

FIBRE CONTRACTION PRINCIPLE IN COMPLETE b -METRIC SPACES*

Adrian Petruşel[†] Gabriela Petruşel[‡] Jen-Chih Yao[§]

Dedicated to the memory of Professor Mihail Megan

DOI 10.56082/annalsarscimath.2026.3.47

Abstract

In the framework of a complete b -metric space, we will discuss existence and uniqueness results for the fixed points of triangular operators. Using various classical metric conditions on the operators, we will obtain a generalization of the well-known fibre contraction principle in complete metric spaces.

Keywords: fixed point, complete b -metric space, contraction, Picard operator, Ulam-Hyers stability, well-posedness in the sense of Reich-Zaslavski.

MSC: 47H10, 54H25.

1 Introduction

Let (X, d, s) be a b -metric space with coefficient $s \geq 1$. (see [1], [3], [4], [18]). The only difference between a metric and a b -metric is given by the triangle inequality. In the b -metric case, the third axiom of the metric (the triangle inequality) takes the following form:

$$d(x, y) \leq s(d(x, z) + d(z, y)), \text{ for all } x, y, z \in X.$$

*Accepted for publication on March 10, 2026

[†]adrian.petrusel@ubbcluj.ro, Department of Mathematics, Babeş-Bolyai University Cluj-Napoca and Academy of Romanian Scientists, 50044 Bucharest, Romania

[‡]gabriela.petrusel@ubbcluj.ro, Department of Business, Babeş-Bolyai University Cluj-Napoca, Romania

[§]yaojc@mail.cmu.edu.tw, Center for General Education, China Medical University, Taichung 404, Taiwan and Academy of Romanian Scientists, 50044 Bucharest, Romania

A special case of b -metric spaces is the strong b -metric space, when the triangle inequality has the following relaxed form:

$$d(x, y) \leq d(x, z) + sd(z, y), \text{ for all } x, y, z \in X.$$

For example, if (X, d) is a metric space and $p > 1$, then d^p is a b -metric with constant 2^p , i.e., $(X, d^p, 2^p)$ is a b -metric space, see [1]. Another well-known example is the space $L_p([0, 1])$, $0 < p < 1$, where

$$L_p([0, 1]) = \left\{ f : [0, 1] \rightarrow \mathbb{R} \text{ measurable} : \int_0^1 |f(t)|^p dt < \infty \right\}.$$

Define $d_p(f, g) = (\int_0^1 |f(t) - g(t)|^p dt)^{\frac{1}{p}}$. Then d_p is a b -metric on $L_p([0, 1])$ with coefficient $2^{\frac{1}{p}}$. For other examples, see [7].

In this paper, we will prove a fibre contraction principle in the framework of a complete b -metric space. A saturated form of this principle is also presented. Some stability properties are also obtained. Our results are in connection to the research directions proposed in [10]. For more details on fibre contraction principle in metric structures (theory and applications), see [5], [11], [15]–[19].

2 Preliminary notions and results

Throughout this paper, $\mathbb{N}$ is the set of natural numbers including 0, while $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The symbol $\mathbb{R}_+$ denotes the set of all real non-negative numbers.

Let (X, d, s) be a b -metric space and $f : X \rightarrow X$ be an operator. Then, we denote by

$$Fix(f) := \{x \in X : x = f(x)\} \text{ the fixed point set of } f,$$

and by $f^n := f \circ \dots \circ f$ (n -times) the n -th iteration of f . For $x_0 \in X$, the sequence $x_n := f^n(x_0)$, $n \in \mathbb{N}, n \geq 1$ is called the sequence of Picard iterates starting from x_0 .

In the case of a complete b -metric spaces, a (nonlinear) variant of the Banach-Caccioppoli Contraction Principle was given by Czerwik in [3]. The correction of a gap from the original proof was pointed out in [6]. We present here a linear form of the Czerwik's Principle with an additional conclusion

Theorem 1. *Let (X, d, s) be a b -complete metric space and $f : X \rightarrow X$ be an α -contraction, i.e., $\alpha \in (0, 1)$ and the following relation takes place*

$$d(f(x), f(y)) \leq \alpha d(x, y), \text{ for every } x, y \in X.$$

Then, the following conclusions hold:

- (1) $\text{Fix}(f) = \{x^*\}$;
- (2) for each $x_0 \in X$, the sequence $x_n := f^n(x_0), n \in \mathbb{N}$ of Picard iterates starting from x_0 converges to x^* ;
- (3) if additionally, $\alpha < \frac{1}{s}$, then the following retraction-displacement condition holds

$$d(x_0, x^*) \leq \frac{s}{1 - s\alpha} d(x_0, f(x_0)), \text{ for each } x_0 \in X.$$

Definition 1. Let (X, d, s) be a b -metric space and $f : X \rightarrow X$ be an operator. Then, f is called a Picard operator if:

- (1) $\text{Fix}(f) = \{x^*\}$;
- (2) for each $x_0 \in X$, the sequence $x_n := f^n(x_0), n \in \mathbb{N}$ of Picard iterates starting from x_0 converges to x^* .

A Picard operator for which there exists $c > 0$ such that

$$d(x_0, x^*) \leq cd(x_0, f(x_0)), \text{ for each } x_0 \in X,$$

is called a c -Picard operator.

A Picard operator for which there exists a map $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ increasing, continuous in 0 with $\Psi(0) = 0$, such that

$$d(x_0, x^*) \leq \Psi(d(x_0, f(x_0))), \text{ for each } x_0 \in X,$$

is called a Ψ -Picard operator.

Remark 1. In terms of the Picard operator theory, the first two conclusions of the above result can be stated in a concise form as f is a Picard operator. All three conclusions of the above theorem can be stated by the sentence "Then f is a $\frac{s}{1-s\alpha}$ -Picard operator."

The following lemma (see [18]) is important for the proof of our main theorems.

Lemma 1. (Cauchy-Toeplitz Lemma) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}_+$, such that the series $\sum_{n \geq 0} a_n$ is convergent and let $(b_n)_{n \in \mathbb{N}}$ be a sequence with nonnegative terms, such that $\lim_{n \rightarrow \infty} b_n = 0$. Then

$$\lim_{n \rightarrow \infty} \left(\sum_{k=0}^n a_{n-k} b_k \right) = 0.$$

3 Main results

The following general problem is well-known. Let X, Y be two nonempty sets and

$$f : X \times Y \rightarrow X \times Y, f(x, y) = (f_1(x, y), f_2(x, y))$$

be an operator.

The problem is: if $f : X \times Y \rightarrow X \times Y$ satisfies the following conditions:

- (h1) $f_1(\cdot, y) : X \rightarrow X$ has at least one fixed point for all $y \in Y$;
- (h2) $f_2(x, \cdot) : Y \rightarrow Y$ has at least one fixed point for all $x \in X$,

under which conditions the map f has at least one fixed point?

It is known that if $(X, d), (Y, \rho)$ are two complete metric spaces there may be mappings

$$f : X \times Y \rightarrow X \times Y, f(x, y) = (f_1(x, y), f_2(x, y))$$

with no fixed points, even if

- (a1) $f_1(\cdot, y) : X \rightarrow X$ is a contraction, for all $y \in Y$;
- (a2) $f_2(x, \cdot) : Y \rightarrow Y$ is a contraction, for all $x \in X$.

For example, take $f(x, y) = (\frac{x+y}{2} + 1, \frac{x+y}{2} + 1), x, y \in \mathbb{R}$, see [8].

The following results are partial answers to the above problem (see [8]).

Theorem 2 (S.B. Nadler Jr. (1968)). *Let (X, d) be a complete metric space and let (Y, ρ) be a metric space. Let $f : X \times Y \rightarrow X \times Y$ be a mapping. Suppose:*

- (i) *f is uniformly continuous on $X \times Y$ (in particular, if f is Lipschitz);*
- (ii) *f is a contraction mapping in the first variable.*

Then f has at least one fixed point.

Theorem 3 (S.B. Nadler Jr. (1968)). *Let (X, d) be a complete metric space and let (Y, ρ) be a metric space with the fixed point property. If $f : X \times Y \rightarrow X \times Y$ is a contraction mapping in each variable separately, then f has at least one fixed point.*

A particular form of the above mentioned general problem is the case of triangular operators. We formulate the problem in the context of b -metric spaces. Let (X_1, d, s) and (X_2, ρ, t) be two b -metric spaces. Given $t_1 : X_1 \rightarrow X_1$ and $t_2 : X_1 \times X_2 \rightarrow X_2$ we define the following triangular operator

$$t : X_1 \times X_2 \rightarrow X_1 \times X_2 \text{ given by } t(x_1, x_2) := (t_1(x_1), t_2(x_1, x_2)).$$

Under which conditions on t_1 and t_2 the operator t has at least one fixed point?

We present now the first main result of this paper, which is an extension to the case of b -metric spaces of the classical result of Hirsch and Pugh [5].

Theorem 4. *Let (X_1, d_1, r) be a b -metric space with coefficient $r \geq 1$ and $t_1 : X_1 \rightarrow X_1$ be an operator having an attractive fixed point $x_1^* \in X_1$. Let (X_2, d_2, s) be a complete b -metric space with coefficient $s \geq 1$ and let $t_2 : X_1 \times X_2 \rightarrow X_2$. Suppose:*

- (1) *there exists $\alpha \in (0, \frac{1}{s})$ such that $t_2(x_1, \cdot) : X_2 \rightarrow X_2$ is an α -contraction, for all $x_1 \in X_1$. Denote by x_2^* the unique fixed point of $t_2(x_1^*, \cdot)$;*
- (2) *the operator $t : X_1 \times X_2 \rightarrow X_1 \times X_2$, $t(x_1, x_2) := (t_1(x_1), t_2(x_1, x_2))$ is continuous.*

Then (x_1^, x_2^*) is an attractive fixed point of t .*

In fact, the following more general result can be proved.

Theorem 5. *Let (X_1, d_1, r) be a b -metric space with coefficient $r \geq 1$ and (X_2, d_2, s) be a complete b -metric space with coefficient $s \geq 1$. Let $t_1 : X_1 \rightarrow X_1$ and $t_2 : X_1 \times X_2 \rightarrow X_2$ be two operators. Suppose:*

- (1) *$t_1 : X_1 \rightarrow X_1$ is a Picard operator (denote by x_1^* its unique fixed point);*
- (2) *there exists $\alpha \in (0, \frac{1}{s})$ such that the operator $t_2(x_1, \cdot) : X_2 \rightarrow X_2$ is an α -contraction, for all $x_1 \in X_1$ (denote by x_2^* the unique fixed point of $t_2(x_1^*, \cdot)$);*
- (3) *the operator $t : X_1 \times X_2 \rightarrow X_1 \times X_2$, $t(x_1, x_2) := (t_1(x_1), t_2(x_1, x_2))$ is continuous.*

Then t is a Picard operator and $\text{Fix}(t) = (x_1^, x_2^*)$.*

Proof. Let $(x_0^1, x_0^2) \in X_1 \times X_2$ be arbitrary. Define the sequences

$$x_{n+1}^1 = t_1(x_n^1) \text{ and } x_{n+1}^2 = t_2(x_n^1, x_n^2), n \in \mathbb{N}.$$

By (1), we know that $Fix(t_1) = \{x_1^*\}$ and the sequence $(x_n^1)_{n \in \mathbb{N}}$ converges to x_1^* . We will show that the sequence $(x_n^2)_{n \in \mathbb{N}}$ converges to x_2^* . We successively have:

$$\begin{aligned}
d_2(x_{n+1}^2, x_2^*) &= d_2(t_2(x_n^1, x_n^2), t_2(x_1^*, x_2^*)) \\
&\leq s [d_2(t_2(x_n^1, x_n^2), t_2(x_n^1, x_2^*)) + d_2(t_2(x_n^1, x_2^*), t_2(x_1^*, x_2^*))] \\
&\leq s\alpha d_2(x_n^2, x_2^*) + sd_2(t_2(x_n^1, x_2^*), t_2(x_1^*, x_2^*)) \\
&\leq s\alpha [s\alpha d_2(x_{n-1}^2, x_2^*) + sd_2(t_2(x_{n-1}^1, x_2^*), t_2(x_1^*, x_2^*))] + sd_2(t_2(x_n^1, x_2^*), t_2(x_1^*, x_2^*)) \\
&= (s\alpha)^2 d_2(x_{n-1}^2, x_2^*) + s^2\alpha d_2(t_2(x_{n-1}^1, x_2^*), t_2(x_1^*, x_2^*)) + sd_2(t_2(x_n^1, x_2^*), t_2(x_1^*, x_2^*)) \\
&\leq \dots \leq (s\alpha)^{n+1} d_2(x_0^2, x_2^*) + s(s\alpha)^n d_2(t_2(x_0^1, x_2^*), t_2(x_1^*, x_2^*)) \\
&\quad + \dots + s(s\alpha) d_2(t_2(x_{n-1}^1, x_2^*), t_2(x_1^*, x_2^*)) + sd_2(t_2(x_n^1, x_2^*), t_2(x_1^*, x_2^*)) \\
&\leq (s\alpha)^{n+1} d_2(x_0^2, x_2^*) + s[(s\alpha)^n d_2(t_2(x_0^1, x_2^*), t_2(x_1^*, x_2^*)) \\
&\quad + \dots + (s\alpha) d_2(t_2(x_{n-1}^1, x_2^*), t_2(x_1^*, x_2^*)) + d_2(t_2(x_n^1, x_2^*), t_2(x_1^*, x_2^*))] \rightarrow 0, n \rightarrow \infty,
\end{aligned}$$

where the convergence to 0 follows by the Cauchy-Toeplitz Lemma 1. $\square$

A similar result takes place if we replace the contraction condition with a Ćirić-Kannan-Reich-Rus type condition (see [2], [12], [18]). We recall first a Kannan type fixed point theorem in complete b -metric spaces.

Theorem 6. *Let (X, d, s) be a b -complete metric space and $f : X \rightarrow X$ be a Kannan α -contraction, i.e., $\alpha \in (0, \max\{\frac{1}{2}, \frac{1}{1+s}\})$ and the following relation takes place*

$$d(f(x), f(y)) \leq \alpha(d(x, f(x)) + d(y, f(y))), \text{ for every } x, y \in X.$$

Then, the following conclusions hold:

- (1) $Fix(f) = \{x^*\}$;
- (2) for each $x_0 \in X$, the sequence $x_n := f^n(x_0), n \in \mathbb{N}$ of Picard iterates starting from x_0 converges to x^* ;
- (3) if we denote $\gamma := \frac{\alpha}{1-\alpha} \in (0, 1)$, then the following retraction-displacement condition holds

$$d(x_0, x^*) \leq \frac{s}{1-s\gamma} d(x_0, f(x_0)), \text{ for each } x_0 \in X.$$

Remark 2. *In the language of Picard operator theory the conclusion of the above theorem is that f is a $\frac{s}{1-s\gamma}$ -Picard operator.*

Theorem 7. Let (X_1, d_1, r) be a b -metric space with coefficient $r \geq 1$ and (X_2, d_2, s) be a complete b -metric space with coefficient $s \geq 1$. Let $t_1 : X_1 \rightarrow X_1$ and $t_2 : X_1 \times X_2 \rightarrow X_2$ be two operators. Suppose:

(1) $t_1 : X_1 \rightarrow X_1$ is a Picard operator (denote by x_1^* its unique fixed point);

(2) there exists $\alpha \in (0, \frac{1}{2s})$ and $\beta > 0$ such that

$$d_2(t_2(x_1, x_2)t_2(y_1, y_2)) \leq \beta d_1(x_1, y_1) + \alpha [d_2(t_2(x_1, x_2), x_2) + d_2(t_2(y_1, y_2), y_2)],$$

for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$;

(3) the operator $t : X_1 \times X_2 \rightarrow X_1 \times X_2$, $t(x_1, x_2) := (t_1(x_1), t_2(x_1, x_2))$ is continuous.

Then t is a Picard operator and $\text{Fix}(t) = (x_1^*, x_2^*)$.

Proof. Let $(x_0^1, x_0^2) \in X_1 \times X_2$ be arbitrary. Define the sequences

$$x_{n+1}^1 = t_1(x_n^1) \text{ and } x_{n+1}^2 = t_2(x_n^1, x_n^2), n \in \mathbb{N}.$$

By (1), we know that $\text{Fix}(t_1) = \{x_1^*\}$ and the sequence $(x_n^1)_{n \in \mathbb{N}}$ converges to x_1^* as $n \rightarrow \infty$. We notice first that, using (2), we obtain that $t_2(x_1^*, \cdot) : X_2 \rightarrow X_2$ satisfies the following relation

$$d_2(t_2(x_1^*, x_2), t_2(x_1^*, y_2)) \leq \alpha [d_2(t_2(x_1^*, x_2), x_2) + d_2(t_2(x_1^*, y_2), y_2)],$$

for every $x_2, y_2 \in X_2$. Thus, $t_2(x_1^*, \cdot) : X_2 \rightarrow X_2$ satisfies the assumptions of Kannan's fixed point theorem (see Theorem 6) and, as a consequence, there exists $x_2^* \in X_2$ such that $t_2(x_1^*, x_2^*) = x_2^*$. We will show now that the sequence $(x_n^2)_{n \in \mathbb{N}}$ converges to x_2^* . We successively have:

$$\begin{aligned} d_2(x_{n+1}^2, x_2^*) &= d_2(t_2(x_n^1, x_n^2), t_2(x_1^*, x_2^*)) \\ &\leq \beta d_1(x_n^1, x_1^*) + \alpha [d_2(t_2(x_n^1, x_n^2), x_n^2) + d_2(t_2(x_1^*, x_2^*), x_2^*)] \\ &\leq \beta d_1(x_n^1, x_1^*) + s\alpha [d_2(x_{n+1}^2, x_2^*) + d_2(x_2^*, x_n^2)]. \end{aligned}$$

From here, for each $n \in \mathbb{N}$, we get

$$d_2(x_{n+1}^2, x_2^*) \leq \frac{\beta}{1-s\alpha} d_1(x_n^1, x_1^*) + \frac{s\alpha}{1-s\alpha} d_2(x_n^2, x_2^*). \quad (1)$$

Then, we successively have

$$\begin{aligned} &d_2(x_{n+1}^2, x_2^*) \\ &\leq \frac{\beta}{1-s\alpha} d_1(x_n^1, x_1^*) + \frac{\beta}{1-s\alpha} \frac{s\alpha}{1-s\alpha} d_1(x_{n-1}^1, x_1^*) + \left(\frac{s\alpha}{1-s\alpha}\right)^2 d_2(x_{n-1}^2, x_2^*) \end{aligned}$$

$$\begin{aligned} &\leq \cdots \leq \frac{\beta}{1-s\alpha} d_1(x_n^1, x_1^*) + \frac{\beta}{1-s\alpha} \left[\frac{s\alpha}{1-s\alpha} d_1(x_{n-1}^1, x_1^*) \right. \\ &\quad \left. + \cdots + \left(\frac{s\alpha}{1-s\alpha} \right)^n d_1(x_0^1, x_1^*) \right] + \left(\frac{s\alpha}{1-s\alpha} \right)^{n+1} d_1(x_0^2, x_2^*). \end{aligned}$$

Hence, by (1) and the Cauchy-Toeplitz Lemma 1, the right hand side of the above relation tends to 0. The proof is complete. $\square$

We will conclude our study by analyzing some stability properties (see [9], [13], [14]) for the above problem.

Theorem 8. *Let (X_1, d_1, r) be a complete b -metric space with coefficient $r \geq 1$ and (X_2, d_2, s) be a complete b -metric space with coefficient $s \geq 1$. Let $t_1 : X_1 \rightarrow X_1$ and $t_2 : X_1 \times X_2 \rightarrow X_2$ be two operators. Suppose:*

(1) $t_1 : X_1 \rightarrow X_1$ is a k -contraction with $k \in (0, \frac{1}{r})$ (denote by x_1^* its unique fixed point);

(2) there exists $\alpha \in (0, \frac{1}{s^2})$ such that the operator $t_2(x_1, \cdot) : X_2 \rightarrow X_2$ is an α -contraction, for all $x_1 \in X_1$ (denote by x_2^* the unique fixed point of the mapping $t_2(x_1^*, \cdot)$);

(3) there exists $L > 0$ such that the operator $t_2(\cdot, x_2) : X_1 \rightarrow X_2$ is an L -Lipschitz, for all $x_2 \in X_2$.

Then the fixed point problem for t is Ulam-Hyers stable and has the well-posedness property in the sense of Reich and Zaslavski.

Proof. By Theorem 4 we know that $Fix(t) = \{(x_1^*, x_2^*)\}$, where x_2^* is the unique fixed point of the mapping $t_2(x_1^*, \cdot) : X_2 \rightarrow X_2$.

For the Ulam-Hyers property, take any $\epsilon > 0$ and let $(\tilde{x}_1, \tilde{x}_2) \in X_1 \times X_2$ satisfying the relation

$$d_\infty((\tilde{x}_1, \tilde{x}_2), t(\tilde{x}_1, \tilde{x}_2)) \leq \epsilon.$$

Then, we have

$$\max\{d_1(\tilde{x}_1, t_1(\tilde{x}_1)), d_2(\tilde{x}_2, t_2(\tilde{x}_1, \tilde{x}_2))\} \leq \epsilon.$$

We must now estimate the following value

$$d_\infty((\tilde{x}_1, \tilde{x}_2), (x_1^*, x_2^*)) = \max\{d_1(\tilde{x}_1, x_1^*), d_2(\tilde{x}_2, x_2^*)\}.$$

First, we have

$$\begin{aligned} d_1(\tilde{x}_1, x_1^*) &\leq r [d_1(\tilde{x}_1, t_1(\tilde{x}_1)) + d_1(t_1(\tilde{x}_1), t_1(x_1^*))] \\ &\leq r\epsilon + rk d_1(\tilde{x}_1, x_1^*). \end{aligned}$$

Thus, we conclude

$$d_1(\tilde{x}_1, x_1^*) \leq \frac{r}{1-rk} \epsilon. \quad (2)$$

Next, we have

$$\begin{aligned} d_2(\tilde{x}_2, x_2^*) &\leq s [d_2(\tilde{x}_2, t_2(\tilde{x}_1, \tilde{x}_2)) + d_2(t_2(\tilde{x}_1, \tilde{x}_2), t_2(x_1^*, x_2^*))] \\ &\leq s\epsilon + s^2 [d_2(t_2(\tilde{x}_1, \tilde{x}_2), t_2(x_1^*, \tilde{x}_2)) + d_2(t_2(x_1^*, \tilde{x}_2), t_2(x_1^*, x_2^*))] \\ &\leq s\epsilon + s^2 L d_1(\tilde{x}_1, x_1^*) + s^2 \alpha d_2(\tilde{x}_2, x_2^*). \end{aligned}$$

As a consequence, we have

$$d_1(\tilde{x}_2, x_2^*) \leq \frac{s}{1-s^2\alpha} \left[1 + \frac{s^2 L}{1-sk} \right] \epsilon. \quad (3)$$

Thus,

$$d_\infty((\tilde{x}_1, \tilde{x}_2), (x_1^*, x_2^*)) \leq C\epsilon,$$

where $C := \max\{\frac{r}{1-rk}, \frac{s}{1-s^2\alpha} \left[1 + \frac{s^2 L}{1-sk} \right]\}$. This shows that the fixed point problem for t is Ulam-Hyers stable.

For the well-posedness property, consider $((x_n^1, x_n^2))_{n \in \mathbb{N}}$ in $X_1 \times X_2$ with the property that $\lim_{n \rightarrow \infty} d_\infty((x_n^1, x_n^2), t(x_n^1, x_n^2)) = 0$. We will show that

$$\lim_{n \rightarrow \infty} d_\infty((x_n^1, x_n^2), (x_1^*, x_2^*)) = 0. \quad (4)$$

Indeed, we have

$$\begin{aligned} d_1(x_n^1, x_1^*) &\leq r [d_1(x_n^1, t_1(x_n^1)) + d_1(t_1(x_n^1), t_1(x_1^*))] \\ &\leq r [d_1(x_n^1, t_1(x_n^1)) + k d_1(x_n^1, x_1^*)]. \end{aligned}$$

As a consequence, we have

$$d_1(x_n^1, x_1^*) \leq \frac{r}{1-rk} d_1(x_n^1, t_1(x_n^1)) \rightarrow 0, n \rightarrow \infty. \quad (5)$$

Similarly, we get

$$\begin{aligned} d_2(x_n^2, x_2^*) &\leq s [d_2(x_n^2, t_2(x_n^1, x_n^2)) + d_2(t_2(x_n^1, x_n^2), t_2(x_1^*, x_2^*))] \\ &\leq s d_2(x_n^2, t_2(x_n^1, x_n^2)) + s^2 [d_2(t_2(x_n^1, x_n^2), t_2(x_1^*, x_n^2)) + d_2(t_2(x_1^*, x_n^2), t_2(x_1^*, x_2^*))] \\ &\leq s d_2(x_n^2, t_2(x_n^1, x_n^2)) + s^2 L d_1(x_n^1, x_1^*) + s^2 \alpha d_2(x_n^2, x_2^*). \end{aligned}$$

As a consequence, we have

$$d_2(x_n^2, x_2^*) \leq \frac{s}{1-s^2\alpha} [d_2(x_n^2, t_2(x_n^1, x_n^2)) + s L d_1(x_n^1, x_1^*)] \rightarrow 0, n \rightarrow \infty. \quad (6)$$

By (5) and (6) we obtain (4). $\square$

In fact, the following saturated form of the fibre contraction principle in b -metric spaces can be established.

Theorem 9. *Let (X_1, d_1, r) be a b -metric space with coefficient $r \geq 1$ and (X_2, d_2, s) be a complete b -metric space with coefficient $s \geq 1$. Let $t_1 : X_1 \rightarrow X_1$ and $t_2 : X_1 \times X_2 \rightarrow X_2$ be two operators. Suppose:*

(1) $t_1 : X_1 \rightarrow X_1$ is a c_1 -Picard operator (denote by x_1^* its unique fixed point);

(2) there exists $\alpha \in (0, \frac{1}{s^2})$ such that the operator $t_2(x_1, \cdot) : X_2 \rightarrow X_2$ is an α -contraction, for all $x_1 \in X_1$ (denote by x_2^* the unique fixed point of the mapping $t_2(x_1^*, \cdot)$);

(3) there exists $L > 0$ such that the operator $t_2(\cdot, x_2) : X_1 \rightarrow X_2$ is an L -Lipschitz, for all $x_2 \in X_2$.

Then t is a Ψ -Picard operator with $\Psi(t) := \max\{c_1 t, \frac{s}{1-s^2\alpha}t + sLc_1 t\}$.

4 Conclusions

In this paper, two fibre contraction principles in b -metric spaces were established. The approach is based on some fixed point results for self operators in complete b -metric spaces (Banach-Caccioppoli contractions and, respectively Kannan contractions) and the Cauchy-Toeplitz Lemma. Some stability results (well-posedness in the sense of Reich and Zaslavski and Ulam-Hyers stability) are also proved.

References

- [1] B. Ali, Ş. Cobzaş and M. Mabula, Fixed points and completeness of strong b -metric spaces, *J. Nonlinear Convex Anal.* 26 (2025), 1155-1171.
- [2] Lj. Ćirić, A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.* 45 (1974), 267-273.
- [3] S. Czerwik, Contraction mappings in b -metric spaces, *Acta Math. Inf. Univ. Ostraviensis* 1 (1993), 511.
- [4] S. Czerwik, Nonlinear set-valued contraction mappings in b -metric spaces, *Atti del Seminario Matematico e Fisico dell' Universit di Modena* 46 (1998), 263-276.

- [5] W.M. Hirsch and C.C. Pugh, Stable manifolds and hyperbolic sets, *Proc. Symp. Pure Math.* 14 (1970), 133-163.
- [6] S. Kajántó and A. Lukács, A note on the paper "Contraction mappings in b -metric spaces" by Czerwik, *Acta Univ. Sapientiae Math.* 10 (2018), 85-89.
- [7] W. Kirk and N. Shahzad, *Fixed Point Theory in Distance Spaces*, Springer Cham, 2014.
- [8] S.B. Nadler Jr. Sequences of contractions and fixed points, *Pacific J. Math.* 27 (1968), 579-585.
- [9] T.P. Petru, A. Petruşel and J.-C. Yao, Ulam-Hyers stability for operatorial equations and inclusions via nonself operators, *Taiwanese J. Math.* 15 (2011), 2195-2212.
- [10] A. Petruşel, I.A. Rus and M.A. Şerban, Some research directions in fibre contraction theory and its applications, *Mathematica* 67 (2025), 112-129.
- [11] A. Petruşel, I.A. Rus and M.A. Şerban, Some variants of the fibre contraction principle and applications: from existence to the convergence of successive approximations, *Fixed Point Theory* 22 (2021), 795-808.
- [12] S. Reich, Fixed points of contractive functions, *Boll. Un. Mat. Ital.* 4 (1972), 26-42.
- [13] S. Reich and A.J. Zaslavski, Well-posedness of fixed point problems, *Far East J. Math. Sci.* Part III (2011), 393-401.
- [14] S. Reich and A.J. Zaslavski, *Genericity in Nonlinear Analysis*, Springer, New York, 2014.
- [15] I.A. Rus, A fibre generalized contraction theorem and applications, *Mathematica* 41 (1999), 85-90.
- [16] I.A. Rus, Fibre Picard operators and applications, *Studia Univ. Babeş-Bolyai Math.* 44 (1999), 89-98.
- [17] I.A. Rus, Fibre Picard operators on generalized metric spaces and applications, *Scripta Scientiarum Mathematicarum* 1 (1999), 326-334.
- [18] I.A. Rus, A. Petruşel and G. Petruşel, *Fixed Point Theory*, Cluj Univ. Press, Cluj-Napoca, 2008.

- [19] M.A. Şerban, Saturated fibre contraction principle, *Fixed Point Theory* 18 (2017), 729-740.