mathcal{X}, \mathcal{X})$, which is a Banach algebra with the identity operator $I \in \mathcal{B}(\mathcal{X})$. The kernel, the range and the spectrum of $T \in \mathcal{B}(\mathcal{X})$ will be denoted by $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\sigma(T)$,

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[†]laurians2002@yahoo.com, Lucian Blaga University of Sibiu, Department of Mathematics and Informatics, Sibiu, Romania

respectively. For a subspace $\mathcal{M} \subset \mathcal{X}$, $\overline{\mathcal{M}}$ stands for the norm closure of $\mathcal{M}$ into $\mathcal{X}$. Also, $\mathcal{X}^*$ and $T^* \in \mathcal{B}(\mathcal{X}^*)$ denote the dual space of $\mathcal{X}$ and the adjoint operator of T , respectively. The *ascent* of T , denoted $\text{asc}(T)$ is the smallest nonnegative integer n with $\mathcal{N}(T^n) = \mathcal{N}(T^{n+1})$.

For such an operator T let $\text{co}(T)$ be the (infinite) convex hull of the powers T^n ($n \geq 0$) of T . More exactly, an element $\tilde{T} \in \text{co}(T)$ has the form

$$\tilde{T} = \sum_{j \geq 0} a_j T^j, \quad a_j \geq 0, \quad \sum_{j \geq 0} a_j = 1, \quad (1)$$

such that if $a_j > 0$ for infinitely many values of j , then the series is norm convergent in $\mathcal{B}(\mathcal{X})$.

In this paper we study some local ergodic properties for the sequences $\{T_n\}_{n \geq 0} \subset \text{co}(T)$ for an operator $T \in \mathcal{B}(\mathcal{X})$. So, we continue a study in this sense begun in [1, 14, 16, 17]. These investigations are in connection with other ergodic results obtained by [6, 7, 9–13] and many other authors.

The results of the paper are applied to the *Cesàro means of higher order* denoted $M_n^{(p)}(T)$, for $T \in \mathcal{B}(\mathcal{X})$ and integers $p, n \geq 0$. We put $M_0^{(p)}(T) = I$, $M_n^{(0)}(T) = T^n$, and for $n, p \geq 1$ are defined by

$$\begin{aligned} M_n^{(p)}(T) &= \frac{p}{(n+1)\dots(n+p)} \sum_{j=0}^n \frac{(j+p-1)!}{j!} M_j^{(p-1)}(T) \\ &= \frac{p}{(n+1)\dots(n+p)} \sum_{j=0}^n \frac{(n-j+p-1)!}{(n-j)!} T^j \\ &= \frac{p}{n+p} \sum_{j=0}^n \left(1 - \frac{j}{n+1}\right) \left(1 - \frac{j}{n+2}\right) \dots \left(1 - \frac{j}{n+p-1}\right) T^j. \end{aligned} \quad (2)$$

It is easy to see that these satisfy for $p, n \geq 1$ the following relations:

$$M_n^{(p)}(T)(T - I) = \frac{p}{n+1} (M_{n+1}^{(p-1)}(T) - I), \quad (3)$$

$$TM_n^{(p)}(T) = \frac{n+p+1}{n+1} M_{n+1}^{(p)}(T) - \frac{p}{n+1} I, \quad (4)$$

$$\frac{n+p+1}{n+1} M_{n+1}^{(p)}(T) - M_n^{(p)}(T) = \frac{p}{n+1} M_{n+1}^{(p-1)}(T), \quad (5)$$

$$\|M_n^{(p)}(T)\| \leq \max_{0 \leq j \leq n} \|M_j^{(p-1)}(T)\| \leq \max_{0 \leq j \leq n} \|T^j\|. \quad (6)$$

Remark by the spectral mapping theorem that, if $1 \in \sigma(T)$, then $1 \in \sigma(M_n^{(p)}(T))$ for all n, p , hence $\|M_n^{(p)}(T)\| \geq 1$ in this case.

The operator T is *p-Cesàro ergodic*, or *p-Cesàro bounded* if the sequence $\{M_n^{(p)}(T)\}$ is strongly convergent, or it is bounded, in $\mathcal{B}(\mathcal{X})$, respectively. When T is *p-Cesàro ergodic* the limit operator is the bounded projection $P \in \mathcal{B}(\mathcal{X})$ onto $\mathcal{R}(P) = \mathcal{N}(T - I)$ with $\mathcal{N}(P) = \overline{\mathcal{R}(T - I)}$ (see, e.g., [8, Corollary VIII. 5.2, p. 662], [12, Theorem 1.3, p. 73]).

In this paper we study some local ergodic properties for the sequences $\{T_n\} \subset co(T)$ which are not necessary bounded, but they satisfy local boundedness conditions.

So, in Section 2 we refer to the case when $\{T_n\}$ is bounded on $\overline{\mathcal{R}(T - I)}$, or only on $\mathcal{R}((T - I)^p)$ for $p > 1$, in order to investigate the subspaces on which $\{T_n\}$ and $\{T_n(T - I)\}$ converge to zero. In the same sense, we assume some boundedness and regularity conditions in vectors from $\mathcal{X}$, to obtain the convergence of the corresponding operator sequences in those vectors. Our investigations are based on the intertwining of T with an operator on another space, which has good ergodic properties.

In Section 3 we explore the similar boundedness and regularity conditions for sequences $\{T_n\} \subset co(T)$, when T acts on a Banach algebra $\mathcal{X}$. We assume such condition with respect to characters of $\mathcal{X}$ (the linear and multiplicative functionals on $\mathcal{X}$). These conditions permit to obtain the convergence to zero of the corresponding sequences, with respect to the respective functionals in $\mathcal{X}^*$ and the vectors from $\mathcal{X}$. Other consequences regarding the means $M_n^{(p)}(T)$ are derived from these results.

2 Ergodic properties for sequences in the convex hull of an operator

In what follows we investigate ergodic properties from the context of Cesàro means for more general operator means. Namely, we refer to sequences from the convex hull of the powers of an operator on a Banach space $\mathcal{X}$. We begin with the following

Theorem 1. *Let $T \in \mathcal{B}(\mathcal{X})$ and $\{T_n\} \subset co(T)$ be such that*

$$\|T_n|_{\overline{\mathcal{R}(T-I)}}\| = O(1) \quad \text{as } n \rightarrow \infty. \quad (7)$$

Then there exists a Banach space $\tilde{\mathcal{X}}$ and the operators $Q \in \mathcal{B}(\mathcal{X}, \tilde{\mathcal{X}})$ and $V \in \mathcal{B}(\tilde{\mathcal{X}})$ satisfying $QT = VQ$, $\mathcal{R}(Q) = \tilde{\mathcal{X}}$ and

$$\mathcal{N}(Q) = \{x \in \mathcal{X} : T_n(T - I)x \rightarrow 0, n \rightarrow \infty\}. \quad (8)$$

Moreover, for every integer $m \geq 1$ we have

$$\mathcal{X}_0 := \{x \in \mathcal{X} : T_n x \rightarrow 0, n \rightarrow \infty\} \subset \mathcal{N}(Q) \cap \overline{\mathcal{R}((T - I)^m)}, \quad (9)$$

and the following conditions are equivalent:

- (i) $\mathcal{R}((T - I)^m) \subset \mathcal{N}(Q)$;
- (ii) $(V - I)^m = 0$;
- (iii) $\mathcal{X}_0 = \overline{\mathcal{R}((T - I)^{m+1})}$.

Proof. The subspace $\mathcal{X}_0$ of $\mathcal{X}$ from (9) is invariant for T and T_n because $TT_n = T_n T$ and $\{T_n\}$ satisfies (7). Let $S = T|_{\mathcal{X}_0}$, $S_n = T_n|_{\mathcal{X}_0}$ for $n \in \mathbb{N}$. As usually, $\mathcal{X}_0^*$ denotes the dual space of $\mathcal{X}_0$. Then for every $x \in \mathcal{X}_0$ and $f \in \mathcal{X}_0^*$ we have (by (9))

$$(S_n^* f)x = f(S_n x) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

But S_n^* are convex combinations of powers of S^* (by the choice $\{T_n\} \subset co(T)$). So, if $f \in \mathcal{N}(S^* - I)$ then $S_n^* f = f$, and by the above convergence one obtains $f(x) = 0$ for $x \in \mathcal{X}_0$, that is $f = 0$. Thus $\mathcal{N}(S^* - I) = \{0\}$ which implies $\mathcal{N}((S^* - I)^m) = \{0\}$ for every integer $m \geq 1$. Hence we have

$$\mathcal{X}_0 = \overline{\mathcal{R}((S - I)^m)} \subset \overline{\mathcal{R}((T - I)^m)}, \quad \text{for } m \geq 1. \quad (10)$$

Since $\sup_{n \in \mathbb{N}} \|T_n(T - I)x\| < \infty$ for $x \in \mathcal{X}$ one can define the (continuous) seminorm γ on $\mathcal{X}$ by

$$\gamma(x) = \limsup_{n \rightarrow \infty} \|T_n(T - I)x\|, \quad x \in \mathcal{X}.$$

So $\mathcal{N}(\gamma) = \{x \in \mathcal{X} : T_n(T - I)x \rightarrow 0, n \rightarrow \infty\}$ is a closed subspace of $\mathcal{X}$, which is invariant for T and T_n , because $T_n T = T T_n$ ($n \in \mathbb{N}$).

Let now $\tilde{\mathcal{X}}$ be the completion of the normed quotient space $\mathcal{X}/\mathcal{N}(\gamma)$ with the norm

$$\|\tilde{x}\| = \gamma(x), \quad \tilde{x} = x + \mathcal{N}(\gamma) \in \tilde{\mathcal{X}}.$$

Let Q be the quotient mapping of $\mathcal{X}$ into $\tilde{\mathcal{X}}$, that is $Qx = \tilde{x}$ for $x \in \mathcal{X}$. Then $\overline{\mathcal{R}(Q)} = \tilde{\mathcal{X}}$, $\mathcal{N}(Q) = \mathcal{N}(\gamma)$, as in (8), and clearly $\mathcal{X}_0 \subset \mathcal{N}(\gamma)$ which together with (10) give the inclusion (9).

On the other hand we have by the above relations that

$$\|QTx\| = \|\tilde{T}x\| = \gamma(Tx) \leq \|T\|\gamma(x) \leq \|T\|\|Qx\|$$

for all $x \in \mathcal{X}$. So, there exists an operator $V \in \mathcal{B}(\tilde{\mathcal{X}})$ such that $QT = VQ$. This implies $Q(T - I)^m = (V - I)^m Q$ for every integer $m \geq 1$. Therefore $\mathcal{R}((T - I)^m) \subset \mathcal{N}(Q)$ if and only if $(V - I)^m = 0$, which means the equivalence of (i) with (ii). But the inclusion from (i) implies that $\overline{\mathcal{R}((T - I)^{m+1})} \subset \mathcal{X}_0$, and later by (10) this implies that $\mathcal{X}_0 = \overline{\mathcal{R}((T - I)^{m+1})}$, that is (iii). Hence the conditions (i) and (iii) are equivalent, what completes the proof. $\square$

Corollary 1. *Let T, T_n and $\mathcal{X}_0$ be as in Theorem 1. If*

$$\|T_n(T - I)^m x\| = o(1) \quad \text{as } n \rightarrow \infty$$

for an integer $m \geq 1$ and each $x \in \mathcal{X}$, then $\mathcal{X}_0 = \overline{\mathcal{R}((T - I)^m)}$ and $\text{asc}(T^ - I) \leq m$.*

Corollary 2. *Let T and Q be as in Theorem 1, and let $S = T|_{\mathcal{N}(Q)}$. Then $\text{asc}(S^* - I) \leq 1$.*

Proof. Preserving the notation from the proof of Theorem 1 and denoting $S_n = T_n|_{\mathcal{N}(Q)}$ for $n \in \mathbb{N}$, we have for $f \in \mathcal{N}(Q)^*$ and $x \in \mathcal{N}(Q)$,

$$(S_n^*(S^* - I)f)x = f(S_n(S - I)x) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Now, if $f \in \mathcal{N}((S^* - I)^2)$ then $(S^* - I)f \in \mathcal{N}(S^* - I)$, and so for $x \in \mathcal{N}(Q)$ we have

$$((S^* - I)f)x = (S^*(S^* - I)f)x = (S_n^*(S^* - I)f)x,$$

taking into account that $S_n^* \in \text{co}(S^*)$ because $T_n \in \text{co}(T)$. Thus by the above convergence we obtain $(S^* - I)f = 0$, that is $f \in \mathcal{N}(S^* - I)$. Having in view the choice of f we infer that $\mathcal{N}((S^* - I)^2) = \mathcal{N}(S^* - I)$, this meaning that $\text{asc}(S^* - I) \leq 1$. $\square$

Remark 1. Concerning Theorem 1 it is possible to have

$$\mathcal{X}_0 = \overline{\mathcal{R}((T - I)^2)} \neq \overline{\mathcal{R}(T - I)},$$

in general. For example, let $T \in \mathcal{B}(\mathcal{X})$ with $(T - I)^2 = 0$, $T \neq I$. Then

$$M_n(T)(T - I) = nM_{n-1}^{(2)}(T)(T - I)^2 + T - I = T - I \neq 0, \quad n \geq 1$$

and $\mathcal{X}_0 = \mathcal{R}(T - I)^2 = \{0\}$ when $T_n = M_n(T)$. This sequence just strongly converges on $\mathcal{R}(T - I)$, but it is unbounded on $\mathcal{X}$ because

$$M_n(T) = n(n-1)M_{n-2}^{(3)}(T)(T - I)^2 + n(T - I) + I = n(T - I) + I, \quad n \geq 2.$$

Since $M_n(T)(T-I)^2 = 0$ we have $\overline{\mathcal{R}(T-I)} \subset \mathcal{N}(Q)$, the inclusion being strict in this case (as $T-I \neq 0$). Also, we remark that

$$\mathcal{R}(T-I) \cap \mathcal{N}(T-I) = \mathcal{R}(T-I) \neq \{0\},$$

a condition which is not compatible with ergodic theory.

Corollary 3. *Let $T \in \mathcal{B}(\mathcal{X})$ and $\{T_n\} \subset co(T)$ be a bounded sequence. Then there exists a Banach space $\hat{\mathcal{X}}$ and the operators $Q_0 \in \mathcal{B}(\mathcal{X}, \hat{\mathcal{X}})$, $V_0 \in \mathcal{B}(\hat{\mathcal{X}})$ such that $Q_0T = V_0Q_0$, $\mathcal{R}(Q_0) = \hat{\mathcal{X}}$ and*

$$\mathcal{N}(Q_0) = \{x \in \mathcal{X} : T_n x \rightarrow 0, n \rightarrow \infty\} \subset \overline{\mathcal{R}((T-I)^m)}$$

for every integer $m \geq 1$.

Moreover, $\mathcal{N}(Q_0) = \overline{\mathcal{R}(T-I)}$ if and only if $V_0 = I$, and in this case $asc(T^* - I) \leq 1$.

Proof. Since $\{T_n\}$ is bounded one can also define the seminorm γ_0 on $\mathcal{X}$ by

$$\gamma_0(x) = \limsup_{n \rightarrow \infty} \|T_n x\|, \quad x \in \mathcal{X}.$$

Proceeding as in the proof of Theorem 1 we get a Banach space $\hat{\mathcal{X}}$ and the operators Q_0 and V_0 satisfying $Q_0T = V_0Q_0$, $\mathcal{R}(Q_0) = \hat{\mathcal{X}}$ and

$$\mathcal{N}(Q_0) = \{x \in \mathcal{X} : T_n x \rightarrow 0, n \rightarrow \infty\} \subset \overline{\mathcal{R}((T-I)^m)}, \quad m \geq 1.$$

Since $Q_0(T-I) = (V_0 - I)Q_0$ it follows that $\overline{\mathcal{R}(T-I)} \subset \mathcal{N}(Q_0)$ if and only if $V_0 = I$. In this case from the above inclusion we infer $\mathcal{N}(Q_0) = \overline{\mathcal{R}((T-I)^m)}$ for $m \geq 1$. This gives $\overline{\mathcal{R}(T-I)} = \overline{\mathcal{R}((T-I)^2)}$ that is $asc(T^* - I) \leq 1$. $\square$

Remark 2. Notice that only the boundedness condition of a sequence $\{T_n\} \subset co(T)$ cannot give information concerning the convergence of $\{T_n\}$, in any way. More precisely, under this condition we cannot describe the whole set of (strongly) convergence of $\{T_n\}$, neither to provide the equality in the inclusion

$$(A) \quad \mathcal{X}_0 \oplus \mathcal{N}(T-I) \subset \{x \in \mathcal{X} : \{T_n x\} \text{ converges in } \mathcal{X}\}.$$

When the sequence $\{T_n\}$ is *ergodic* that is $\mathcal{X}_0 = \overline{\mathcal{R}(T-I)}$, the equality in (A) holds, which means $V_0 = I$ by Corollary 3, that is an extreme case.

We want to see if the equality occurs by other conditions on $\{T_n\}$ and V_0 , for instance V_0 to be an isometry. This idea leads to another ergodic concept, as follows.

We say that a sequence $\{T_n\} \subset co(T)$ is (*uniformly*) *regular* if it is bounded and there exists an integer $n_0 \geq 0$ such that $\{TT_n - T_{n+n_0}\}$ strongly (respectively, uniformly) converges to zero. In this case, $\{T_n\}$ is called (*uniformly*) *regular of order* n_0 . Clearly, the ergodicity means regularity of order $n_0 = 0$. But $\{T_n\}$ can be regular of an order $n_0 \geq 1$ without being ergodic. Cesàro averages of order p form a regular sequence of order 1 (see relation (4) below) but these means are not necessary ergodic sequences, in general.

Other examples can be provided as follows

Example 1. Let $T \in \mathcal{B}(\mathcal{X})$ be Cesàro bounded such that $\{\frac{1}{n}T^n\}$ does not strongly converge to zero. (Such operators are provided in [6], for instance; or on $\mathbb{C}^2$ can be considered the Assani's example $T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$).

Let $\{T_n\} \in co(T)$ given by $T_n = \frac{1}{2}(M_n(T) + M_{n+1}(T))$, $n \in \mathbb{N}$. Then $\{T_n\}$ is bounded and by the relation (4) we have

$$TT_n - T_{n+1} = \frac{1}{2}(TM_n(T) - M_{n+1}(T) + TM_{n+1}(T) - M_{n+2}(T)) \rightarrow 0,$$

hence $\{T_n\}$ is regular of order 1. Clearly, by recurrence we can find such a sequence for any order $n_0 > 1$.

On the other hand one has

$$T_n(T - I) = \frac{1}{2} \left(\frac{T^{n+1}}{n+1}(T + I) - \frac{T^{n+2}}{(n+1)(n+2)} - \frac{2n+3}{(n+1)(n+2)}I \right).$$

As $\{M_n(T)\}$ is bounded, the second term in the right side uniformly converges to zero.

Thus, assuming that $\mathcal{R}(T + I) = \mathcal{X}$, under the above assumption on T it follows from this relation that $\{T_n(T - I)\}$ does not strongly converge to zero, that is $\{T_n\}$ is not ergodic.

As another case, we can consider T an invertible isometry on $\mathcal{X}$ such that $1, -1 \notin \sigma(T)$ and $T_n = \frac{1}{2}(T^n + T^{n+1})$, $n \in \mathbb{N}$. Here $TT_n - T_{n+1} = 0$ therefore $\{T_n\}$ is regular of order 1, but $T_n(T - I) = \frac{1}{2}T^n(T^2 - I)$ does not converge strongly to zero, hence it is not ergodic.

Notice that in [17] the regularity condition was used to derive certain ergodic properties for the Cesàro means of higher order, as well as to obtain two extensions of the Esterle-Katznelson-Tzafriri theorem for some sequences in the convex hull of an operator.

Now we present results on local convergence for some sequences of operators which satisfy local regularity conditions.

Theorem 2. Let $T \in \mathcal{B}(\mathcal{X})$ with $\sigma(T) \subset \mathbb{D} \cup \{1\}$ and $\{T_n\} \subset co(T)$ such that $\|T_n x_0\| = O(1)$ and $\|(TT_n - T_{n+n_0})x_0\| = o(1)$ as $n \rightarrow \infty$, for some $x_0 \in \mathcal{X}$ and $n_0 \geq 1$. Then $\|T_n(T - I)x_0\| = o(1)$ as $n \rightarrow \infty$.

Proof. Since the sequence $\{T_n x_0\}$ is bounded, we can define the seminorm Γ on $\mathcal{B}(\mathcal{X})$ by

$$\Gamma(S) = \limsup_{n \rightarrow \infty} \|ST_n x_0\|, \quad S \in \mathcal{B}(\mathcal{X}). \quad (11)$$

In order to apply [2, Theorem 5] to the operator T in the Banach algebra $\mathcal{B}(\mathcal{X})$ relative to Γ , we firstly note that Γ is continuous on $\mathcal{B}(\mathcal{X})$, and it is needed to verify that $\Gamma(T^m) = O(1)$ as $m \rightarrow \infty$. We write $TT_n = (TT_n - T_{n+n_0}) + T_{n+n_0}$, and so for every $m \geq 1$ we have

$$\begin{aligned} \Gamma(T^m) &= \limsup_{n \rightarrow \infty} \|T^m T_n x_0\| \\ &\leq \|T^{m-1}\| \lim_{n \rightarrow \infty} \|(TT_n - T_{n+n_0})x_0\| + \limsup_{n \rightarrow \infty} \|T^{m-1} T_{n+n_0} x_0\| \\ &= \limsup_{n \rightarrow \infty} \|T^{m-1} T_n x_0\| = \Gamma(T^{m-1}). \end{aligned}$$

Hence $\Gamma(T^m) \leq \Gamma(T^{m-1}) \leq \Gamma(T) \leq \Gamma(I)$ for $m \geq 1$. Since $\sigma(T) \subset \mathbb{D} \cup \{1\}$, by the above quoted theorem we obtain $\Gamma(T^{m+1} - T^m) \rightarrow 0$ as $m \rightarrow \infty$.

On the other hand, we observe that

$$T^2 T_n = T(TT_n - T_{n+n_0}) + (TT_{n+n_0} - T_{n+2n_0}) + T_{n+2n_0},$$

and by induction, for every $m \geq 1$ we get the relation

$$T^m T_n = \sum_{j=0}^{m-1} T^{m-j-1} (TT_{n+jn_0} - T_{n+(j+1)n_0}) + T_{n+mn_0}. \quad (12)$$

Having in view this expression of $T^m T_n$, we get the inequality

$$\|(T^{m+1} - T^m)T_n x_0\| \geq \|T_{n+mn_0}(T - I)x_0\| -$$

$$\sum_{j=0}^{m-1} \|T^{m-j-1}(T - I)\| \|(TT_{n+jn_0} - T_{n+(j+1)n_0})x_0\|. \quad (13)$$

Finally, from (11), (12) and (13) we infer the following relations for $m \geq 1$,

$$\limsup_{n \rightarrow \infty} \|T_n(T - I)x_0\| = \limsup_{n \rightarrow \infty} \|T_{n+mn_0}(T - I)x_0\|$$

$$\begin{aligned}
&\leq \Gamma(T^{m+1} - T^m) + \sum_{j=0}^{m-1} \|T^{m-j-1}(T - I)\| \lim_{n \rightarrow \infty} \|(TT_{n+jn_0} - T_{n+(j+1)n_0})x_0\| \\
&= \Gamma(T^{m+1} - T^m).
\end{aligned}$$

Clearly, here we also used that $(TT_n - T_{n+n_0})x_0 \rightarrow 0$ as $n \rightarrow \infty$. Since $\Gamma(T^{m+1} - T^m) \rightarrow 0$ as $m \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} \|T_n(T - I)x_0\| = 0$, which ends the proof. $\square$

From this theorem we derive a version concerning the local convergence for Cesàro means, of the well-known result of Hille [11], this later being a strengthening of Gelfand's theorem involving the doubly power bounded operators with the spectrum reduced to $\{1\}$.

For an invertible operator T in $\mathcal{B}(\mathcal{X})$ we denote $M_{-n}^{(p)}(T) = M_n^{(p)}(T^{-1})$ for $n, p \in \mathbb{N}$.

Corollary 4. *Let $T \in \mathcal{B}(\mathcal{X})$ with $0 \notin \sigma(T) \subset \mathbb{D} \cup \{1\}$, and let $x_0 \in \mathcal{X}$ such that $\|M_n^{(p)}(T)x_0\| = O(1)$ as $|n| \rightarrow \infty$, for some integer $p \geq 1$. Then $\|M_n^{(p-1)}(T)x_0\| = o(|n|)$ and $\|T^n x_0\| = o(|n|^p)$, as $|n| \rightarrow \infty$.*

Moreover, if $\sigma(T) = \{1\}$ and $\|M_n^{(p)}(T)\| = O(1)$ as $|n| \rightarrow \infty$ then $T = I$.

Proof. We apply Theorem 2 for the operators $T_n = M_n^{(p)}(T)$ and $S_n = M_n^{(p)}(T^{-1})$ ($n \in \mathbb{N}$) respectively, because by the relation (4) and the boundedness of $\{T_n\}$ and $\{S_n\}$, we have $(TT_n - T_{n+1})x_0 \rightarrow 0$ and $(SS_n - S_{n+1})x_0 \rightarrow 0$, as $n \rightarrow \infty$. So, we obtain $M_n^{(p)}(T)(T - I)x_0 \rightarrow 0$ as $|n| \rightarrow \infty$, which means by (3) that $\frac{1}{n}M_n^{(p-1)}(T)x_0 \rightarrow 0$, and finally $\frac{1}{n^p}T^n x_0 \rightarrow 0$, as $|n| \rightarrow \infty$. This gives the first assertion.

Now, if the sequence $\{M_n^{(p)}(T)\}_{n \in \mathbb{Z}}$ is bounded, then $\|M_n^{(p)}(T)\| = o(n)$ as $|n| \rightarrow \infty$, and by (3) it follows that $\|M_n(T)\| = o(n^p)$ as $|n| \rightarrow \infty$. By Drissi-Zemánek theorem in [7], this means $(T - I)^p = 0$. But the boundedness of $\{M_n^{(p)}(T)\}$ ensures $\text{asc}(T - I) \leq 1$, that is $\mathcal{N}(T - I) = \mathcal{N}((T - I)^p) = \mathcal{X}$. Hence $T = I$, which ends the proof. $\square$

Let us recall (see [9]) that a positive operator T on a Banach lattice for which the Cesàro means of an (arbitrary) order $p \geq 2$ are bounded is, in fact, Cesàro bounded. Thus, in this case Theorem 2 shows that $\frac{1}{n}T^n \rightarrow 0$ strongly, if $\sigma(T) \subset \mathbb{D} \cup \{1\}$ and $\|M_n^{(p)}(T)\| = O(1)$ as $n \rightarrow \infty$, for an integer $p \geq 1$. But, in this context we have the following more general result, which gives also another version of Gelfand's theorem. This theorem can be related to similar results of de Pagter-Bernau-Huijsmans in [5] for power bounded operators.

Theorem 3. *Let $\mathcal{X}$ be a Banach lattice and $T \in \mathcal{B}(\mathcal{X})$ be a positive operator with $\sigma(T) = \{1\}$. Let $\{T_n\} \subset co(T)$ and $x_0 \in \mathcal{X}$ be as in Theorem 2. Then $\|T_n x_0\| = o(1)$ as $n \rightarrow \infty$. Moreover, if $\|T_n\| = O(1)$ and $\|TT_n - T_{n+n_0}\| = o(1)$ as $n \rightarrow \infty$, for some $n_0 \geq 1$, then $T = I$.*

Proof. Let $r \in (0, 1)$ fixed. Then by a result of Meyer-Nieberg [15] there exists an integer $N \geq 0$ such that $T^N = (1 - r)^N I$.

Let $\{T_n\}$ and $x_0 \in \mathcal{X}$ be as in Theorem 2. From the proof of this theorem we have $\Gamma(T^{Nm}) \leq \Gamma(I)$ for $m \geq 1$, and since $\sigma(T^N) = \{1\}$, by [2, Theorem 5] we obtain

$$\Gamma(T^{Nm}(T^N - I)) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

But $I - T^N = (1 - \delta)I$ where $\delta = (1 - r)^N \in (0, 1)$, therefore $\Gamma(T^{Nm}) \rightarrow 0$ as $m \rightarrow \infty$. Then using the expression of $T^{Nm}T_n$ given by (12) as well as the convergence condition from hypothesis, we get for $m \geq 1$ (as in the proof of Theorem 2)

$$\limsup_{n \rightarrow \infty} \|T_n x_0\| = \limsup_{n \rightarrow \infty} \|T_{n+Nm n_0} x_0\| \leq \Gamma(T^{Nm}),$$

hence $\lim_{n \rightarrow \infty} \|T_n x_0\| = 0$. This gives the first assertion of theorem.

Suppose now that $\|T_n\| = O(1)$ and $\|TT_n - T_{n+n_0}\| = o(1)$, as $n \rightarrow \infty$. Consider the (continuous) seminorm Γ_0 on $\mathcal{B}(\mathcal{X})$ given by

$$\Gamma_0(S) = \limsup_{n \rightarrow \infty} \|ST_n\|, \quad S \in \mathcal{B}(\mathcal{X}).$$

Then we have (as above) for every $m \in \mathbb{N}$,

$$\begin{aligned} \Gamma_0(T^{Nm}) &= \limsup_{n \rightarrow \infty} \|T^{Nm}T_n\| \leq \limsup_{n \rightarrow \infty} \|T_{n+Nm n_0}\| \\ &+ \sum_{j=0}^{Nm-1} \|T^{Nm-j-1}\| \lim_{n \rightarrow \infty} \|TT_{n+jn_0} - T_{n+(j+1)n_0}\| \leq \sup_{n \in \mathbb{N}} \|T_n\| < \infty. \end{aligned}$$

This assures as above (using [2, Theorem 5]) that $\Gamma_0(T^{Nm}) \rightarrow 0$ as $m \rightarrow \infty$. Latterly, using the relation (12) and the condition of convergence from our assumption, we obtain $\|T_n\| \rightarrow 0$, $n \rightarrow \infty$. But this ensures that $T = I$. Indeed, by contrary, we firstly obtain by Theorem 1 that $\mathcal{X} = \overline{\mathcal{R}(T - I)}$ and $\mathcal{N}(T - I) = \{0\}$. On the other hand, as $N \geq 1$ (by the assumption $T \neq I$) from the relation $I - T^N = (1 - \delta)I$ follows that $\mathcal{R}(T - I) = \mathcal{X}$. Thus, $T - I$ is invertible in $\mathcal{B}(\mathcal{X})$ that is $1 \notin \sigma(T)$, which contradicts the hypothesis. We conclude $T = I$, and this ends the proof. $\square$

Corollary 5. *Let T be a positive operator on a Banach lattice $\mathcal{X}$ with $\sigma(T) = \{1\}$ such that $\|M_n(T)x_0\| = O(1)$ as $n \rightarrow \infty$ for some $x_0 \in \mathcal{X}$. Then $\|M_n(T)x_0\| = o(1)$ as $n \rightarrow \infty$. In addition, if T is Cesàro bounded then $T = I$.*

3 Ergodic conditions involving characters of Banach algebras

The above results show that some ergodic properties cannot occur for the operators $T \neq I$, when $T - I$ is not nilpotent of order 1. So, from the ergodic point of view it is important to know when $T - I$ is nilpotent of an order $k \geq 2$, or more general, to have information about the behavior of the sequence $\|(T - I)^k x\|$ for some (any) $x \in \mathcal{X}$. Clearly, a nilpotent operator of an order $k \geq 2$ cannot be p -Cesàro bounded unless it is identity, but such operators can satisfy certain conditions of pointwise boundedness of the Cesàro means, even of their powers.

Recall that some results in this context are given by Atzmon [3, 4] in connection to the invariant subspace problem. In another sense, Esterle and Zarrabi in [10, Theorem 1.1] studied the behavior of $f((T - I)^k x)$ for $x \in \mathcal{X}$ and $f \in \mathcal{X}^*$. They proved that if $|f(T^n x)| = o(n^{k+1})$ as $n \rightarrow \infty$ then either $\limsup_{n \rightarrow \infty} n|f((T - I)^n x)|^{1/n} > 0$, or $f((T - I)^n x) = 0$ for $n \geq k + 1$.

Notice that some theorems for Cesàro means give conditions which assure the hypothesis of this result. Other local properties concerning the sequences in the convex hull of the powers of an operator, will be given below.

Theorem 4. *Let $\mathcal{X}$ be a Banach algebra with unit element e , and let $T \in \mathcal{B}(\mathcal{X})$ with $\sigma(T) \subset \mathbb{D} \cup \{1\}$ and $\{T_n\} \subset co(T)$. If $|f(T_n x)| = O(1)$ and $|f((TT_n - T_{n+n_0})x)| = o(1)$ as $n \rightarrow \infty$ for some $x \in \mathcal{X}$, f a character of $\mathcal{X}$ and $n_0 \geq 1$, then $|f(T_n(T - I)x)| = o(1)$ as $n \rightarrow \infty$.*

Proof. Based on the boundedness condition from hypothesis, we can define the seminorm γ on $\mathcal{X}$ by

$$\gamma(y) = \limsup_{n \rightarrow \infty} |f(yT_n x)|, \quad y \in \mathcal{X}.$$

We prove that $\{\gamma(T^m e)\}$ is bounded. Indeed, as in the proof of Theorem

2 we have for $m \geq 1$,

$$\begin{aligned}
\gamma(T^m e) &= \limsup_{n \rightarrow \infty} |f(T^m T_n x)| \\
&\leq \sup_{0 \leq k \leq m-1} |f(T^k e)| \sum_{j=0}^{m-1} \lim_{n \rightarrow \infty} |f((T T_{n+jn_0} - T_{n+(j+1)n_0})x)| \\
&\quad + \limsup_{n \rightarrow \infty} |f(T_{n+mn_0} x)| \\
&= \gamma(e).
\end{aligned}$$

Here we used the assumption that f is linear and multiplicative on $\mathcal{X}$, and the assumption that $f((T T_n - T_{n+n_0})x) \rightarrow 0$ as $n \rightarrow \infty$. Also, as f is continuous on $\mathcal{X}$ it follows $\gamma(y) \leq \gamma(e)\|y\|$, for $y \in \mathcal{X}$, hence γ is continuous on $\mathcal{X}$. So, by [2, Theorem 5] we infer $\gamma((T^{m+1} - T^m)e) \rightarrow 0$ as $m \rightarrow \infty$. This leads (using a similar argument as in the proof of Theorem 2) to the conclusion that $f((T_n(T - I)x) \rightarrow 0$ as $n \rightarrow \infty$, which ends the proof. $\square$

The conclusion of this theorem is also true for $f \in \mathcal{X}^*$ (without being multiplicative), under the stronger condition $(T T_n - T_{n+n_0})x \rightarrow 0$ as $n \rightarrow \infty$, for some $n_0 \geq 1$ and $x \in \mathcal{X}$.

Corollary 6. *Let $\mathcal{X}$ be a Banach algebra with unit e , and $T \in \mathcal{B}(\mathcal{X})$ with $\sigma(T) \subset \mathbb{D} \cup \{1\}$ such that $|f(M_n^{(p)}(T)x)| = O(1)$ as $n \rightarrow \infty$, for some $x \in \mathcal{X}$, f a character of $\mathcal{X}$ and $p \geq 1$. Then $|f(T^n x)| = o(n^p)$ as $n \rightarrow \infty$. Moreover, one of the following statements hold :*

- (i) $f(T^n x) = 0$ for any $n \geq 0$, or
- (ii) $\limsup_{n \rightarrow \infty} n |f((T - I)^n x)|^{1/n} > 0$, or
- (iii) $f((T - I)^n x) = 0$ for $n \geq p$ and $\limsup_{n \rightarrow \infty} n |f(T^n x)|^{1/n} > 0$.

Proof. We apply Theorem 4 to T and $T_n = M_n^{(p)}(T)$, $n \in \mathbb{N}$. So, we obtain

$$f(M_n^{(p)}(T)(T - I)x) \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

which implies (f being a character)

$$|f(M_n^{(p)}(T)(T - I)^p x)| = |f((T - I)^{p-1} e) f(M_n^{(p)}(T)(T - I)x)| \rightarrow 0.$$

But, by (3) this yields that $\frac{1}{n^p} f(T^n x) \rightarrow 0$. Now by [10, Theorem 1.1] we obtain: either $\limsup_{n \rightarrow \infty} n |f((T - I)^n x)|^{1/n} > 0$ that is the statement (ii), or $f((T - I)^n x) = 0$ for $n \geq p$.

In the second case, we can also apply the same theorem for the operator $T - I$ instead of T . We conclude that either $\limsup_{n \rightarrow \infty} n|f(T^n x)|^{1/n} > 0$, or $f(T^n x) = 0$ for $n \geq 1$. The first alternative just means the statement (iii), while the last equality together with the relation $f((T - I)^p x) = 0$ of above give $f(x) = 0$. Hence $f(T^j x) = 0$, for any $j \geq 0$ which is the statement (i). This ends the proof. $\square$

Notice that Theorem 4 is a version of Theorem 2 in the context of Banach algebras. Some conditions in the hypothesis of Theorem 4 can be modified for Banach spaces, in order to obtain a weaker conclusion, as follows.

Theorem 5. *Let $T \in \mathcal{B}(\mathcal{X})$ with $\sigma(T) \cap \mathbb{T} \subset \{1\}$. Let $\{T_n\} \subset \overline{\text{co}(T)}$ and $f \in \mathcal{X}^*$ such that $|f(T_n y)| = O(1)$ as $n \rightarrow \infty$ for any $y \in \overline{\mathcal{R}(T - I)}$, and $|f((TT_n - T_{n+n_0})x)| = o(1)$ as $n \rightarrow \infty$, for some $n_0 \geq 1$ and any $x \in \mathcal{X}$. Then either $|f(T_n(T - I)x)| = o(1)$ as $n \rightarrow \infty$ for any $x \in \mathcal{X}$, or $\sigma_p(T^*) = \{1\}$ and $|f(T_n(T - I)^2 x)| = o(1)$ as $n \rightarrow \infty$, for any $x \in \mathcal{X}$.*

Moreover, the first conclusion certainly occurs when $\text{asc}(T^ - I) \leq 1$.*

Proof. By hypothesis the sequence $\{f(T_n(T - I)x)\}$ is bounded, for $x \in \mathcal{X}$, so one can define the seminorm γ on $\mathcal{X}$ by

$$\gamma(x) = \limsup_{n \rightarrow \infty} |f(T_n(T - I)x)|, \quad x \in \mathcal{X}.$$

Since $f((TT_n - T_{n+n_0})y) \rightarrow 0$ for any $y \in \mathcal{X}$, we obtain

$$\begin{aligned} \gamma(Tx) &= \limsup_{n \rightarrow \infty} |f((TT_n - T_{n+n_0})(T - I)x) + f(T_{n+n_0}(T - I)x)| \\ &= \limsup_{n \rightarrow \infty} |f(T_{n+n_0}(T - I)x)| = \gamma(x). \end{aligned}$$

Hence γ induces an isometry V on a Banach space (as in the proof of Theorem 1). Since $\sigma(V) \subset \sigma(T)$, and $\sigma(T) \cup \mathbb{T} \subset \{1\}$, it follows $\sigma(V) \subset \{1\}$. If $1 \notin \sigma(V)$ then necessary $\mathcal{X} = \mathcal{N}(\gamma)$, so $f(T^n(T - I)x) \rightarrow 0$ for any $x \in \mathcal{X}$. In the case that $\sigma(V) = \{1\}$ one has $V = I$. This gives $\overline{\mathcal{R}(T - I)} \subset \mathcal{N}(\gamma)$ that is $f(T_n(T - I)^2 x) \rightarrow 0$ for any $x \in \mathcal{X}$. In this case we have also $\sigma_p(T^*) = \{1\}$, by the spectral condition from hypothesis.

Now denote $S_n = T_n|_{\overline{\mathcal{R}(T - I)}}$, $f_0 = f|_{\overline{\mathcal{R}(T - I)}}$. Then the previous conclusion means $(f_0 \circ S_n)(T - I)^2 x \rightarrow 0$ for $x \in \mathcal{X}$. Since the sequence $\{f_0 \circ S_n\}$ is bounded, it follows that $(f_0 \circ S_n)y \rightarrow 0$ for any $y \in \overline{\mathcal{R}((T - I)^2)} = \overline{\mathcal{R}(T - I)}$, this equality being assured under the assumption $\text{asc}(T^* - I) \leq 1$. We conclude in this case that $f(T_n(T - I)x) \rightarrow 0$ for any $x \in \mathcal{X}$, which ends the proof. $\square$

Corollary 7. *Let $T \in \mathcal{B}(\mathcal{X})$ with $\sigma(T) \cap \mathbb{T} \subset \{1\}$ and $f \in \mathcal{X}^*$ satisfying (for some $p \geq 1$) one of the conditions :*

(a) *$|f(M_n^{(p)}(T)x)| = O(1)$ as $n \rightarrow \infty$ for $x \in \overline{\mathcal{R}(T - I)}$ and $\text{asc}(T^* - I) \leq 1$, or*

(b) *$|f(M_n^{(p)}(T)x)| = O(1)$ as $n \rightarrow \infty$ for any $x \in \mathcal{X}$, and $\sigma(T) \subset \mathbb{D} \cup \{1\}$.*

Then $|f(T^n x)| = o(n^p)$ as $n \rightarrow \infty$, for any $x \in \mathcal{X}$. Moreover, for every $x \in \mathcal{X}$ one of the statements (i), (ii) or (iii) of Corollary 6 hold.

Proof. If condition (a) is satisfied, an argument similar to the one in the proof of Theorem 5 can be used, with the seminorm γ given by

$$\gamma(x) = \limsup_{n \rightarrow \infty} \frac{1}{n} |f(M_n^{(p-1)}(T)x)|, \quad x \in \mathcal{X}.$$

So, by (4) it is immediate that $\gamma(Tx) = \gamma(x)$ for $x \in \mathcal{X}$, and as in the previous proof we obtain one of the assertions:

(i) either $\frac{1}{n} f(M_n^{(p-1)}(T)x) \rightarrow 0$ for any $x \in \mathcal{X}$, or

(ii) $\sigma_p(T^*) = \{1\}$ and $\frac{1}{n} f(M_n^{(p-1)}(T)(T - I)x) \rightarrow 0$ for any $x \in \mathcal{X}$.

The last convergence in (ii) means $f(M_n^{(p)}(T)(T - I)^2 x) \rightarrow 0$ for $x \in \mathcal{X}$, which together with the conditions of (a) imply that $f(M_n^{(p)}(T)(T - I)x) \rightarrow 0$ for $x \in \mathcal{X}$. In both cases we infer by (3) that $\frac{1}{n^p} f(T^n x) \rightarrow 0$, for $x \in \mathcal{X}$.

Now we suppose the condition (b) satisfied. This means $|(T_n f)x| = O(1)$ as $n \rightarrow \infty$ for any $x \in \mathcal{X}$, that is $\|T_n f\| = O(1)$ as $n \rightarrow \infty$, where $T_n = M_n^{(p)}(T^*)$. Since we have also

$$\|(TT_n - T_{n+1})f\| = \frac{p}{n+1} (T_{n+1} - I)f \rightarrow 0,$$

by Theorem 2 we obtain $T_n(T^* - I)f \rightarrow 0$. This implies $f(M_n^{(p)}(T)(T - I)x) \rightarrow 0$ for every $x \in \mathcal{X}$, which leads to $\frac{1}{n^p} f(T^n x) \rightarrow 0$ for all $x \in \mathcal{X}$.

Having in view the conclusion obtained in both cases (a) and (b), the alternatives (i), (ii) and (iii) mentioned in Corollary 7 follows as in the proof of Corollary 6. $\square$

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