

FISHER-LIKE METRICS ASSOCIATED WITH ϕ -DEFORMED NAUDTS ENTROPIES II - THE WEIGHTED CASE*

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Dedicated to the memory of Professor Mihail Megan

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Abstract

The paper continues our previous study on semi-Riemannian generalized Fisher metrics and Fisher-like metrics [26], in the framework of weighted ϕ -deformed (Naudts) entropy. The associated scalar curvatures carry information related to the local structure of the space of the distributions of probability. The addition of some related group logarithms extends and refines results from the previously quoted paper on the non-weighted case.

Keywords: weighted ϕ -deformed (Naudts) entropy, weighted divergence, weighted relative group entropy, weighted generalized Fisher metric, weighted Fisher-like metric.

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1 Introduction

Modeling stochastic phenomena involves many different types of entropies and divergences, ranking from "simpler" to "generalized" ones [5, 7, 11, 14, 20–22, 26, 28, 30, 31, 34–36] (see also [4, 6, 9, 25, 28, 29, 38] for further details and references).

One direction uses differential geometric tools ([1, 2, 20]), such as the Fisher (semi-Riemannian) metrics, inspired by the Fisher Information matrices. The under-construction dictionary between Statistics and (Riemannian) Geometry is expected to enrich both domains with new insights ([6, 20, 40]).

The context can be generalized by considering weights associated to entropies and divergences [7, 8]. These weights allow additional refinements and new applications, for both entropies and divergences, which impact all derived notions, in particular the associated Fisher metrics. A synthesis concerning the weighting extension method can be found in our previous paper [8].

The ϕ -deformed (Naudts) entropy is a newcomer in the panorama of modern entropies ([15, 16]). Its versatility qualifies it for various applications in Physics and Information theory, beyond the classical entropies like BGS, Kaniadakis, Tsallis ones. The Fisher metrics associated with it and the related divergences were intensively studied ([12, 16–19, 39]). For additional historical notes and more details on generalized entropy and generalized divergence topics, including some related to the ϕ -deformed (Naudts) entropy, we refer to our previous paper [26].

1.1 The content of the paper

In Section 2, we recall the framework for the generalized entropies and divergences defined in [8, 26], such as the (classical) ϕ -deformed (Naudts) entropy and the generalized group entropy functional.

In Section 3, we recall the generalized Fisher-like metrics associated with generalized entropies and divergences, defined and studied in [6, 26].

The core of the paper is Section 4; it contains the definition of the weighted generalized Fisher-like metrics, associated with group relative entropies based on weighted ϕ -deformed (Naudts) entropies and divergences. Their coefficients depend on the PDFs, on the ϕ -deformed logarithm, on the weighting function and (eventually) on a group logarithm too. A comparison with the non-weighted case is pointed out.

Examples of the new metrics are provided in Section 5, based on exponential PDFs. We calculate their scalar curvatures functions and we do some geometrically oriented remarks.

1.2 Conventions

We suppose, implicitly, that the integrals are correctly defined and that they commute with their derivatives. All differential objects are considered to be differentiable ("smooth"). Unless otherwise stated, the metrics are supposed to be (only) non-degenerated, i.e. semi-Riemannian ones.

2 Preliminaries: the ϕ -deformed (Naudts) entropy, the formal group logarithm and the generalized divergences

Let $X \subset \mathbb{R}^m$ be a domain and $\rho = \rho(x)$ a fixed probability density function (PDF) on X (i.e. $\rho(x) \geq 0$ and $\int_X \rho(x)dx = 1$). Let φ be a fixed real valued differentiable function on X . We defined ([26]) a φ -dependent generalized (normalized) entropy of X , by

$$H[\rho] = - \int_X \rho(x) \varphi(\rho(x)) dx. \quad (1)$$

Let $F : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ a smooth function and σ an additional fixed PDF. We define

$$D(\rho, \sigma) := \int_X F(\rho(x), \sigma(x)) dx. \quad (2)$$

Implicitly, we suppose that $D(\rho, \sigma) \geq 0$ and $D(\rho, \sigma) = 0$ if and only if $\rho = \sigma$. The number $D(\rho, \sigma)$ is called the (generalized) divergence between ρ and σ and measures to what extent σ influences ρ , as governed by F (see [26]).

Example 1. *With the previous notations, we give two examples; for more examples, comments and details, see [23, 24, 26, 32].*

(i) *Let $\phi : (0, \infty) \rightarrow \mathbb{R}$ be a positive, differentiable, strictly-increasing function. The ϕ -deformed (Naudts) logarithm is defined by (see [16, 26])*

$$\log_\phi^N(y) := \int_1^y \frac{1}{\phi(z)} dz. \quad (3)$$

The function $\varphi_\phi^N := \log_\phi^N$ defines the ϕ -deformed (Naudts) entropy.

(ii) Let $G = G(t)$ be a formal group logarithm, which is a differentiable real valued function with some special algebraic properties, inspired from the formal series linking Lie groups to Lie algebras. More precisely,

$$G(t) := \sum_{i=0}^{\infty} c_i \frac{t^{i+1}}{i+1},$$

where $c_0 = 1$ and $c_i \in \mathbb{Q}$. Its inverse is

$$F(s) := \sum_{i=0}^{\infty} \gamma_i \frac{s^{i+1}}{i+1},$$

where $\gamma_i \in \mathbb{Q}$, $\gamma_0 = 1$, $\gamma_1 = -c_1$, $\gamma_2 = \frac{3}{2}c_1^2 - c_2$ and so on. (We refer to [6, 37, 38] for details about these functions). The simplest example is $G(t) = t$.

We defined [26] the generalized group entropy functional (GGEF) associated with (1), by

$$S_G(\rho) := \int_X \rho(x) G(\varphi \circ \rho(x)) dx. \quad (4)$$

In particular, for $\varphi := -\log$, we recover the well-known group entropy functional ([6, 37]) associated with (1)

$$S_G(\rho) := \int_X \rho(x) G(\log \rho(x)^{-1}) dx. \quad (5)$$

Similar GGEFs can be provided by replacing the Neperian logarithm by other “generalized” logarithms. For example, in Section 3, we shall consider the geometries associated with the GGEF, based on ϕ -deformed (Naudts) entropies, by using the generalized logarithm $\log_\phi^N$ from (3).

Example 2. (i) Consider the previous framework; we recall now the two main types of divergence from [26].

The generalized (quotient) relative entropy (a.k.a. generalized divergence) between ρ and σ is given by (see [3, 8])

$$\tilde{D}(\rho \parallel \sigma) := \int_X \rho(x) \varphi\left(\frac{\rho(x)}{\sigma(x)}\right) dx. \quad (6)$$

In this case, the previous function $F(z, y) := z\varphi(\frac{z}{y})$.

The generalized (difference) relative entropy between ρ and σ is given by

$$D(\rho \parallel \sigma) := \int_X \rho(x) [\varphi(\rho(x)) - \varphi(\sigma(x))] dx. \quad (7)$$

In this case, the function $F(z, y) := z[\varphi(z) - \varphi(y)]$.

(ii) Following the Example 1 (vi), possible generalized divergences as relative group entropies may use formal group logarithms G , φ -likelihood functions and quotient/ difference operation upon two PDFs. For example, from (7) we can derive

$$D_G(\rho \parallel \sigma) := \int_X \rho(x) \cdot G(\varphi(\rho(x)) - \varphi(\sigma(x))) dx.$$

(iii) Let ρ_1 and ρ_2 be two fixed PDFs. Consider a fixed convex differentiable function $\psi : \mathbb{R} \rightarrow \mathbb{R}$. Then, the Bregman divergence is given by

$$D_\psi(\rho_1 \parallel \rho_2) := \int_X \{\psi(\rho_1(x)) - \psi(\rho_2(x)) - (\rho_1(x) - \rho_2(x))\psi'(\rho_2(x))\} dx. \quad (8)$$

(iv) Let $w : X \rightarrow \mathbb{R}$ a differentiable "weighting" function (usually supposed non-negative). The weighted entropies and divergences functionals [7, 8] are obtained from the formulas of the non-weighted ones, by multiplying the integrand with w . We remark that any φ -generalized entropy from (1) and any generalized divergence from (2) can be put in the form of a weighted entropy or a weighted divergence, by means of other functions φ and F , respectively.

For example, the ϕ -deformed (Naudts) entropy is (see Example 1 (i))

$$H[\rho] = - \int_X \rho(x) \cdot \log_\phi^N(\rho(x)) dx$$

and the weighted ϕ -deformed (Naudts) entropy is

$$H^w[\rho] = - \int_X w(x) \cdot \rho(x) \cdot \log_\phi^N(\rho(x)) dx. \quad (9)$$

3 Preliminaries: Fisher-like metrics associated with generalized entropies and divergences

We recall, following [6, 8, 26], the notion of Fisher metric associated with (generalized) entropies and divergences, defined on the space of parameters of an arbitrary PDF.

Let Θ be an open set of $\mathbb{R}^n$ and $\rho : X \times \Theta \rightarrow \mathbb{R}$, $\rho = \rho(x, \theta)$ be a family of PDFs, parameterized by $\theta \in \Theta$. Fix $\varphi : \mathbb{R} \rightarrow \mathbb{R}$, $\varphi = \varphi(y)$ a differentiable ("controlling") function. By analogy with formula (1), we derived [26] the generalized entropy function $H : \Theta \rightarrow \mathbb{R}$

$$H(\theta) = - \int_X \rho(x, \theta) \cdot \varphi(\rho(x, \theta)) dx. \quad (10)$$

Generalized divergence functions can be defined accordingly, by extending formula (2) and the similar ones.

We defined [26] the *generalized Fisher metrics of type 1 and type 2* (denoted *GFM1* and *GFM2*, respectively), with components:

$$g_{ij}(\theta) := - \int_X \rho(x, \theta) \frac{\partial^2 \varphi(\rho(x, \theta))}{\partial \theta^i \partial \theta^j} dx, \quad i, j = \overline{1, n} \quad (11)$$

and

$$\tilde{g}_{ij}(\theta) := \int_X \rho(x, \theta) \frac{\partial \varphi(\rho(x, \theta))}{\partial \theta^i} \cdot \frac{\partial \varphi(\rho(x, \theta))}{\partial \theta^j} dx, \quad i, j = \overline{1, n}. \quad (12)$$

(Recall that, implicitly, we suppose the matrices $(g_{ij})_{i,j=\overline{1,n}}$ and $(\tilde{g}_{ij})_{i,j=\overline{1,n}}$ are non-degenerated and of constant index on Θ .)

If non-degenerated, the Hessian matrix

$$h_{ij}(\theta) := \frac{\partial^2 H(\theta)}{\partial \theta^i \partial \theta^j} \quad (13)$$

provides another semi-Riemannian metric on Θ .

In [26], we defined other two Fisher-like semi-Riemannian metrics α and β on Θ , with components:

$$\alpha_{ij}(\theta) := \int_X \frac{\partial^2 \rho(x, \theta)}{\partial \theta^i \partial \theta^j} \cdot \varphi(\rho(x, \theta)) dx \quad (14)$$

and

$$\beta_{ij}(\theta) := \int_X \left\{ \frac{\partial \rho(x, \theta)}{\partial \theta^i} \cdot \frac{\partial \varphi(\rho(x, \theta))}{\partial \theta^j} + \frac{\partial \rho(x, \theta)}{\partial \theta^j} \cdot \frac{\partial \varphi(\rho(x, \theta))}{\partial \theta^i} \right\} dx \quad (15)$$

(under the same non-degeneracy implicit hypothesis).

Remark 1. (i) Consider a weighting function w on $X \times \Theta$. There are several possibilities to construct weighted Fisher-like metrics. For example, we can define the w -weighted counterpart of the metric (11):

$$g_{ij}(\theta) := - \int_X w(x, \theta) \cdot \rho(x, \theta) \frac{\partial^2 \varphi(\rho(x, \theta))}{\partial \theta^i \partial \theta^j} dx, \quad i, j = \overline{1, n}. \quad (16)$$

In what follows, we shall use this weighting convention.

(ii) In [26], we studied the previous Fisher-like metrics, in the particular case of the ϕ -deformed (Naudts) entropy, i.e. for $\varphi := \log_\phi^N$. In Section 4, we shall extend this inquiry in the weighted ϕ -deformed (Naudts) entropies case.

4 Fisher metrics associated with GGEFs based on weighted ϕ -deformed (Naudts) entropies and divergences

We particularize now the results from Section 3, in the case of the weighted ϕ -deformed (Naudts) entropies.

Consider ϕ a positive, differentiable and strictly-increasing function as in Example 1 (v) and the ϕ -deformed (Naudts) logarithm $\log_\phi^N$ defined in Formula (3). Let $\rho : X \times \Theta \rightarrow \mathbb{R}$, $\rho = \rho(x, \theta)$ be a family of parameterized PDFs, as in Section 3. The associated *GFM1* metric g and the *GFM2* metric $\tilde{g}$ are obtained as particular cases from (11) and (12):

$$g_{ij}(\theta) := - \int_X \rho(x, \theta) \frac{\partial^2 \log_\phi^N(\rho(x, \theta))}{\partial \theta^i \partial \theta^j} dx, \quad i, j = \overline{1, n} \quad (17)$$

and

$$\tilde{g}_{ij}(\theta) := \int_X \rho(x, \theta) \frac{\partial \log_\phi^N(\rho(x, \theta))}{\partial \theta^i} \cdot \frac{\partial \log_\phi^N(\rho(x, \theta))}{\partial \theta^j} dx, \quad i, j = \overline{1, n}. \quad (18)$$

We suppose, as usual, that g and $\tilde{g}$ are non-degenerated and that $\tilde{g}$ has a constant index on X .

Consider, in addition, a weighting function w on $X \times \Theta$. The associated w -weighted *GFM1* metric g^w and the w -weighted *GFM2* metric $\tilde{g}^w$ write:

$$g_{ij}^w(\theta) := - \int_X w(x, \theta) \rho(x, \theta) \frac{\partial^2 \log_\phi^N(\rho(x, \theta))}{\partial \theta^i \partial \theta^j} dx, \quad i, j = \overline{1, n} \quad (19)$$

and

$$\tilde{g}_{ij}^w(\theta) := \int_X w(x, \theta) \rho(x, \theta) \frac{\partial \log_\phi^N(\rho(x, \theta))}{\partial \theta^i} \cdot \frac{\partial \log_\phi^N(\rho(x, \theta))}{\partial \theta^j} dx, \quad i, j = \overline{1, n}. \quad (20)$$

We also consider, via (13), the associated Hessian metric $h = h(\theta)$

$$h_{ij}(\theta) = -\frac{\partial^2}{\partial \theta^i \partial \theta^j} \left\{ \int_X \rho(x, \theta) \cdot \log_\phi^N(\rho(x, \theta)) dx \right\}, \quad i, j = \overline{1, n}. \quad (21)$$

The Hessian metric associated to the w -weighted entropy function H^w is

$$h_{ij}^w(\theta) = -\frac{\partial^2}{\partial \theta^i \partial \theta^j} \left\{ \int_X w(x, \theta) \rho(x, \theta) \cdot \log_\phi^N(\rho(x, \theta)) dx \right\}, \quad i, j = \overline{1, n}. \quad (22)$$

Proposition 1. *With the previous notations, for every $i, j = \overline{1, n}$, we have*

$$g_{ij}^w(\theta) = \int_X w(x, \theta) \rho(x, \theta) \left\{ \frac{\partial \rho(x, \theta)}{\partial \theta^i} \cdot \frac{\partial \rho(x, \theta)}{\partial \theta^j} \cdot \phi^{-2}(\rho(x, \theta)) \cdot \phi'(\rho(x, \theta)) \right. \quad (23)$$

$$\left. - \frac{\partial^2 \rho(x, \theta)}{\partial \theta^i \partial \theta^j} \cdot \phi^{-1}(\rho(x, \theta)) \right\} dx,$$

$$\tilde{g}_{ij}^w(\theta) := \int_X w(x, \theta) \rho(x, \theta) \cdot \frac{\partial \rho(x, \theta)}{\partial \theta^i} \cdot \frac{\partial \rho(x, \theta)}{\partial \theta^j} \cdot \phi^{-2}(\rho(x, \theta)) dx, \quad (24)$$

and

$$h_{ij}^w(\theta) = \int_X w(x, \theta) \left\{ \rho(x, \theta) \frac{\partial \rho(x, \theta)}{\partial \theta^i} \cdot \frac{\partial \rho(x, \theta)}{\partial \theta^j} \cdot \phi^{-2}(\rho(x, \theta)) \cdot \phi'(\rho(x, \theta)) \right. \quad (25)$$

$$\begin{aligned} & - \frac{\partial^2 \rho(x, \theta)}{\partial \theta^i \partial \theta^j} \cdot \log_\phi^N(\rho(x, \theta)) - 2 \frac{\partial \rho(x, \theta)}{\partial \theta^i} \cdot \frac{\partial \rho(x, \theta)}{\partial \theta^j} \cdot \phi^{-1}(\rho(x, \theta)) \\ & - \rho(x, \theta) \cdot \frac{\partial^2 \rho(x, \theta)}{\partial \theta^i \partial \theta^j} \cdot \phi^{-1}(\rho(x, \theta)) \Big\} dx \\ & - \int_X \left\{ \frac{\partial^2 w(x, \theta)}{\partial \theta^i \partial \theta^j} \rho(x, \theta) \log_\phi^N(\rho(x, \theta)) \right. \\ & \left. + \left(\frac{\partial w(x, \theta)}{\partial \theta^j} \frac{\partial \rho(x, \theta)}{\partial \theta^i} + \frac{\partial w(x, \theta)}{\partial \theta^i} \frac{\partial \rho(x, \theta)}{\partial \theta^j} \right) (\log_\phi^N(\rho(x, \theta)) + \rho(x, \theta) \phi^{-1}(\rho(x, \theta))) \right\} dx. \end{aligned}$$

In this case, α^w and β^w are given by

$$\alpha_{ij}^w(\theta) := \int_X \frac{\partial^2(w(x, \theta)\rho(x, \theta))}{\partial\theta^i\partial\theta^j} \cdot \log_\phi^N(\rho(x, \theta))dx$$

and

$$\begin{aligned} & \beta_{ij}^w(\theta) \\ &:= \int_X \left\{ \frac{\partial(w(x, \theta)\rho(x, \theta))}{\partial\theta^i} \cdot \frac{\partial \log_\phi^N(\rho(x, \theta))}{\partial\theta^j} + \frac{\partial(w(x, \theta)\rho(x, \theta))}{\partial\theta^j} \cdot \frac{\partial \log_\phi^N(\rho(x, \theta))}{\partial\theta^i} \right\} dx. \end{aligned}$$

Corollary 1. *In a condensed form, we have the following relation*

$$h^w = g^w - \alpha^w - \beta^w.$$

We consider now, in addition, a fixed formal group logarithm G , as in Example 1 (vi). Let $\sigma := \rho(x, \theta_0)$ be the associated parameterized PDFs and $D_{G,\phi} = D_{G,\phi}(\rho \parallel \sigma)(\theta, \theta_0)$ be the generalized (difference) group relative entropy (a.k.a. the generalized (difference) group divergence), as particularization from (7) and Example 2.2 (i), (ii), written as

$$D_{G,\phi}(\rho \parallel \sigma)(\theta, \theta_0) = \int_X \rho(x, \theta) \cdot G\left(\log_\phi^N(\rho(x, \theta)) - \log_\phi^N(\rho(x, \theta_0))\right)dx.$$

Let

$$D_{G,\phi}^w(\rho \parallel \sigma)(\theta, \theta_0) = \int_X w(x, \theta)\rho(x, \theta) \cdot G\left(\log_\phi^N(\rho(x, \theta)) - \log_\phi^N(\rho(x, \theta_0))\right)dx.$$

Denote the generalized group Fisher metric associated with $D_{G,\phi}$ by

$$\hat{g}_{jk}(\theta_0) := \frac{\partial^2 D_{G,\phi}(\rho \parallel \sigma)(\theta, \theta_0)}{\partial\theta^j\partial\theta^k} \Big|_{\theta=\theta_0}. \quad (26)$$

Denote the w -weighted generalized group Fisher metric associated with $D_{G,\phi}^w$ by

$$\hat{g}_{jk}^w(\theta_0) := \frac{\partial^2 D_{G,\phi}^w(\rho \parallel \sigma)(\theta, \theta_0)}{\partial\theta^j\partial\theta^k} \Big|_{\theta=\theta_0}. \quad (27)$$

This Hessian-type metric will be calculated in the next result.

Proposition 2. *With the previous notations, we have the relation*

$$\begin{aligned}
\hat{g}_{jk}^w(\theta_0) = & G'(0) \cdot \left\{ \int_X w(x, \theta_0) \frac{\partial^2 \rho(x, \theta_0)}{\partial \theta^j \partial \theta^k} \cdot \frac{\rho(x, \theta_0)}{\phi(\rho(x, \theta_0))} dx \right. \\
& + 2 \int_X w(x, \theta_0) \phi(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^j} \log_\phi^N(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^k} \log_\phi^N(\rho(x, \theta_0)) dx \\
& - \int_X w(x, \theta_0) \rho(x, \theta_0) \cdot \phi'(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^j} \log_\phi^N(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^k} \log_\phi^N(\rho(x, \theta_0)) dx \Big\} \\
& + G''(0) \cdot \int_X w(x, \theta_0) \rho(x, \theta_0) \cdot \frac{\partial}{\partial \theta^j} \log_\phi^N(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^k} \log_\phi^N(\rho(x, \theta_0)) dx \\
& + G'(0) \int_X \left(\frac{\partial w(x, \theta_0)}{\partial \theta^k} \frac{\partial \rho(x, \theta_0)}{\partial \theta^j} + \frac{\partial w(x, \theta_0)}{\partial \theta^j} \frac{\partial \rho(x, \theta_0)}{\partial \theta^k} \right) \frac{\rho(x, \theta_0)}{\phi(\rho(x, \theta_0))} dx,
\end{aligned} \tag{28}$$

which may be re-written as depending only on ϕ and ρ , in

$$\begin{aligned}
\hat{g}_{jk}^w(\theta_0) = & G'(0) \cdot \left\{ \int_X w(x, \theta_0) \cdot \frac{\partial^2 \rho(x, \theta_0)}{\partial \theta^j \partial \theta^k} \cdot \frac{\rho(x, \theta_0)}{\phi(\rho(x, \theta_0))} dx \right. \\
& + 2 \int_X w(x, \theta_0) \cdot \phi^{-1}(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^j} \rho(x, \theta_0) \cdot \frac{\partial}{\partial \theta^k} \rho(x, \theta_0) dx \\
& - \int_X w(x, \theta_0) \cdot \rho(x, \theta_0) \cdot \phi'(\rho(x, \theta_0)) \cdot \phi^{-2}(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^j} \rho(x, \theta_0) \cdot \frac{\partial}{\partial \theta^k} \rho(x, \theta_0) dx \\
& + \int_X \left(\frac{\partial w}{\partial \theta^k}(x, \theta_0) \cdot \frac{\partial \rho}{\partial \theta^j}(x, \theta_0) + \frac{\partial w}{\partial \theta^j}(x, \theta_0) \cdot \frac{\partial \rho}{\partial \theta^k}(x, \theta_0) \right) \cdot \rho(x, \theta_0) \cdot \phi^{-1}(\rho(x, \theta_0)) dx \Big\} \\
& + G''(0) \cdot \int_X w(x, \theta_0) \cdot \rho(x, \theta_0) \cdot \phi^{-2}(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^j} \rho(x, \theta_0) \cdot \frac{\partial}{\partial \theta^k} \rho(x, \theta_0) dx.
\end{aligned} \tag{29}$$

The proof is similar with the previous ones and will be skipped.

Suppose, moreover, that $G(t) = t$. Then, we have

$$\begin{aligned}
\hat{g}_{jk}^w(\theta_0) = & \int_X w(x, \theta_0) \cdot \frac{\partial^2 \rho(x, \theta_0)}{\partial \theta^j \partial \theta^k} \cdot \frac{\rho(x, \theta_0)}{\phi(\rho(x, \theta_0))} dx \\
& + 2 \int_X w(x, \theta_0) \cdot \phi^{-1}(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^j} \rho(x, \theta_0) \cdot \frac{\partial}{\partial \theta^k} \rho(x, \theta_0) dx
\end{aligned} \tag{30}$$

$$\begin{aligned}
& - \int_X w(x, \theta_0) \cdot \rho(x, \theta_0) \cdot \phi'(\rho(x, \theta_0)) \cdot \phi^{-2}(\rho(x, \theta_0)) \cdot \frac{\partial}{\partial \theta^j} \rho(x, \theta_0) \cdot \frac{\partial}{\partial \theta^k} \rho(x, \theta_0) dx \\
& + \int_X \left(\frac{\partial w}{\partial \theta^k}(x, \theta_0) \cdot \frac{\partial \rho}{\partial \theta^j}(x, \theta_0) + \frac{\partial w}{\partial \theta^j}(x, \theta_0) \cdot \frac{\partial \rho}{\partial \theta^k}(x, \theta_0) \right) \cdot \rho(x, \theta_0) \cdot \phi^{-1}(\rho(x, \theta_0)) dx.
\end{aligned}$$

By analogy, starting with a generalized (quotient) group relative entropy (a.k.a. the generalized (quotient) group divergence) $\tilde{D}_{G,\phi} = \tilde{D}_{G,\phi}(\rho \parallel \sigma)(\theta, \theta_0)$, as particularization from (6), we shall obtain, in the sequel, other Fisher-like metrics, similar to the ones in Proposition 2.

Denote the generalized group Fisher metric associated with $\tilde{D}_{G,\phi}$ by

$$\bar{g}_{jk}(\theta_0) := \frac{\partial^2 \tilde{D}_{G,\phi}(\rho \parallel \sigma)(\theta, \theta_0)}{\partial \theta^j \partial \theta^k} \Big|_{\theta=\theta_0}. \quad (31)$$

The previous considerations relative to $D_{G,\phi}$, $D_{G,\phi}^w$, $\hat{g}$, $\hat{g}^w$ can be adapted, by analogy, for $\tilde{D}_{G,\phi}$, $\tilde{D}_{G,\phi}^w$ and associated metrics $\bar{g}$, $\bar{g}^w$ as well. A result similar to Proposition 2 can be easily derived and proved as in the non-weighted case [26].

Remark 2. *We constructed a large family of associated Riemannian metrics, in terms of the functions ϕ and w . Examples (with specific calculations focusing on the scalar curvature) will be given in the next section.*

5 Examples

We particularize now the results from Section 4, for the case when ρ is an exponential PDF and $m = 1$, $n = 2$. The deforming function ϕ and the weighting function w will be chosen conveniently, in order to be able to compute the integrals. For simplicity, we will omit (whenever possible) writing the function variables.

Let $X := \mathbb{R}$ and $\rho : \mathbb{R} \times \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ be the exponential (normal) PDF given by

$$\rho(x; \theta^1, \theta^2) = \frac{1}{\sqrt{2\pi}\theta^2} \cdot e^{-\frac{(x-\theta^1)^2}{2(\theta^2)^2}}. \quad (32)$$

We denote the first order and the second order partial derivatives of ρ , with respect to the variables θ^1 and θ^2 , by ρ_1 , ρ_2 , ρ_{11} , ρ_{12} , ρ_{22} . We have the formulas ([8]):

$$\begin{aligned}
\rho_1 &= \frac{x - \theta^1}{(\theta^2)^2} \cdot \rho, & \rho_2 &= \left\{ \frac{(x - \theta^1)^2}{(\theta^2)^3} - \frac{1}{\theta^2} \right\} \cdot \rho, \\
\rho_{11} &= \left\{ \frac{(x - \theta^1)^2}{(\theta^2)^4} - \frac{1}{(\theta^2)^2} \right\} \cdot \rho, & \rho_{12} &= \left\{ \frac{(x - \theta^1)^3}{(\theta^2)^5} - \frac{3(x - \theta^1)}{(\theta^2)^3} \right\} \cdot \rho, \\
\rho_{22} &= \left\{ \frac{(x - \theta^1)^4}{(\theta^2)^6} - \frac{5(x - \theta^1)^2}{(\theta^2)^4} + \frac{2}{(\theta^2)^2} \right\} \cdot \rho.
\end{aligned}$$

The classical Fisher metric g^0 has the coefficients $g_{11}^0 = (\theta^2)^{-2}$, $g_{12}^0 = g_{21}^0 = 0$ and $g_{22}^0 = 2(\theta^2)^{-2}$ ([8]).

Remark 3. Let A, k_1, k_2 be fixed real constants, with $k_1 \neq 0, k_2 \neq 0$. Then, the semi-Riemannian metric

$$y^{-A} \cdot \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

on the set $y \neq 0$ in $\mathbb{R}^2$ has the scalar curvature $-A \cdot (k_2)^{-1} \cdot y^{A-2}$ ([8]).

In the sequel, we give examples of semi-Riemannian metrics from Proposition 1, under various particular hypothesis. Consider a 2-parameter family of weighting functions $w = w_{a,b} : \mathbb{R} \times \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$, of the form

$$w(x, \theta^1, \theta^2) = (x - \theta^1)^{2a} \cdot (\theta^2)^{2b}.$$

(Here, $a > 0$ and b are fixed arbitrary parameters.)

Suppose $\phi(t) := t^c$, with $c \in (0, 2)$ a fixed arbitrary parameter. From Formula (23), we calculate the coefficients:

$$g_{11}^w = K_1(a, c) \cdot (\theta^2)^{2a+2b+c-3}, \quad g_{12}^w = g_{21}^w = 0,$$

$$g_{22}^w = (\sqrt{2\pi})^{c-2} \cdot (\theta^2)^{2b+c-8} \cdot [(c-1)I_{a+2} + (5-2c)(\theta^2)^2 I_{a+1} + (c-2)(\theta^2)^4 I_a],$$

where

$$K_1(a, c) = (2-c)^{-a-\frac{3}{2}} \cdot (\sqrt{2\pi})^{c-2} \cdot 2^{a+\frac{1}{2}} \cdot \Gamma(a + \frac{1}{2}) \cdot [(c-1)(2a+1) + 2-c]$$

and $I_a = I_a(c, \theta^2)$ is given by

$$I_a = (\theta^2)^{2a+1} \cdot (2-c)^{-a-\frac{1}{2}} \cdot 2^{a+\frac{1}{2}} \cdot \Gamma(a + \frac{1}{2}).$$

As the coefficients of g^w depend on the gamma-integrals, it is difficult to derive specific invariant objects from it. This is why, in the sequel, we suppose a be a positive integer. The previous formulas become:

$$\begin{aligned} g_{11}^w &= K_1(a, c) \cdot (\theta^2)^{2a+2b+c-3}, & g_{12}^w &= g_{21}^w = 0, \\ g_{22}^w &= K_2(a, c) \cdot (\theta^2)^{2a+2b+c-3}, \end{aligned} \quad (33)$$

where

$$K_1(a, c) = (2-c)^{-a-\frac{3}{2}} \cdot (\sqrt{2\pi})^{c-1} \cdot (2a-1)!! \cdot [(c-1)(2a+1) + 2 - c],$$

$$\begin{aligned} K_2(a, c) &= (2-c)^{-a-\frac{5}{2}} \cdot (\sqrt{2\pi})^{c-1} \cdot (2a-1)!! \cdot \\ &\cdot [(c-1)(2a+1)(2a+3) + (c-2)(2a+1)(2c-5) + (c-2)^3]. \end{aligned}$$

Denote by M the set of all pairs $(a, c) \in \mathbb{N}^* \times (0, 2)$, satisfying the (compatible !) system of inequalities

$$(c-1)(2a+1) + 2 - c > 0$$

$$(c-1)(2a+1)(2a+3) + (c-2)(2a+1)(2c-5) + (c-2)^3 > 0.$$

For each pair in M , g^w given in (5.2) is a Riemannian metric. In this case, as a consequence of Remark 3, the scalar curvature S^w of g^w is

$$S^w = \frac{(2-c)^{a+\frac{5}{2}} \cdot (\sqrt{2\pi})^{1-c} \cdot (2a+2b+c-3) \cdot (\theta^2)^{1-2a-2b-c}}{[(c-1)(2a+1)(2a+3) + (c-2)(2a+1)(2c-5) + (c-2)^3] \cdot (2a-1)!!}. \quad (34)$$

Remark 4. (i) Similar examples of metrics can be found also from $\tilde{g}^w$, h^w , α^w , β^w , $\hat{g}^w$ and $\bar{g}^w$ (see [26] for the non-weighted case).

(ii) We have

$$\text{sgn} S^w = \text{sgn}(2a + 2b + c - 3).$$

The scalar curvature S^w is constant if and only if $2a + 2b + c = 1$; in this case, it is negative (as expected).

(iii) The metrics g^w given in (5.2) are globally conformal with the Euclidean metric on the space of parameters (θ^1, θ^2) . The (explicit) conformal factors are independent of θ^1 , even if the weighting function w depends on both parameters θ^1 and θ^2 .

The scalar curvature function S^w in (5.3) is also dependent only on the "standard deviation" of the PDF modeled by θ^2 . Statistical applications dependent only on the standard deviation may be found, for example, in [10, 13, 27, 33, 41]. The statistical interpretation of the sectional curvature of the Fisher-like metrics studied in the weighted case is similar to that in the non-weighted case (for the later, see our comments in [26] and references therein).

The independence from θ^1 , mentioned earlier, is probably related to the fact that the parameter b is not subject to additional algebraic restrictions, as are the parameters a and c .

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