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INVERTIBLE CHEBYSHEV–BINOMIAL TRANSFORMATIONS*

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Dedicated to Professor Constantin Fetecau on the occasion of his 80th anniversary

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Abstract

We show that explicit expansions of the Chebyshev polynomials of the first kind naturally induce several invertible triangular transformations involving classical binomial arrays and closely related ballot-number arrays. Using a unified coefficient-extraction method, we derive direct and inverse formulas relating the coefficients $\binom{2n}{n-k}$ and the ballot-number differences $\binom{2n}{n+k}-\binom{2n}{n+k+1}$ to the sequences $\binom{n}{k}2^{n-k}$ and $\binom{n}{k}4^{n-k}$. The resulting identities define explicit mutually inverse triangular transformations. The closed forms appearing on their right-hand sides admit natural combinatorial interpretations in terms of disjoint subsets and block decompositions. We also introduce a signed refinement of the classical enumeration of pairs of disjoint subsets, which yields an additional Chebyshev–binomial transformation and its explicit inverse.

Keywords: Chebyshev polynomials, symmetric Laurent polynomials, binomial transformations, combinatorial interpretations.

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1 Introduction

Let n and k be nonnegative integers with $n \geq k$. The quantity

$$\binom{n}{k} 2^{n-k}$$

admits several well-known combinatorial interpretations.

A classical interpretation is that it counts the pairs (A, B) of disjoint subsets of an n-set X satisfying $|A| = k$. Indeed, one first chooses the subset A in $\binom{n}{k}$ ways, and then each of the remaining $n - k$ elements may either belong to B or remain outside $A \cup B$, giving 2^{n-k} possibilities.

A second interpretation arises from perfect matchings [4]. Let

$$X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$$

be a partition of a $2n$-set into 2-element blocks. Then

$$\binom{n}{k} 2^{n-k}$$

is precisely the number of $(n + k)$-subsets of X intersecting every block X_i. Indeed, such a subset must contain at least one element from each block, and exactly k blocks contribute both of their elements.

The same quantity also appears in the classical identity

$$\sum_{i=k}^{n} \binom{i}{k} \binom{n}{i} = \binom{n}{k} 2^{n-k},$$

which follows immediately from the elementary relation

$$\binom{i}{k} \binom{n}{i} = \binom{n}{k} \binom{n-k}{i-k}$$

and the binomial theorem.

The binomial coefficients

$$\binom{2n}{n-k}$$

count walks of length $2n$ on the integer line, starting at the origin, with steps $+1$ and -1, and ending at $2k$. Consequently,

$$z^{-n}(1+z)^{2n} = \sum_{k=-n}^{n} \binom{2n}{n-k} z^k$$

is the corresponding generating function.

The main idea of this paper is to work in the space of symmetric Laurent polynomials

$$\mathcal{A}_{\text{sym}} := \{f(z) \in \mathbb{C}[z, z^{-1}] : f(z) = f(z^{-1})\}.$$

Every element of this space can be written uniquely as

$$c_0 + \sum_{d=1}^{N} c_d \left(z^d + z^{-d}\right)$$

(cf. [5, p. 2]). Hence,

$$\{1\} \cup \{z^d + z^{-d}\}_{d \geq 1}$$

is its natural monomial basis. Upon setting

$$u = z + z^{-1},$$

we have

$$z^d + z^{-d} = 2T_d\Big(\frac{u}{2}\Big), \qquad d \geq 1.$$

Thus, the transition from the monomial symmetric Laurent basis to the ordinary monomial basis in u is governed by the coefficients of the scaled Chebyshev polynomials $2T_d(u/2)$. Combined with coefficient extraction, this transition induces an invertible triangular transformation between

$$\binom{2n}{n-k} \qquad \text{and} \qquad \binom{n}{k} 2^{n-k}.$$

The direct transformation is given in Theorem 1, while its inverse appears in Corollary 2.

Explicit representations of Chebyshev polynomials and the corresponding connection coefficients between the Chebyshev and monomial bases are classical; see, for example, [6, 7]. These connections have been used in several contexts involving binomial sums. In particular, Cigler [3] studied the relation between the moments of Chebyshev polynomials and central binomial coefficients, while Bacher [2] obtained a Pascal-type triangular array from identities involving Chebyshev polynomials. More recently, Adegoke, Frontczak and Goy [1] used explicit Chebyshev representations to derive binomial sums involving Fibonacci and Lucas numbers.

The present work follows a different direction. Starting from the substitutions

$$u = z + z^{-1} \qquad \text{and} \qquad v = z - z^{-1},$$

we use the resulting Chebyshev identities as explicit connection formulas that rewrite the symmetric and antisymmetric Laurent expansions arising below as polynomial expansions in u or v. We then combine these formulas systematically with coefficient extraction. This procedure yields three explicit pairs of mutually inverse triangular transformations. They relate, respectively,

$$\binom{2n}{n-k} \quad \text{to} \quad \binom{n}{k}2^{n-k},$$

$$\binom{2n}{n-k} \quad \text{to} \quad \binom{n}{k}4^{n-k},$$

and

$$\binom{2n}{n-k} - \binom{2n}{n-k-1} \quad \text{to} \quad \binom{n}{k}4^{n-k}.$$

To the best of our knowledge, these particular pairs of mutually inverse triangular transformations have not been recorded explicitly in the forms presented here. Thus, our contribution lies not in the underlying Chebyshev expansions, which are classical, but in using a common Laurent coefficient-extraction framework to derive three pairs of mutually inverse triangular transformations and to connect their closed forms with natural combinatorial interpretations. We further show that the same mechanism extends to a signed refinement of the first transformation.

Theorem 1. *For every $n \geq k \geq 0$,*

$$\binom{2n}{n-k} + \sum_{m=1}^{\lfloor (n-k)/2 \rfloor} (-1)^m \frac{k+2m}{k+m}\binom{k+m}{m}\binom{2n}{n-k-2m} = \binom{n}{k}2^{n-k}.$$

The direct transformation of Theorem 1 admits the following inverse form.

Corollary 2. *For every $n \geq k \geq 0$,*

$$\binom{2n}{n-k} = \sum_{m=0}^{\lfloor (n-k)/2 \rfloor} \binom{k+2m}{m}\binom{n}{k+2m}2^{n-k-2m}.$$

For $d \geq 1$, the coefficients appearing in Theorem 1 are precisely those occurring in the classical explicit expansion [6, Chap. 2]

$$2T_d\left(\frac{x}{2}\right) = \sum_{m=0}^{\lfloor d/2 \rfloor} (-1)^m \frac{d}{d-m}\binom{d-m}{m}x^{d-2m} \tag{1}$$

of the Chebyshev polynomials of the first kind.

In fact, Theorem 1 is only the first member of a broader family of Chebyshev–binomial identities. The even-indexed and odd-indexed Chebyshev families give rise to analogous summation formulas whose evaluations are remarkably simple and involve the factor $\binom{n}{k}4^{n-k}$. We state two representative examples below.

Theorem 3. *For every $n \geq k \geq 0$,*

$$\binom{2n}{n-k} + \sum_{m=1}^{n-k} \frac{2k+2m}{2k+m}\binom{2k+m}{m}\binom{2n}{n-k-m} = \binom{n}{k}4^{n-k}.$$

As in the case of Theorem 1, the transformation of Theorem 3 also admits an explicit inverse.

Corollary 4. *For every $n \geq k \geq 0$,*

$$\binom{2n}{n-k} = \sum_{m=0}^{n-k}(-1)^m\binom{2k+2m}{m}\binom{n}{k+m}4^{n-k-m}.$$

Theorem 3 is induced by the even-indexed Chebyshev polynomials

$$2T_{2r}\left(\frac{iv}{2}\right),$$

whereas Theorem 5 arises from the odd-indexed polynomials

$$2T_{2r+1}\left(\frac{iv}{2}\right), \qquad v = z - z^{-1}.$$

The imaginary argument converts the alternating coefficients in the standard Chebyshev expansions into the positive coefficients occurring in Lemmas 7 and 8.

Theorem 5. *For every $n \geq k \geq 0$,*

$$\sum_{m=0}^{n-k} \frac{(2k+2m+1)^2}{(2k+m+1)(n+k+m+1)}\binom{2k+m+1}{m}\binom{2n}{n+k+m} = \binom{n}{k}4^{n-k}.$$

Theorem 5 is accompanied by the following inverse transformation.

Corollary 6. *For every $n \geq k \geq 0$,*

$$\frac{2k+1}{n+k+1}\binom{2n}{n+k} = \sum_{m=0}^{n-k}(-1)^m\binom{2k+2m+1}{m}\binom{n}{k+m}4^{n-k-m}.$$

The common value

$$\binom{n}{k}4^{n-k}$$

in Theorems 3 and 5 has an independent combinatorial interpretation. If a $4n$-element set is partitioned into n blocks of size 4, then this quantity counts the $(n+3k)$-subsets that intersect every block and contain exactly k complete blocks. One chooses the complete blocks in $\binom{n}{k}$ ways and one element from each remaining block in 4^{n-k} ways.

The remainder of the paper is organized as follows. Section 2 proves Theorem 1 by combining the classical expansion of the Chebyshev polynomials with a coefficient-extraction argument. Sections 3 and 4 establish Theorems 3 and 5, respectively, by deriving analogous expansions for the even and odd Chebyshev families and applying the same coefficient-extraction principle. In Section 5 we introduce a signed refinement of the classical enumeration of pairs of disjoint subsets. This refinement yields a combinatorial interpretation of

$$\frac{n-2k}{n}\binom{n}{k}2^{n-k},$$

and, through the same Laurent coefficient-extraction mechanism, yields an additional pair of mutually inverse Chebyshev–binomial transformations.

2 First Chebyshev–binomial identity

In this section, we prove Theorem 1 by expressing the elements of the standard monomial symmetric Laurent basis

$$\{1\} \cup \{z^d + z^{-d}\}_{d\geq 1}$$

as scaled Chebyshev polynomials of the first kind in $u = z + z^{-1}$. The explicit expansion of these polynomials provides the coefficients occurring in the left-hand side of Theorem 1, while a coefficient-extraction argument yields the desired closed form. We conclude the section by proving the inverse transformation stated in Corollary 2.

Proof of Theorem 1. Let $T_d(x)$ denote the Chebyshev polynomial of the first kind, defined by

$$T_d(\cos\theta) = \cos(d\theta).$$

Equivalently, if

$$u = z + z^{-1},$$

then

$$z^d + z^{-d} = 2T_d\Big(\frac{u}{2}\Big).$$

We shall use the classical explicit expansion of the Chebyshev polynomials of the first kind (1). Consequently, for $k \geq 1$, $m \geq 0$, and $d = k + 2m$, the coefficient of u^k in $z^d + z^{-d}$ is

$$[u^k]\big(z^{k+2m} + z^{-(k+2m)}\big) = (-1)^m \frac{k+2m}{k+m}\binom{k+m}{m}.$$

Now observe that

$$z^{-n}(1+z)^{2n} = (z + 2 + z^{-1})^n = (u+2)^n.$$

On the other hand,

$$z^{-n}(1+z)^{2n} = \sum_{d=-n}^{n} \binom{2n}{n+d} z^d.$$

Grouping the terms corresponding to d and $-d$, we obtain

$$(u+2)^n = \binom{2n}{n} + \sum_{d=1}^{n} \binom{2n}{n-d}\left(z^d + z^{-d}\right).$$

Taking the coefficient of u^k, with $k \geq 1$, gives

$$[u^k](u+2)^n = \sum_{\substack{k \leq d \leq n \\ d \equiv k \pmod 2}} \binom{2n}{n-d} [u^k](z^d + z^{-d}).$$

Writing $d = k + 2m$, we obtain

$$[u^k](u+2)^n = \sum_{m=0}^{\lfloor (n-k)/2 \rfloor} (-1)^m \frac{k+2m}{k+m}\binom{k+m}{m}\binom{2n}{n-k-2m}.$$

Since

$$[u^k](u+2)^n = \binom{n}{k} 2^{n-k},$$

the desired identity follows for $k \geq 1$.

It remains to consider $k = 0$. From

$$(u+2)^n = \binom{2n}{n} + \sum_{d=1}^{n} \binom{2n}{n-d}(z^d + z^{-d}),$$

only the terms with even d contribute to the constant term in u. For $d = 2m$, $m \geq 1$, the Chebyshev expansion gives

$$[u^0](z^{2m} + z^{-2m}) = 2(-1)^m.$$

Therefore,

$$[u^0](u+2)^n = \binom{2n}{n} + 2\sum_{m=1}^{\lfloor n/2 \rfloor} (-1)^m \binom{2n}{n-2m}.$$

On the other hand,

$$[u^0](u+2)^n = 2^n.$$

Hence

$$\binom{2n}{n} + 2\sum_{m=1}^{\lfloor n/2 \rfloor} (-1)^m \binom{2n}{n-2m} = 2^n,$$

which is precisely the case $k = 0$ of Theorem 1. This completes the proof. □

Proof of Corollary 2. Using

$$u = z + z^{-1},$$

we have

$$(u+2)^n = (z + z^{-1} + 2)^n = z^{-n}(1+z)^{2n}.$$

Therefore,

$$[z^k](u+2)^n = [z^{n+k}](1+z)^{2n} = \binom{2n}{n+k} = \binom{2n}{n-k}.$$

On the other hand,

$$(u+2)^n = \sum_{r=0}^{n} \binom{n}{r} 2^{n-r} u^r,$$

and

$$u^r = (z + z^{-1})^r = \sum_{j=0}^{r} \binom{r}{j} z^{r-2j}.$$

Hence the coefficient of z^k in u^r is nonzero only when

$$r \equiv k \pmod 2 \qquad \text{and} \qquad r \geq k.$$

Writing

$$r = k + 2m,$$

we obtain

$$[z^k]u^{k+2m} = \binom{k+2m}{m}.$$

Consequently,

$$\begin{aligned}(u+2)^n &= \sum_{m=0}^{\lfloor (n-k)/2 \rfloor} \binom{n}{k+2m} 2^{n-k-2m}[z^k]u^{k+2m} \\ &= \sum_{m=0}^{\lfloor (n-k)/2 \rfloor} \binom{k+2m}{m}\binom{n}{k+2m} 2^{n-k-2m}.\end{aligned}$$

Comparing this expression with

$$[z^k](u+2)^n = \binom{2n}{n-k}$$

gives the desired identity. □

For fixed n, define

$$a_k := \binom{2n}{n-k}, \qquad b_k := \binom{n}{k}2^{n-k}, \qquad 0 \le k \le n.$$

Theorem 1 has the form

$$b_k = \sum_{\substack{d=k \\ d\equiv k \pmod 2}}^{n} C_{k,d}\, a_d,$$

with

$$C_{k,k} = 1.$$

Consequently, after arranging the even and odd indices separately, the coefficient matrix becomes block diagonal, and each diagonal block is upper unitriangular. It is therefore invertible. Equivalently, in the natural ordering $0, 1, \ldots, n$, the full matrix is upper triangular with unit diagonal and with zero entries whenever $d \not\equiv k \pmod 2$.

Corollary 2 provides the inverse transformation explicitly. Thus, the direct and inverse identities form a pair of mutually inverse triangular transformations between

$$\left\{\binom{2n}{n-k}\right\}_{k=0}^{n} \quad \text{and} \quad \left\{\binom{n}{k}2^{n-k}\right\}_{k=0}^{n}.$$

3 Even Chebyshev family

As in Section 2, the expansion used below is induced by the Chebyshev polynomials of the first kind, now evaluated at an imaginary argument. Indeed, from

$$2T_d\left(\frac{w+w^{-1}}{2}\right) = w^d + w^{-d},$$

upon setting $w = iz$ and $v = z - z^{-1}$, we obtain

$$2T_d\left(\frac{iv}{2}\right) = (iz)^d + (iz)^{-d}.$$

In particular,

$$z^{2r} + z^{-2r} = (-1)^r\, 2T_{2r}\left(\frac{iv}{2}\right).$$

Thus, the coefficients occurring in the expansion of $z^{2r} + z^{-2r}$ in powers of v^2 are obtained from the even-indexed Chebyshev polynomials of the first kind.

We first derive the corresponding explicit expansion. It is then combined with the identity

$$(z + z^{-1})^{2n} = (v^2 + 4)^n$$

to obtain Theorem 3 by coefficient extraction.

Lemma 7. *Let $v = z - z^{-1}$. For every $r \geq 0$, we have*

$$z^{2r} + z^{-2r} = \begin{cases} 2, & r = 0, \\ \displaystyle\sum_{j=0}^{r} \frac{2r}{r+j}\binom{r+j}{r-j} v^{2j}, & r > 0. \end{cases}$$

Proof. For $r = 0$, the identity is immediate. Let $r \geq 1$. Since

$$z^{2r} + z^{-2r} = (-1)^r\, 2T_{2r}\left(\frac{iv}{2}\right),$$

formula (1) gives

$$z^{2r} + z^{-2r} = (-1)^r \sum_{m=0}^{r} (-1)^m \frac{2r}{2r-m}\binom{2r-m}{m}(iv)^{2r-2m}.$$

Setting $j = r - m$, and observing that

$$(-1)^r(-1)^{r-j}i^{2j} = 1,$$

we obtain

$$z^{2r}+z^{-2r}=\sum_{j=0}^{r}\frac{2r}{r+j}\binom{r+j}{r-j}v^{2j},$$

as required. □

Proof of Theorem 3. Since

$$v^2+4=(z-z^{-1})^2+4=(z+z^{-1})^2,$$

we obtain

$$(v^2+4)^n=(z+z^{-1})^{2n}.$$

Expanding the right-hand side gives

$$\begin{aligned}(z+z^{-1})^{2n}&=z^{-2n}(1+z^2)^{2n}\\&=z^{-2n}\sum_{m=0}^{2n}\binom{2n}{m}z^{2m}\\&=\sum_{r=-n}^{n}\binom{2n}{n+r}z^{2r}\\&=\binom{2n}{n}+\sum_{r=1}^{n}\binom{2n}{n-r}(z^{2r}+z^{-2r}).\end{aligned}$$

Applying Lemma 7, we obtain

$$(v^2+4)^n=\binom{2n}{n}+\sum_{r=1}^{n}\binom{2n}{n-r}\sum_{j=0}^{r}\frac{2r}{r+j}\binom{r+j}{r-j}v^{2j}.$$

First suppose that $k\geq 1$. Extracting the coefficient of v^{2k} gives

$$[v^{2k}](v^2+4)^n=\sum_{r=k}^{n}\binom{2n}{n-r}\frac{2r}{r+k}\binom{r+k}{r-k}.$$

Replacing r by $r+k$, we obtain

$$[v^{2k}](v^2+4)^n=\sum_{r=0}^{n-k}\binom{2n}{n-k-r}\frac{2k+2r}{2k+r}\binom{2k+r}{r}.$$

On the other hand, the binomial expansion

$$(v^2+4)^n=\sum_{r=0}^{n}\binom{n}{r}4^{n-r}v^{2r}$$

gives

$$[v^{2k}](v^2+4)^n = \binom{n}{k}4^{n-k}.$$

Comparing the two expressions proves the identity for $k \geq 1$.

It remains to consider $k = 0$. Since the term corresponding to $r = 0$ in the Laurent expansion was separated, we have

$$[v^0](v^2+4)^n = \binom{2n}{n} + \sum_{r=1}^{n} \binom{2n}{n-r}[v^0](z^{2r} + z^{-2r}).$$

By Lemma 7,

$$[v^0](z^{2r} + z^{-2r}) = \frac{2r}{r}\binom{r}{r} = 2, \qquad r \geq 1.$$

Hence

$$[v^0](v^2+4)^n = \binom{2n}{n} + 2\sum_{r=1}^{n} \binom{2n}{n-r}.$$

On the other hand,

$$[v^0](v^2+4)^n = 4^n.$$

Therefore,

$$\binom{2n}{n} + 2\sum_{r=1}^{n} \binom{2n}{n-r} = 4^n,$$

which is precisely the case $k = 0$ of Theorem 3. This completes the proof. □

Proof of Corollary 4. Let

$$v = z - z^{-1}.$$

Since

$$v^2 + 4 = (z + z^{-1})^2,$$

we have

$$(v^2+4)^n = (z + z^{-1})^{2n}.$$

Therefore,

$$[z^{2k}](v^2+4)^n = [z^{2k}](z + z^{-1})^{2n} = \binom{2n}{n+k} = \binom{2n}{n-k}.$$

On the other hand,

$$(v^2+4)^n = \sum_{j=0}^{n} \binom{n}{j} 4^{n-j} v^{2j},$$

while

$$v^{2j} = (z - z^{-1})^{2j} = \sum_{\ell=0}^{2j} (-1)^{\ell} \binom{2j}{\ell} z^{2j-2\ell}.$$

For $j \geq k$, the coefficient of z^{2k} in v^{2j} is

$$[z^{2k}]v^{2j} = (-1)^{j-k} \binom{2j}{j-k}.$$

Consequently,

$$[z^{2k}](v^2+4)^n = \sum_{j=k}^{n} (-1)^{j-k} \binom{2j}{j-k} \binom{n}{j} 4^{n-j}.$$

Replacing j by $k+m$ gives

$$[z^{2k}](v^2+4)^n = \sum_{m=0}^{n-k} (-1)^m \binom{2k+2m}{m} \binom{n}{k+m} 4^{n-k-m}.$$

Comparing the two expressions for $[z^{2k}](v^2+4)^n$ proves the result. □

As in Section 2, the coefficient matrix in Theorem 3 is upper triangular with unit diagonal, and is therefore invertible. Corollary 4 gives its inverse explicitly. Hence these two identities define mutually inverse triangular transformations between

$$\left\{ \binom{2n}{n-k} \right\}_{k=0}^{n} \quad \text{and} \quad \left\{ \binom{n}{k} 4^{n-k} \right\}_{k=0}^{n}.$$

4 Odd Chebyshev family

The odd family is obtained from the same Chebyshev substitution. Indeed, setting $w = iz$ in

$$2T_d\left(\frac{w+w^{-1}}{2}\right) = w^d + w^{-d}$$

and writing $v = z - z^{-1}$, we obtain

$$2T_d\left(\frac{iv}{2}\right) = (iz)^d + (iz)^{-d}.$$

For $d = 2r + 1$, this gives

$$2T_{2r+1}\left(\frac{iv}{2}\right) = i(-1)^r\left(z^{2r+1} - z^{-2r-1}\right),$$

or equivalently,

$$z^{2r+1} - z^{-2r-1} = \frac{(-1)^r}{i}\, 2T_{2r+1}\left(\frac{iv}{2}\right).$$

Hence the expansion used in this section is induced by the odd-indexed Chebyshev polynomials of the first kind.

After establishing the corresponding explicit expansion, we combine it with

$$(z - z^{-1})(z + z^{-1})^{2n} = v(v^2 + 4)^n$$

and compare the coefficients of v^{2k+1}.

Lemma 8. *Let $v = z - z^{-1}$. For every $r \geq 0$, we have*

$$z^{2r+1} - z^{-2r-1} = \sum_{j=0}^{r} \frac{2r+1}{r+j+1}\binom{r+j+1}{r-j} v^{2j+1}.$$

Proof. From

$$z^{2r+1} - z^{-2r-1} = \frac{(-1)^r}{i}\, 2T_{2r+1}\left(\frac{iv}{2}\right)$$

and (1), we obtain

$$z^{2r+1} - z^{-2r-1} = \frac{(-1)^r}{i} \sum_{m=0}^{r} (-1)^m \frac{2r+1}{2r+1-m}\binom{2r+1-m}{m}(iv)^{2r+1-2m}.$$

Setting $j = r - m$, the signs cancel, and therefore

$$z^{2r+1} - z^{-2r-1} = \sum_{j=0}^{r} \frac{2r+1}{r+j+1}\binom{r+j+1}{r-j} v^{2j+1}.$$

□

Proof of Theorem 5. Since

$$v = z - z^{-1} \qquad \text{and} \qquad v^2 + 4 = (z + z^{-1})^2,$$

we have

$$v(v^2+4)^n = (z - z^{-1})(z + z^{-1})^{2n}.$$

Expanding the right-hand side, we obtain

$$\begin{aligned}(z - z^{-1})(z + z^{-1})^{2n} &= (z - z^{-1}) \sum_{r=-n}^{n} \binom{2n}{n+r} z^{2r} \\ &= \sum_{r=0}^{n} \left(\binom{2n}{n+r} - \binom{2n}{n+r+1} \right) (z^{2r+1} - z^{-2r-1}).\end{aligned}$$

Since

$$\binom{2n}{n+r} - \binom{2n}{n+r+1} = \frac{2r+1}{n+r+1} \binom{2n}{n+r},$$

it follows that

$$v(v^2+4)^n = \sum_{r=0}^{n} \frac{2r+1}{n+r+1} \binom{2n}{n+r} (z^{2r+1} - z^{-2r-1}).$$

Applying Lemma 8, we get

$$v(v^2+4)^n = \sum_{r=0}^{n} \frac{2r+1}{n+r+1} \binom{2n}{n+r} \sum_{j=0}^{r} \frac{2r+1}{r+j+1} \binom{r+j+1}{r-j} v^{2j+1}.$$

Extracting the coefficient of v^{2k+1} gives

$$[v^{2k+1}]\, v(v^2+4)^n = \sum_{r=k}^{n} \frac{(2r+1)^2}{(r+k+1)(n+r+1)} \binom{r+k+1}{r-k} \binom{2n}{n+r}.$$

Replacing r by $r+k$, we obtain

$$[v^{2k+1}]\, v(v^2+4)^n = \sum_{r=0}^{n-k} \frac{(2k+2r+1)^2}{(2k+r+1)(n+k+r+1)} \binom{2k+r+1}{r} \binom{2n}{n+k+r}.$$

On the other hand,

$$v(v^2+4)^n = v \sum_{r=0}^{n} \binom{n}{r} 4^{n-r} v^{2r},$$

and therefore

$$[v^{2k+1}]\, v(v^2+4)^n = \binom{n}{k} 4^{\,n-k}.$$

Comparing the two expressions for the coefficient of v^{2k+1} completes the proof. □

Proof of Corollary 6. Let

$$v = z - z^{-1}.$$

Since

$$v^2 + 4 = (z + z^{-1})^2,$$

we have

$$v(v^2+4)^n = (z - z^{-1})(z + z^{-1})^{2n}.$$

Expanding the right-hand side, we obtain

$$(z - z^{-1})(z + z^{-1})^{2n} = (z - z^{-1}) \sum_{r=-n}^{n} \binom{2n}{n+r} z^{2r}.$$

Therefore,

$$\begin{aligned} v(v^2+4)^n &= \binom{2n}{n+k} - \binom{2n}{n+k+1} \\ &= \frac{2k+1}{n+k+1} \binom{2n}{n+k}. \end{aligned}$$

On the other hand,

$$v(v^2+4)^n = \sum_{j=0}^{n} \binom{n}{j} 4^{n-j} v^{2j+1},$$

while

$$v^{2j+1} = (z - z^{-1})^{2j+1} = \sum_{\ell=0}^{2j+1} (-1)^{\ell} \binom{2j+1}{\ell} z^{2j+1-2\ell}.$$

For $j \geq k$, the coefficient of z^{2k+1} in v^{2j+1} is

$$[z^{2k+1}] v^{2j+1} = (-1)^{j-k} \binom{2j+1}{j-k}.$$

Consequently,

$$[z^{2k+1}]\, v(v^2+4)^n = \sum_{j=k}^{n} (-1)^{j-k} \binom{2j+1}{j-k} \binom{n}{j} 4^{n-j}.$$

Replacing j by $k+m$, we obtain

$$[z^{2k+1}]\, v(v^2+4)^n = \sum_{m=0}^{n-k} (-1)^m \binom{2k+2m+1}{m} \binom{n}{k+m} 4^{n-k-m}.$$

Comparing the two expressions for $[z^{2k+1}]\, v(v^2+4)^n$ proves the result. □

For fixed n, set

$$a_k^{\text{odd}} := \frac{2k+1}{n+k+1}\binom{2n}{n+k} = \binom{2n}{n+k} - \binom{2n}{n+k+1},$$

and

$$b_k := \binom{n}{k} 4^{n-k}.$$

Then Theorem 5 and Corollary 6 define mutually inverse triangular transformations between the arrays

$$\left\{a_k^{\text{odd}}\right\}_{k=0}^{n} \quad \text{and} \quad \{b_k\}_{k=0}^{n}.$$

The corresponding coefficient matrices are triangular with unit diagonal.

5 Signed refinement of a classical enumeration

Recall that

$$\binom{n}{k} 2^{n-k}$$

counts the ordered pairs (A, B) of disjoint subsets of an n-element set X satisfying $|A| = k$.

We now introduce a signed refinement of this enumeration. Let x be a distinguished element of X. Instead of counting every pair (A, B) with weight 1, we assign a sign according to whether the distinguished element belongs to A. More precisely, define

$$w_x(A, B) = \begin{cases} 1, & x \notin A, \\ -1, & x \in A. \end{cases}$$

The following result shows that the resulting signed enumeration is given by a simple modification of the classical counting formula.

Theorem 9. *Let $n \geq 1$ and $0 \leq k \leq n$ be integers. Let X be an n-element set, and let $x \in X$ be a distinguished element. Then*

$$\sum_{\substack{A,B\subseteq X\\ A\cap B=\varnothing,\ |A|=k}} w_x(A,B) = \frac{n-2k}{n}\binom{n}{k}2^{n-k}.$$

Proof. The pairs with positive weight are precisely those for which $x \notin A$. There are

$$\binom{n-1}{k}$$

ways to choose A, and after A has been chosen, each of the remaining $n-k$ elements may either belong to B or remain outside $A \cup B$. Hence the total positive contribution equals

$$\binom{n-1}{k}2^{n-k}.$$

Similarly, the pairs with negative weight are those for which $x \in A$. In this case the remaining $k-1$ elements of A are chosen from the other $n-1$ elements, giving

$$\binom{n-1}{k-1}2^{n-k}$$

such pairs.

Therefore,

$$\sum_{\substack{A,B\subseteq X\\ A\cap B=\varnothing,\ |A|=k}} w_x(A,B) = \left(\binom{n-1}{k} - \binom{n-1}{k-1}\right)2^{n-k}.$$

Finally, using

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k},$$

we obtain

$$\sum_{\substack{A,B\subseteq X\\ A\cap B=\varnothing,\ |A|=k}} w_x(A,B) = \binom{n}{k}\frac{n-2k}{n}\,2^{n-k},$$

as claimed. □

Combining the signed enumeration of Theorem 9 with the coefficient-extraction argument underlying Theorem 1 leads to the following identity.

Theorem 10. *For every $n \geq k \geq 1$,*

$$\sum_{m=0}^{\lfloor (n-k)/2 \rfloor} (-1)^m \frac{k+2m}{k+m} \binom{k+m}{m} \frac{n-2(k+2m)^2}{n(2n-1)} \binom{2n}{n-k-2m} = \frac{n-2k}{n} \binom{n}{k} 2^{n-k}.$$

Proof. Let

$$u = z + z^{-1}.$$

Then

$$u + 2 = z + z^{-1} + 2 = \frac{(1+z)^2}{z}$$

and

$$2 - u = 2 - z - z^{-1} = -\frac{(z-1)^2}{z}.$$

Hence

$$(2-u)(u+2)^{n-1} = -\frac{(z-1)^2(1+z)^{2n-2}}{z^n}.$$

We expand the right-hand side as follows:

$$\begin{aligned}(2-u)(u+2)^{n-1} &= -z^{-n}(z^2-2z+1)(1+z)^{2n-2} \\ &= -\sum_{j=0}^{2n} \left(\binom{2n-2}{j-2} - 2\binom{2n-2}{j-1} + \binom{2n-2}{j} \right) z^{j-n},\end{aligned}$$

where, as usual, $\binom{2n-2}{r} = 0$ if $r < 0$ or $r > 2n-2$.

Putting $s = j - n$, we get

$$(2-u)(u+2)^{n-1} = \sum_{s=-n}^{n} c_s z^s,$$

where

$$c_s = -\left(\binom{2n-2}{n+s-2} - 2\binom{2n-2}{n+s-1} + \binom{2n-2}{n+s} \right).$$

By the symmetry of binomial coefficients, $c_s = c_{-s}$. Therefore

$$(2-u)(u+2)^{n-1} = c_0 + \sum_{s=1}^{n} c_s \left(z^s + z^{-s} \right).$$

For $0 \le s \le n$, we express the three binomial coefficients in terms of $\binom{2n}{n+s}$. We have

$$\binom{2n-2}{n+s-2} = \frac{(n+s)(n+s-1)}{2n(2n-1)}\binom{2n}{n+s},$$

$$\binom{2n-2}{n+s-1} = \frac{(n+s)(n-s)}{2n(2n-1)}\binom{2n}{n+s},$$

and

$$\binom{2n-2}{n+s} = \frac{(n-s)(n-s-1)}{2n(2n-1)}\binom{2n}{n+s}.$$

Therefore,

$$c_s = \frac{\binom{2n}{n+s}}{2n(2n-1)}\Big(-(n+s)(n+s-1) + 2(n+s)(n-s) - (n-s)(n-s-1)\Big).$$

The expression in parentheses simplifies to

$$2n - 4s^2 = 2(n-2s^2).$$

Hence

$$c_s = \frac{n-2s^2}{n(2n-1)}\binom{2n}{n+s}.$$

Finally, by the symmetry of the binomial coefficients,

$$\binom{2n}{n+s} = \binom{2n}{n-s},$$

and thus

$$c_s = \frac{n-2s^2}{n(2n-1)}\binom{2n}{n-s}.$$

Thus

$$(2-u)(u+2)^{n-1} = c_0 + \sum_{s=1}^{n} \frac{n-2s^2}{n(2n-1)}\binom{2n}{n-s}\left(z^s + z^{-s}\right).$$

Now let $1 \le k \le n$. Since the constant Laurent term has no contribution to the coefficient of u^k, we obtain

$$[u^k](2-u)(u+2)^{n-1} = \sum_{\substack{k \le s \le n \\ s \equiv k \pmod 2}} c_s\,[u^k](z^s + z^{-s}).$$

Writing $s = k + 2m$, we have

$$[u^k](z^{k+2m} + z^{-k-2m}) = (-1)^m \frac{k+2m}{k+m}\binom{k+m}{m}.$$

Therefore,

$$(2-u)(u+2)^{n-1} = \sum_{m=0}^{\lfloor (n-k)/2 \rfloor} (-1)^m \frac{k+2m}{k+m}\binom{k+m}{m} \times \frac{n-2(k+2m)^2}{n(2n-1)}\binom{2n}{n-k-2m}.$$

On the other hand,

$$(2-u)(u+2)^{n-1} = 2(u+2)^{n-1} - u(u+2)^{n-1},$$

and hence

$$\begin{aligned}(2-u)(u+2)^{n-1} &= 2\binom{n-1}{k}2^{n-1-k} - \binom{n-1}{k-1}2^{n-k} \\ &= \left(\binom{n-1}{k} - \binom{n-1}{k-1}\right)2^{n-k}.\end{aligned}$$

Using

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k},$$

we obtain the desired identity. □

Remark. The case $k = 0$ requires separate treatment because the factor

$$\frac{k+2m}{k+m}$$

is undefined at $k = m = 0$. Extracting the constant term in u from

$$(2-u)(u+2)^{n-1} = c_0 + \sum_{s=1}^{n} c_s(z^s + z^{-s}),$$

we obtain

$$[u^0](2-u)(u+2)^{n-1} = c_0 + \sum_{m=1}^{\lfloor n/2 \rfloor} c_{2m}[u^0](z^{2m} + z^{-2m}).$$

Since

$$c_0 = \frac{1}{2n-1}\binom{2n}{n}$$

and

$$[u^0](z^{2m} + z^{-2m}) = 2(-1)^m, \qquad m \geq 1,$$

it follows that

$$\frac{1}{2n-1}\binom{2n}{n} + 2\sum_{m=1}^{\lfloor n/2 \rfloor}(-1)^m \frac{n-8m^2}{n(2n-1)}\binom{2n}{n-2m} = 2^n.$$

Indeed,

$$[u^0](2-u)(u+2)^{n-1} = 2^n.$$

Thus, this identity is the constant-term counterpart of Theorem 10.

If the distinguished element is allowed to vary over all elements of X, one obtains a related signed enumeration. Consider triples (A, B, x), where $A, B \subseteq X$, $A \cap B = \varnothing$, $|A| = k$, and $x \in X$. Assign the weight

$$\omega(A, B, x) = \begin{cases} 1, & x \notin A, \\ -1, & x \in A. \end{cases}$$

For each fixed pair (A, B), there are $n - k$ choices of x with positive weight and k choices with negative weight. Hence the total contribution of each pair is $n - 2k$. Since the number of such pairs (A, B) is $\binom{n}{k}2^{n-k}$, we obtain

$$\sum_{\substack{A,B\subseteq X,\ x\in X \\ A\cap B=\varnothing,\ |A|=k}} \omega(A, B, x) = (n-2k)\binom{n}{k}2^{n-k}.$$

This observation allows us to rewrite Theorem 10 in the equivalent form

$$\sum_{m=0}^{\lfloor (n-k)/2 \rfloor}(-1)^m \frac{k+2m}{k+m}\binom{k+m}{m}\frac{n-2(k+2m)^2}{2n-1}\binom{2n}{n-k-2m} = (n-2k)\binom{n}{k}2^{n-k},$$

whose right-hand side admits the signed combinatorial interpretation described above.

Theorem 10, together with its constant-term counterpart given above, defines a triangular transformation for the full range $0 \leq k \leq n$. Reversing the coefficient-extraction argument yields the corresponding inverse transformation.

Corollary 11. *For every $n \geq 1$ and $0 \leq k \leq n$,*

$$\frac{n-2k^2}{2n-1}\binom{2n}{n-k} = \sum_{m=0}^{\lfloor (n-k)/2 \rfloor} (n-2k-4m)\binom{k+2m}{m}\binom{n}{k+2m}2^{n-k-2m}.$$

Proof. Let

$$u = z + z^{-1}$$

and set

$$F_n(u) := (2-u)(u+2)^{n-1}.$$

Expanding $F_n(u)$ in powers of u, we obtain

$$\begin{aligned} F_n(u) &= 2(u+2)^{n-1} - u(u+2)^{n-1} \\ &= \sum_{r=0}^{n} \left(\binom{n-1}{r} - \binom{n-1}{r-1} \right) 2^{n-r}u^r \\ &= \sum_{r=0}^{n} \frac{n-2r}{n}\binom{n}{r}2^{n-r}u^r, \end{aligned}$$

where the usual convention $\binom{n-1}{-1} = \binom{n-1}{n} = 0$ is understood.

On the other hand, the Laurent expansion used in the proof of Theorem 10 gives

$$F_n(u) = c_0 + \sum_{s=1}^{n} c_s(z^s + z^{-s}),$$

where

$$c_s = \frac{n-2s^2}{n(2n-1)}\binom{2n}{n-s}.$$

Hence

$$[z^k]F_n(u) = \frac{n-2k^2}{n(2n-1)}\binom{2n}{n-k}.$$

Since

$$u^r = (z+z^{-1})^r = \sum_{j=0}^{r}\binom{r}{j}z^{r-2j},$$

the coefficient of z^k in u^r is nonzero only when $r \geq k$ and $r \equiv k \pmod 2$. Writing $r = k+2m$, we have

$$[z^k]u^{k+2m} = \binom{k+2m}{m}.$$

Consequently,

$$F_n(u) = \sum_{m=0}^{\lfloor (n-k)/2 \rfloor} \binom{k+2m}{m} \frac{n-2(k+2m)}{n} \binom{n}{k+2m} 2^{n-k-2m}.$$

Comparing the two expressions for $[z^k]F_n(u)$ and multiplying by n proves the result. □

For fixed $n \geq 1$, define

$$a_k := \frac{n-2k^2}{n(2n-1)} \binom{2n}{n-k}, \qquad b_k := \frac{n-2k}{n} \binom{n}{k} 2^{n-k}, \qquad 0 \leq k \leq n.$$

Then Theorem 10, together with its constant-term counterpart, gives the complete direct transformation

$$b_0 = a_0 + 2 \sum_{m=1}^{\lfloor n/2 \rfloor} (-1)^m a_{2m},$$

and, for $1 \leq k \leq n$,

$$b_k = \sum_{m=0}^{\lfloor (n-k)/2 \rfloor} (-1)^m \frac{k+2m}{k+m} \binom{k+m}{m} a_{k+2m}.$$

On the other hand, Corollary 11 is equivalently written as

$$a_k = \sum_{m=0}^{\lfloor (n-k)/2 \rfloor} \binom{k+2m}{m} b_{k+2m}, \qquad 0 \leq k \leq n.$$

Thus, after arranging the even and odd indices separately, the coefficient matrix of the direct transformation becomes block diagonal, and each diagonal block is upper unitriangular. The inverse formula above gives the inverse matrix explicitly. Hence Theorem 10, its constant-term counterpart, and Corollary 11 together define a pair of mutually inverse triangular transformations between

$$\left\{ \frac{n-2k^2}{n(2n-1)} \binom{2n}{n-k} \right\}_{k=0}^{n} \quad \text{and} \quad \left\{ \frac{n-2k}{n} \binom{n}{k} 2^{n-k} \right\}_{k=0}^{n}.$$

6 Concluding remarks

A central outcome of this paper is that the explicit expansions of the Chebyshev polynomials of the first kind naturally induce several invertible triangular transformations between classical families of binomial coefficients. Although these transformations arise from different Chebyshev expansions, they are all obtained by the same coefficient-extraction principle, which provides a unified framework for deriving both direct and inverse identities.

Beyond their algebraic interest, the closed forms occurring in the direct transformations admit natural combinatorial interpretations. They connect the binomial coefficients

$$\binom{2n}{n-k}$$

with the classical combinatorial sequences

$$\binom{n}{k}2^{n-k} \quad \text{and} \quad \binom{n}{k}4^{n-k},$$

whose enumerative meanings arise from disjoint subsets and block decompositions, respectively. Furthermore, the signed refinement introduced in Section 5 shows that the same approach naturally extends to weighted enumerations, yielding an additional pair of direct and inverse Chebyshev–binomial transformations.

The coefficient-extraction approach employed here is not restricted to the particular identities considered in this paper. Whenever suitable explicit connection formulas or Laurent-type representations are available, the same method may produce further invertible triangular transformations together with their associated combinatorial interpretations. It would therefore be interesting to investigate analogous constructions for other classical orthogonal polynomials, particularly the Jacobi polynomials and their Gegenbauer specializations.

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