

FEEDBACK NULL CONTROLLABILITY IN MINIMUM TIME FOR NONLINEAR SYSTEMS*

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Dedicated to the memory of Professor Mihail Megan

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Abstract

For a nonlinear control system, by considering appropriate feedback laws, we prove results on null controllability and regularity of the minimum time function.

Keywords: feedback control, null controllability, minimum time function, nonlinear control system.

MSC: 93B52, 93B05.

1 Introduction

Let X be a Banach space, $A : D(A) \subseteq X \rightrightarrows X$ be a (possible) multivalued m -dissipative operator and $f : \overline{D(A)} \rightarrow X$ a given function. Let U be a Hilbert space, $B : U \rightarrow X$ a linear bounded operator and consider the control system

$$\begin{cases} y'(t) \in Ay(t) + f(y(t)) + Bu(t) \\ y(0) = x, \end{cases} \quad (1)$$

where $x \in \overline{D(A)}$ and $u(\cdot)$ is the control.

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Let $r > 0$ and denote by $\mathcal{U}$ the set of all admissible controls, i.e. measurable functions $u(\cdot)$ satisfying $\|u(t)\| \leq r$ a.e. Denote by $\mathcal{C}(t)$ the null controllable set at time $t > 0$, i.e. the set of points $x \in X$ for which there exist $u \in \mathcal{U}$ and a mild solution $y(\cdot)$ to (1) which transfers x to 0 at time t . Set $\mathcal{C} = \bigcup_{t>0} \mathcal{C}(t)$ called the null controllable set, and define the minimum time function $\mathcal{T} : X \rightarrow [0, +\infty]$ by

$$\mathcal{T}(x) = \begin{cases} \inf \{t \geq 0; x \in \mathcal{C}(t)\}, & \text{if } x \in \mathcal{C} \\ +\infty, & \text{otherwise.} \end{cases}$$

The aim of this paper is to provide a feedback control which leads to null controllability and to get estimates for the minimum time function around the target.

When B is the identity operator, this topic was studied in [5], in Hilbert spaces setting, in [6], in Banach spaces, in the case when A is the generator of a C_0 -semigroup, using feedback controls of type $-ry(t) / \|y(t)\|$. See also [1], [2]. For the semilinear case with B a surjective linear bounded operator, we recall [7], [8], where null controllability and estimates for the associated minimum time function were obtained, considering appropriate feedback laws. For the case when A is m -dissipative and B is the identity operator, we recall [9], where it was used the same feedback as in [5].

In this paper, we extend some results of [9] and [8] to the control system (1), where A is m -dissipative and B is a surjective linear operator, by means of Lyapunov pairs characterizations.

2 Preliminaries

First, we present some concepts and results concerning Lyapunov pairs, which will be used to get the main results of the paper.

Let X be a Banach space, $A : D(A) \subseteq X \rightrightarrows X$ a m -dissipative operator, let M be a nonempty subset of $\overline{D(A)}$, $F : M \rightarrow X$ a function and consider the Cauchy problem

$$\begin{cases} y'(t) \in Ay(t) + F(y(t)) \\ y(0) = \xi, \end{cases} \quad (2)$$

where $\xi \in M$. By a mild solution to (2) on $[0, T]$, with $T > 0$, we mean a function $y : [0, T] \rightarrow M$ which renders $t \mapsto \varphi(t) = F(y(t))$ integrable,

satisfies $y(0) = \xi$ and is an integral (or mild) solution on $[0, T]$ to the equation

$$y'(t) \in Ay(t) + \varphi(t). \quad (3)$$

We recall that $y : [0, T] \rightarrow X$ is an integral solution to (3) on $[0, T]$ if y is continuous on $[0, T]$, $y(t) \in \overline{D(A)}$ for each $t \in [0, T]$, and

$$\|y(t) - x\| \leq \|y(s) - x\| + \int_s^t [y(\tau) - x, \varphi(\tau) + y]_+ d\tau,$$

for each $x \in D(A)$, $y \in Ax$ and $0 \leq s \leq t \leq T$. We denoted by $[x, y]_+$ the right directional derivative of the norm calculated at x in the direction y , i.e.

$$[x, y]_+ = \lim_{h \downarrow 0} \frac{\|x + hy\| - \|x\|}{h}.$$

We recall that $[x, y + ax]_+ = [x, y]_+ + a\|x\|$ for any $x, y \in X$ and $a \in \mathbb{R}$. For more properties of $[\cdot, \cdot]_+$ see, e.g., [11].

It is known that, for each $\xi \in \overline{D(A)}$ and $\varphi \in L^1(0, T; X)$, there exists a unique integral solution $y : [0, T] \rightarrow \overline{D(A)}$ to (3) which satisfies $y(0) = \xi$ (see [11, Theorem 3.6.1]). Also, we recall the following result of Benilan (see, e.g., [12]).

Theorem 1. *Let $A : D(A) \subseteq X \Rightarrow X$ be a m -dissipative operator, let $\varphi, \psi \in L^1(0, T; X)$ and let $y(\cdot), z(\cdot)$ be two integral solutions to (3) corresponding to φ and ψ , respectively, with $y(0) = \xi$ and $z(0) = \eta$, $\xi, \eta \in \overline{D(A)}$. Then*

$$\|y(t) - z(t)\| \leq \|\xi - \eta\| + \int_0^t [y(s) - z(s), \varphi(s) - \psi(s)]_+ ds,$$

for any $t \in [0, T]$.

By a mild solution to (2) on $[0, \tilde{T})$ we mean a function $y : [0, \tilde{T}) \rightarrow M$ which is a mild solution to (2) on $[0, T]$, for each $T \in (0, \tilde{T})$. A solution $y : [0, T) \rightarrow M$ to (2) is called noncontinuable if there is no other solution $\tilde{y} : [0, \tilde{T}) \rightarrow M$ with $T < \tilde{T}$ and $y(t) = \tilde{y}(t)$ on $[0, T)$.

Following [10] and [9], we give the following definition.

Definition 1. The functions $V, g : \overline{D(A)} \rightarrow (-\infty, +\infty]$ form a Lyapunov pair for the problem (2) if $\text{dom}(V) \subseteq M$ and for every $\xi \in \text{dom}(V)$ there exist $T > 0$ and a solution $y : [0, T] \rightarrow M$ to (2) such that $t \mapsto g(y(t))$ is integrable on $[0, T]$ and we have

$$V(y(t)) + \int_0^t g(y(s)) ds \leq V(\xi), \quad (4)$$

for all $t \in [0, T]$.

We denote by $\text{dom}(V)$ the domain of V , i.e.

$$\text{dom}(V) = \left\{ x \in \overline{D(A)}; V(x) < +\infty \right\}.$$

Definition 2. The A -contingent derivative $\underline{D}^A V(x)(v)$ associated with the m -dissipative operator A on a Banach space X of the function $V : \overline{D(A)} \rightarrow (-\infty, +\infty]$ at $x \in \text{dom}(V)$ in the direction $v \in X$ is defined by

$$\underline{D}^A V(x)(v) = \liminf_{t \downarrow 0, w \rightarrow 0} \frac{1}{t} [V(S_{A+v}(t)x + tw) - V(x)],$$

where $S_{A+v}(t) : \overline{D(A)} \rightarrow \overline{D(A)}$, $t \geq 0$, stands for the semigroup of nonexpansive mappings generated by $A + v$.

In [9] it was proved the following result.

Theorem 2. Let X be a Banach space, let $A : D(A) \subseteq X \rightrightarrows X$ be a m -dissipative operator and let $F : M \rightarrow X$ be a locally Lipschitz function, where M is a nonempty subset of $\overline{D(A)}$. Let $g : \overline{D(A)} \rightarrow \mathbb{R}$ be a locally Lipschitz function and let $V : \overline{D(A)} \rightarrow (-\infty, +\infty]$ be a proper function with $\text{dom}(V) \subseteq M$ and having a locally closed epigraph. Then V and g form a Lyapunov pair for the problem (2) if and only if

$$\underline{D}^A V(x) F(x) + g(x) \leq 0,$$

for all $x \in \text{dom}(V)$.

Recall that the epigraph of the function V is defined by

$$\text{epi}(V) = \left\{ (x, \mu) \in \overline{D(A)} \times \mathbb{R}; V(x) \leq \mu \right\}.$$

3 Main results

First, we present the hypotheses we shall refer to in what follows.

- (H1) X is a Banach space and U is a Hilbert space.
- (H2) The operator $A : D(A) \subseteq X \rightrightarrows X$ is m -dissipative and $0 \in A0$.
- (H3) The function $f : \overline{D(A)} \rightarrow X$ is locally Lipschitz and satisfies

$$[x, f(x)]_+ \leq \lambda \|x\|, \tag{5}$$

for any $x \in \overline{D(A)}$.

(H4) $B : U \rightarrow X$ a linear bounded surjective operator.

Remark 1. Since U is a Hilbert space, the operator B admits a right inverse, i.e. there exists $\Lambda : X \rightarrow U$ a linear bounded operator such that $B \circ \Lambda = I_X$ (see, e.g., [4, Theorem 2.12]). Moreover, since Λ is linear and continuous, there exists $\gamma > 0$ such that

$$\|\Lambda x\| \leq \gamma \|x\| \quad (6)$$

for any $x \in X$.

We provide an admissible control that will be used to reach the origin starting from the initial point $x \in \overline{D(A)}$ in some time T , by mild solutions to (1). More exactly, we use a feedback control of the form

$$u(t) = -\frac{r\Lambda y(t)}{\|\Lambda y(t)\|}.$$

First, we prove the following result.

Theorem 3. *Assume (H1)-(H4). Then, for any $x \in \overline{D(A)}$, $x \neq 0$, there exist $T > 0$ and $y : [0, T] \rightarrow \overline{D(A)}$ solution to*

$$\begin{cases} y'(t) \in Ay(t) + f(y(t)) - \frac{ry(t)}{\|\Lambda y(t)\|} \\ y(0) = x \end{cases} \quad (7)$$

which satisfies

$$\|y(t)\| \leq \|x\| - \frac{r}{\gamma}t + \lambda \int_0^t \|y(s)\| ds \quad (8)$$

and $y(t) \neq 0$ for every $t \in [0, T]$.

Proof. Define the functions $V : \overline{D(A)} \rightarrow (-\infty, +\infty]$ by $V(x) = \|x\|$ for $x \in \overline{D(A)} \setminus \{0\}$ and $V(x) = +\infty$ otherwise and $g : \overline{D(A)} \rightarrow \mathbb{R}$, $g(x) = \frac{r}{\gamma} - \lambda \|x\|$ for $x \in \overline{D(A)}$. Clearly, $\text{epi}(V)$ is locally closed and g is Lipschitz on $\overline{D(A)}$.

We shall apply Theorem 2 to prove that V and g form a Lyapunov pair for the problem (2) with $F : \overline{D(A)} \setminus \{0\} \rightarrow X$, defined by $F(x) = f(x) - \frac{rx}{\|\Lambda x\|}$, for $x \in \overline{D(A)} \setminus \{0\}$. Let us remark that the function $x \mapsto \frac{rx}{\|\Lambda x\|}$ is locally Lipschitz continuous on $\overline{D(A)} \setminus \{0\}$. Further, we have to check that

$$\underline{D}^A V(x) \left(f(x) - \frac{rx}{\|\Lambda x\|} \right) + g(x) \leq 0$$

for any $x \in \text{dom}(V) = \overline{D(A)} \setminus \{0\}$. Let $x \in \text{dom}(V)$ and define

$$z(t) = S_{A+f(x)-\frac{rx}{\|\Lambda x\|}}(t)x, \quad t \geq 0.$$

By Theorem 1 and assumption $0 \in A0$, we have

$$\|z(t)\| \leq \|x\| + \int_0^t \left[z(s), f(x) - \frac{rx}{\|\Lambda x\|} \right]_+ ds \quad (9)$$

for all $t \geq 0$. Definition 2, together with (9), (5), (6) and the properties of $[\cdot, \cdot]_+$ implies

$$\begin{aligned} \underline{D}^A V(x) \left(f(x) - \frac{rx}{\|\Lambda x\|} \right) &\leq \liminf_{t \downarrow 0} \frac{1}{t} [\|z(t)\| - \|x\|] \\ &\leq \limsup_{t \downarrow 0} \frac{1}{t} \int_0^t \left[z(s), f(x) - \frac{rx}{\|\Lambda x\|} \right]_+ ds \\ &\leq \left[x, f(x) - \frac{rx}{\|\Lambda x\|} \right]_+ = [x, f(x)]_+ - \frac{r}{\|\Lambda x\|} \|x\| \\ &\leq \lambda \|x\| - \frac{r}{\gamma}. \end{aligned}$$

Therefore, by Theorem 2, V and g form a Lyapunov pair for the problem (7). This proves the existence of a solution $y(\cdot)$ to (7) on some interval $[0, T]$ satisfying (8) for any $t \in [0, T]$. Moreover, $y(t) \in \overline{D(A)} \setminus \{0\}$ for any $t \in [0, T]$. $\square$

By a continuation argument we can prove the following result.

Corollary 1. *Assume (H1)-(H4). Then, for any $x \in \overline{D(A)}$, $x \neq 0$, there exists $y : [0, \sigma) \rightarrow \overline{D(A)} \setminus \{0\}$ a noncontinuable solution to (7) which satisfies (8) for every $t \in [0, \sigma)$. The solution is unique.*

Now we can state the main result of the paper, which leads to null controllability of the system (1).

Theorem 4. *Assume (H1)-(H4). Moreover, assume that f is bounded on bounded sets. The following properties hold.*

(i) *In case $\lambda \leq 0$, for any $x \in \overline{D(A)}$, $x \neq 0$, there exist a control $u(\cdot)$ and a mild solution $y(\cdot)$ to (1) that reaches the origin of X in some time $T \leq \frac{\gamma}{r} \|x\|$ and satisfies*

$$\|y(t)\| \leq \|x\| - \frac{r}{\gamma} t \quad (10)$$

for any $t \in [0, T]$. Therefore,

$$\mathcal{T}(x) \leq \frac{\gamma}{r} \|x\|$$

for any $x \in \overline{D(A)}$.

(ii) In case $\lambda > 0$, for every $x \in \overline{D(A)}$ satisfying $0 < \|x\| < \frac{r}{\gamma\lambda}$, there exist a control $u(\cdot)$ and a mild solution $y(\cdot)$ to (1) that reaches the origin of X in some time

$$T \leq \frac{1}{\lambda} \log \frac{r}{r - \gamma\lambda \|x\|}$$

and satisfies

$$\|y(t)\| \leq e^{\lambda t} \left(\|x\| - \frac{r}{\gamma\lambda} \right) + \frac{r}{\gamma\lambda} \quad (11)$$

for any $t \in [0, T]$. Considering $0 < \rho < \frac{r}{\gamma\lambda}$ and defining $\mu = \frac{\gamma}{r - \gamma\lambda\rho}$ we have

$$\mathcal{T}(x) \leq \mu \|x\| \quad (12)$$

for any $x \in \overline{D(A)}$ with $0 < \|x\| \leq \rho$.

Proof. Since the first situation is simpler, we present the proof for the second case. Let $x \in \overline{D(A)}$ with $0 < \|x\| < \frac{r}{\gamma\lambda}$. From Corollary 1, we get a noncontinuable solution $y(\cdot)$ to (1) on $[0, \sigma)$, with the feedback control $u(t) = -r\Lambda y(t)/\|\Lambda y(t)\|$, which satisfies (8) and $y(t) \neq 0$ for any $t \in [0, \sigma)$. Let us remark that $\sigma < \infty$. Indeed, from (8), using the Gronwall inequality, we obtain that (11) holds for any $t \in [0, \sigma)$, which leads to $\sigma < \frac{1}{\lambda} \log \frac{r}{r - \gamma\lambda\|x\|}$. Moreover, it follows that $y(\cdot)$ is bounded on $[0, \sigma)$.

Let $F(x) = f(x) - \frac{rx}{\|\Lambda x\|}$, for $x \in \overline{D(A)} \setminus \{0\}$. Since f is bounded on bounded sets and $\left\| \frac{rx}{\|\Lambda x\|} \right\| \leq r\|B\|$ for any $x \in \overline{D(A)} \setminus \{0\}$ (we have used that $B \circ \Lambda = I_X$), we obtain that F maps bounded subsets in $\overline{D(A)} \setminus \{0\}$ into bounded subsets in X , hence $F(y(\cdot))$ is bounded on $[0, \sigma)$. So, there exists $\lim_{t \uparrow \sigma} y(t)$ which belongs to $\overline{D(A)}$. If $\lim_{t \uparrow \sigma} y(t) \neq 0$, then, by Theorem 3, we obtain that $y(\cdot)$ can be continued to the right of σ , which is false. Therefore, $\lim_{t \uparrow \sigma} y(t) = 0$. To get (12) we use the elementary inequality $\log(1+z) < z$ for any $z > 0$. $\square$

Proposition 1. Assume (H1)-(H4). Moreover, assume that f is globally Lipschitz of constant $L > 0$.

(i) If $\lambda \leq 0$, then, for any $x, z \in \overline{D(A)}$ we have

$$\mathcal{T}(z) \leq \mathcal{T}(x) + \frac{\gamma}{r} e^{L\mathcal{T}(x)} \|x - z\|.$$

Moreover, $\mathcal{T}(\cdot)$ is locally Lipschitz continuous on $\mathcal{C}$.

(ii) Suppose $\lambda > 0$, let $0 < \rho < \frac{r}{\gamma\lambda}$ and define $\mu = \frac{\gamma}{r - \gamma\lambda\rho}$. If $x \in \mathcal{C}$ and $z \in \overline{D(A)}$ is such that

$$\|z - x\| \leq \rho e^{-L\mathcal{T}(x)},$$

then $z \in \mathcal{C}$ and

$$\mathcal{T}(z) \leq \mathcal{T}(x) + \mu e^{L\mathcal{T}(x)} \|x - z\|.$$

Moreover, $\mathcal{C}$ is open and $\mathcal{T}(\cdot)$ is locally Lipschitz continuous on $\mathcal{C}$.

Proof. We present the proof for (ii). We use the same idea as in [5, Proposition 3.3] (see also [7, Proposition 2.1]). Let $x \in \mathcal{C}$ and $z \in \overline{D(A)}$ such that

$$\|z - x\| \leq \rho e^{-L\mathcal{T}(x)}.$$

To simplify the exposition we assume the existence of an optimal control u for x , i.e. $y(\mathcal{T}(x)) = 0$, where $y(\cdot)$ is the solution to (1) starting from x . Denote by $\tilde{y}(\cdot)$ the solution to (1) with the initial data $\tilde{y}(0) = z$. By Theorem 1, we have that $\|y(t) - \tilde{y}(t)\| \leq e^{Lt} \|x - z\|$, for any $t \geq 0$. Hence,

$$\|\tilde{y}(\mathcal{T}(x))\| \leq e^{L\mathcal{T}(x)} \|x - z\| \leq \rho.$$

Using now Theorem 4, we get that $\tilde{y}(\mathcal{T}(x)) \in \mathcal{C}$ and

$$\mathcal{T}(\tilde{y}(\mathcal{T}(x))) \leq \mu \|\tilde{y}(\mathcal{T}(x))\| \leq \mu e^{L\mathcal{T}(x)} \|x - z\|.$$

From the dynamic programming principle (see [3]), we have that, for any $t \in [0, \mathcal{T}(z)]$,

$$\mathcal{T}(z) \leq t + \mathcal{T}(\tilde{y}(t)).$$

It follows that

$$\mathcal{T}(z) \leq \mathcal{T}(x) + \mathcal{T}(\tilde{y}(\mathcal{T}(x))) \leq \mathcal{T}(x) + \mu e^{L\mathcal{T}(x)} \|x - z\|.$$

Let $x_0 \in \mathcal{C}$ and take $\delta = \frac{\rho}{2} e^{-L(\mathcal{T}(x_0) + \mu\rho)}$. By the first part of the proof, we obtain

$$\mathcal{T}(x_1) - \mathcal{T}(x_2) \leq \mu e^{L(\mathcal{T}(x_0) + \mu\rho)} \|x_1 - x_2\|,$$

for any $x_1, x_2 \in B_X(x_0, \delta)$. $\square$

Remark 2. If X is a Hilbert space too, then we can also take the feedback law $-B^*y(t) / \|B^*y(t)\|$, where B^* is the adjoint operator of B , with a similar proof.

References

- [1] V. Barbu, Optimal feedback controls for a class of nonlinear distributed parameter systems, *SIAM J. Control Optim.* 21 (1983), 871-894.
- [2] V. Barbu, *Analysis and Control of Nonlinear Infinite-Dimensional Systems*, Mathematics in Science and Engineering, vol. 190, Academic Press, Boston, 1993.
- [3] R. Bellman, *Dynamic Programming*, Princeton, NJ: Princeton University Press, 1957.
- [4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.
- [5] P. Cannarsa and O. Cârjă, On the Bellman equation for the minimum time problem in infinite dimensions, *SIAM J. Control Optim.* 43 (2004), 532-548.
- [6] O. Cârjă, On the minimal time function and the minimum energy problem: a nonlinear case, *Syst. Control Lett.* 55 (2006), 543-548.
- [7] O. Cârjă and A.I. Lazu, Feedback null controllability for a class of semilinear control systems, *Carpathian J. Math.* 38 (2022), 1-9.
- [8] O. Cârjă and A.I. Lazu, Feedback null controllability for a class of nonlinear systems in Banach spaces, *Appl. Math. Optim.*, to appear.
- [9] O. Cârjă and D. Motreanu, Characterization of Lyapunov pairs in the nonlinear case and applications, *Nonlinear Anal.* 70 (2009), 352-363.
- [10] M. Kocan and P. Soravia, Lyapunov functions for infinite-dimensional systems, *J. Funct. Anal.* 192 (2002) 342-363.
- [11] V. Lakshmikantham and S. Leela, *Nonlinear Differential Equations in Abstract Spaces*, Pergamon Press, Oxford, 1981.
- [12] I.I. Vrabie, *Compactness Methods for Nonlinear Evolutions*. In: Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 75, Addison-Wesley and Longman, 1995.

