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EXTENDED KANTOROVICH-TYPE RESULTS ON NEWTON'S METHOD FOR SOLVING GENERALIZED EQUATIONS*

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Dedicated to Professor Constantin Fetecau on the occasion of his 80th anniversary

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Abstract

A semi-local convergence analysis is developed for Newton's method in order to approximate a solution of a generalized equations involving Banach space valued operators. The convergence conditions are based on generalized continuity conditions on the Fréchet derivative of the operator involved and the Aubin property. The specialized cases of our results extend earlier ones using the same information. Suitable examples complement the theory.

Keywords: Banach space, semi-local convergence, Newton's method, generalized equation.

MSC: 34A34, 65B99, 65P30, 65H05.

1 Introduction

A plethora of applications from diverse disciplines can be reduced using Mathematical Modelling [1–10,12–14,18–27] to solving the generalized equa-

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tion of the form

$$g(x) + G(x) \ni 0. \tag{1}$$

Here, S_1, S_2 are denoting complete normed spaces, $g : S_1 \longrightarrow S_2$ is a continuously differentiable operator, and $G : S_1 \rightrightarrows S_2$ is a set-valued operator with a closed nonempty graph [12,13]. The local convergence analysis of the Newton's method

$$g(x_n) + g'(x_n)(x_{n+1} - x_n) + G(x_{n+1}) \ni 0,\ n = 0, 1, 2, \ldots \tag{2}$$

for approximating a solution $x_* \in S_1$ of the generalized equation (1) when $G = 0$ or not is given in [1–10, 12–15] under various conditions such as Lipschitz, Hölder continuity on the Fréchet derivative of the operator g. Such conditions are important in the convergence of Newton's method, since they control the derivative [1–10, 12–15].

The following concerns exist with the aforementioned works in general:
(1) The convergence domain is small.
(2) Error estimates on the distances $\|x_n - x_*\|$ are pessimistic, where x_* stands for a solution of the equation (1).
(3) The uniqueness of the convergence ball is small in general.

Items (1)-(3) restrict the applicability of the method. That is why, by developing more precise majorizing sequences and the concept of generalized continuity, we positively address these concerns.

In particular, the semi-local convergence analysis of Newton's method (2) is developed in [12] using the Aubin property which is related to metric regularity as it is shown by Dontchev and Rockafellar in [12, 13]. It turns out that the following advantages are obtained:
(1) Weaker convergence criteria.
(2) Tighter error distances $\|x_n - x_*\|$ and
(3) An at least as precise information on the location of the solution.

It is also worth noting that these advantages are realized with no additional conditions, since the new majorant functions specialize to the ones used in earlier studies [8,9,15]. Moreover, we make a special comment on the notable studies by Cibulka et al. [8, 9] and Ferreira et al. [14, 15], where the semi-local convergence analysis of the method (2) is developed under Lipschitz continuouty conditions and Kantorovich-type assumptions. A direct and favorable to us comparison is given at the end of the article. Moreover, we use generalized Lipschitz-type conditions in our analysis in order to include a larger class of problems.

The rest of the paper includes: Preliminaries in Section 2; the semi-local convergence in Section 3 and the special cases and numerical examples in Section 4.

2 Preliminaries

We assume familiarity with the concepts of graph (gph G), the domain (dom), the range (rge) and the inverse G^{-1} of a set-valued operator G [8–10, 12–15]. Moreover, we use d, e which are the standard notations for the distance and excess, respectively between two subsets of S_1. Let $R > 0$.

Define the linearization error for some continuously differentiable function $f : [0, R) \longrightarrow (-\infty), +\infty)$

$$e_f(v_1, v_2) = f(v_2) - f(v_1) - f'(v_1)(v_2 - v_1)$$

for each $v_1, v_2 \in [0, R)$. Moreover, define analogously

$$E_g(u_1, u_2) = g(u_2) - g(u_2) - g'(u_1)(u_2 - u_1)$$

for each $u_1, u_2 \in S_1$ and

$$E_{g+G,x_*}(v) = g(x) + g'(x)(v - x_*) + G(v)$$

for each $x, v \in S_1$.

The definition of the Aubin property and the following version of the contraction mapping principle are given in order to make the article as self contained as possible. More information can be found in [1–3, 8–10, 12, 13].

The notation $T(x, \alpha)$ is used to denote an open ball centered at $x \in S_1$ and of radius $\alpha > 0$. Moreover, $T[x, \alpha]$ stands for the closure of $T(x, \alpha)$.

Definition 1. *[13] Let $\tilde{y} \in S_2$ for $\tilde{x} \in S_1$. Then, the inverse operator G^{-1} of G is said to have the Aubin property at $(\tilde{y}, \tilde{x})$ with modulus c, provided that $\tilde{x} \in G^{-1}(\tilde{y})$, if $gphG^{-1}$ is locally closed at $(\tilde{y}, \tilde{x})$, for $\tilde{x} \in G^{-1}(\tilde{y})$, and there exist constants $a, b > 0$ so that*

$$e^{-1}(G^{-1}(y) \cap T[\tilde{x}, a],\ G^{-1}(y_1)) \leq c\|y - y_1\|$$

for each $y, y_1 \in T[\tilde{y}, b]$. If in addition the operator $S_1 \ni w \longrightarrow G^{-1}(w) \cap (\tilde{x}, a)$ is not multivalued on S_1, then G is said to be strongly regular. Then, the operator G is Lipschitz continuous on S_1.

Theorem 1. *[10] Let $\Psi : S_1 \rightrightarrows S_1$ be a set-valued operator and $z \in S_1$. Suppose:*
There exist scalars $\xi > 0$ and $\xi_1 \in (0,1)$ so that $gph\Psi \cap (T[z,\xi] \times T[z,\xi])$ is closed and

(i) $d(z, \Psi(z)) \leq \xi(1 - \xi_1)$

(ii) $e(\Psi(x) \cap T[z,\xi], \Psi(z_1)) \leq \xi_1 \|x - z_1\|$ *for each* $\xi_1, z_1 \in T[z,\xi]$.

Then, there exists $z_* \in T[z,\beta]$ so that $z_* \in \Psi(z_*)$, i.e. Ψ admits a fixed point in $T[z,\xi]$.

3 Local convergence

We introduce and compare certain majorant conditions. Let α, q_1 be positive constants and $x_0 \in S_1$.

Definition 2. *Let g be a differentiable operator on the ball $T(x_0,\alpha)$. We say that g' satisfies the center majorant condition on the ball $T(x_0,\alpha)$ if there exists a continuous and nondecreasing (CN) function $v_0 : [0,\alpha) \longrightarrow \mathbb{R}$ so that for each $x \in T(x_0,\alpha)$*

$$\|g'(x) - g'(x_0)\| \leq v_0(\|x - x_0).$$

(c1) Assume that there exists a minimal solution of the equation $qv_0(t) - 1 = 0$ for some $q > 0$ satisfying $\rho \in (0,\alpha]$.
Define the set $S_0 = S_1 \cap T(x_0,\rho)$.

Definition 3. *Let g be a differentiable operator on the ball $T(x_0,\rho)$. We say that g' satisfies the restricted majorant condition on the set S_0 if there exists a CN function $v : [0,\rho) \longrightarrow \mathbb{R}$ so that for each $x, \tilde{x} \in T(x_0,\rho)$*

$$\|g'(\tilde{x}) - g'(x)\| \leq v(\|\tilde{x} - x\|).$$

Definition 4. *Let g be a differentiable operator on the ball $T(x_0,\alpha)$. We say that g' satisfies the majorant condition on the ball $T(x_0,\alpha)$ if there exists a CN $v_1 : [0,\alpha) \to \mathbb{R}$ so that for each $x, \tilde{x} \in T(x_0,\rho)$ $\|g'(\tilde{x}) - g'(x)\| \leq v_1(\|\tilde{x} - x\|)$.*

Remark 1. *These definitions imply that for each $s \in [0,\rho)$*

$$v_0(s) \leq v_1(s) \tag{3}$$

and

$$v(s) \leq v_1(s). \tag{4}$$

Define the sequence $\{s_n\}$ *for* $s_0 = 0$, *some* $s_1 \geq 0$, *by*

$$s_2 = s_1 + \frac{qv_0(s_1)}{1 - qv_0(s_1)}$$

and

$$s_{n+2} = s_{n+1} + \frac{qv(s_{n+1})}{1 - qv_0(s_{n+1})} \text{ for each } n = 1, 2, \ldots. \tag{5}$$

The sequence $\{s_n\}$ *plays the role of a majorizing sequence in the semi-local analysis of convergence for the Newton method (2) (see Theorem 2).*

However, an auxiliary result is useful for the convergence of it.

(c2) There exists $\rho_0 \in [0, \rho)$ so that for each $n = 0, 1, 2, \ldots$ $qv_0(s_n) < 1$ and $s_n \leq \rho_0$.

This condition, the definition (2) and a simple inductive argument that for each $n = 0, 1, 2, \ldots$ $0 \leq s_n \leq s_{n+1} \leq \rho_0$. Thus, the sequence $\{s_n\}$ is nondecreasing and bounded from above by ρ_0 and as such it is convergent to some $s_* \in [0, \rho_0]$. The limit point s_* is the least upper bound of the sequence $\{s_n\}$ which is the only one.

(c3) Let β and q be positive numbers. The operator

$$x \to H(x) = g(x_0) + g'(x_0)(x - x_0) + G(x) \tag{6}$$

is said to be metrically regular at x_0 for y_0 with constant q related to the balls $T(x_0, \alpha)$ and $T(x_0, \beta)$.

(c4) Let $q_1 > q$ and take $s_1 = q_1\|y_0\|$.

Assume:

$s_* \leq \alpha$ and $\|y_0\| + q\left(\int_0^1 v_0((1-\theta)s_*) + \int_0^1 v((1-\theta)s_*)\right) d\theta s_* \leq \beta$.

The main semi-local analysis of convergence for the Newton method (2) follows using the conditions (c1)-(c4) and the preceeding notation.

Theorem 2. *Assume that the conditions (c1)-(c4) are valid, where the functions* v_0 *and* v *are as in the Definitions 2 and 3, respectively. Then, there exists a sequence* $\{x_n\}$ *starting at* x_0, *generated by the Newton method (2) which remains in* $T(x_0, \alpha)$ *and is convergent to a solution* $x_* \in T[x_0, \alpha]$ *of the generalized equation (1).*

Moreover, the error estimate is valid for each $n = 0, 1, 2, \ldots$

$$\|x_n - x^*\| \leq s_* - s_n. \tag{7}$$

In particular, since the operator H *is strongly metrically regular,* $\{x_n\}$ *is the only sequence generated by the Newton method (2) which stays in the ball* $T(x_0, \alpha)$.

Proof. Induction on the natural number m is employed to establish the assertion

$$\|x_{m+1} - x_m\| \quad \leq \quad s_{m+1} - s_m, \tag{8}$$

for some sequence $\{x_m\} \subset T(x_0, \alpha)$. If $y_0 = 0$ and $m = 0$, set $x_1 = x_0$. Let $y_0 \neq 0$. Then, since $0 \in T(y_0, \beta)$ and $y_0 \in H(x_0)$

$$\begin{aligned} & q\|g(x_0) + g'(x_0)(w - x_0) - g(x_m) - g'(x_m)(w - x_m)\| \\ \leq \; & q\left[\int_0^1 v_0((1-\theta)\|w - x_0\|)d\theta\|w - x_0\| \right. \\ & \left. + \int_0^1 v((1-\theta)\|w - x_m\|)d\theta\|w - x_m\|\right] \\ \leq \; & q\int_0^1 \left[v_0((1-\theta)s_*) + v((1-\theta)s_*)\right] d\theta s_* \leq \beta - \|y_0\|, \end{aligned}$$

so

$$g(x_0) + g'(x_0)(w - x_0) - g(x_m) - g'(x_m)(w - x_m) \in T(y_0, \beta).$$

Define the numbers

$$p_0 \quad := \quad \frac{q\int_0^1 v_0((1-\theta)\|x_1 - x_0\|)d\theta\|x_1 - x_0\|}{1 - q(\|x_1 - x_0\|)}$$

and

$$p := \frac{q\int_0^1 v((1-\theta)\|x_m - x_{m-1}\|)d\theta\|x_m - x_{m-1}\|}{1 - qv_0(\|x_m - x_0\|)}$$

for each $n = 0, 1, 2, \ldots$ By the induction hypothesis and (5), we get

$$\begin{aligned} p_0 \quad & \leq \quad \frac{q\int_0^1 v_0((1-\theta)(s_1 - s_0))d\theta(s_1 - s_0)}{1 - qv_0(s_0)} = s_2 - s_1, \\ p \quad & \leq \quad \frac{q\int_0^1 v((1-\theta)(s_m - s_{m-1}))d\theta(s_m - s_{m-1})}{1 - qv_0(s_m)} = s_{m+1} - s_m. \end{aligned}$$

Then, we have, since $x_m \in T(x_0, \alpha)$ is produced by the Newton method (2) given x_{m-1} and for each $n = 0, 1, 2, \ldots$

$$g(x_0) + g'(x_0)(x_m - x_0) - g(x_{m-1}) - g'(x_{m-1})(x_m - x_{m-1}) \in H(x_m). \quad (9)$$

Next, define the set-valued operator

$$\begin{aligned} S_1 \ni w \to \Psi_m(x) &= H^{-1}(g(x_0) + g'(x_0)(w - x_0) - g(x_m) \\ &\quad -g'(x_m)(w - x_m)) \subset S_1. \end{aligned} \quad (10)$$

If $x_m = x_{m-1}$, then set $x_{m+1} = x_m$. Assume that $x_m \neq x_{m-1}$. Then, it follows by the definition of the distance, (9) and (11) in turn that

$$\begin{aligned} dist(x_m, \Psi_m(x_m)) &= dist(x_m, H^{-1}(g(x_0) + g'(x_0)(x_n - x_0) - g(x_m))) \\ &\leq q dist(g(x_0) + g'(x_0)(x_m - x_0) - g(x_m), H(x_m)) \\ &\leq q\|g(x_m) - g(x_{m-1}) - g'(x_{m-1})(x_m - x_{m-1})\| \\ &\leq q\int_0^1 v((1-\theta)\|x_m - x_{m-1}\|)d\theta\|x_m - x_{m-1}\| \\ &< \frac{q\int_0^1 v((1-\theta)\|x_m - x_{m-1}\|)d\theta\|x_m - x_{m-1}\|}{1 - qv_0(\|x_m - x_0\|)} \\ &\quad \times(1 - qv_0(\|x_m - x_0\|)) \\ &= \bar{p}(1 - qv_0(\|x_m - x_0\|)), \end{aligned}$$

where

$$\bar{p} = \begin{cases} p_0, \text{if} & n = 0 \\ p, \text{ if} & n = 1, 2, \ldots \end{cases}$$

Take $w_1, w_2 \in T(x_m, s_* - s_m)$ and $w_3 \in \Psi_m(w_1) \cap T(x_m, s_* - s_m)$.

It follows

$$g(x_0) + g'(x_0)(w_1 - x_0) - g(x_m) - g'(x_m)(w_1 - x_m) \in H(w_3).$$

Consequently, we get in turn that

$$\begin{aligned} & dist(w_3, \psi_m(w_2)) \\ = \; & dist(w_3, H^{-1}(g(x_0) + g'(x_0)(w_2 - x_0) - g(x_m) - g'(x_m)(w_2 - x_m))) \\ \leq \; & q dist(g(x_0) + g'(x_0)(w_2 - x_0) - g(x_m) - g'(x_m)(w_2 - x_m), H(w_3)) \\ \leq \; & q\|g(x_0) + g'(x_0)(w_2 - x_0) - g(x_0) - g'(x_0)(w_2 - x_m) \\ & -(g(x_0) + g'(x_0)(w_1 - x_0) - g(x_m) - g'(x_m)(w_1 - x_m))\| \\ \leq \; & q\|g'(x_0) - g'(x_m)\|\|u_1 - u_2\| \\ \leq \; & qv_0(\|x_m - x_0\|)\|u_1 - u_2\|. \end{aligned}$$

By applying Theorem 1 and since $T(x_m, \bar{p}) \subset T(x_m, s_* - s_m)$ there exists a fixed point $x_{m+1} \in T(x_n, \bar{p})$ of the operator ψ_m. Therefore, we have the metric regularity of the operator (H) in (6) implies $dist(x_0, H^{-1}(0)) \leq q\|y_0\| \leq q_1\|y_0\|$. Thus, there exists $x_1 \in H^{-1}(0)$ so that $\|x_1 - x_0\| < q_1\|y_0\| = s_1 = s_1 - s_0$.

Assume that for some natural number m, we have determined the iterates $x_0, x_1, \ldots x_m \in T(x_0, \alpha)$ so that $\|x_i - x_{i-1}\| \leq s_i - s_{i-1}$ for $i = 0, 1, 2, \ldots, m$.

We can assume without losing any generality that $x_m \neq x_0$.

Otherwise, there is nothing to show. Then, by the induction hypothesis

$$\begin{aligned} \|x_m - x_0\| &\leq \sum_{i=1}^{m} \|x_i - x_{i-1}\| \\ &\leq \sum_{i=1}^{m} (s_i - s_{i-1}) = s_n - s_0 \\ &= s_n - 0 = s_n < s_* < \alpha. \end{aligned}$$

Moreover, for each $w \in T(x_m, s_* - s_m) \subset T(x_0, s_*)$ it follows by the Definition 2, the Definition 3 and $(c4)$ that

$$\begin{aligned} & q\|g(x_0) + g'(x_0)(w - x_0) - g(x_m) - g(x_m) - g'(x_m)(w - x_m)\| \\ \leq\ & q\|g(x) - g(x_0) - g'(x_m)(w - x_0)\| \\ & + q\|g(x) - g(x_m) - g'(x_m)(w - x_m)\| \end{aligned}$$

so $x_{m+1} \in H^{-1}(g(x_0) + g'(x_0)(x_{m+1} - x_0) - g(x_m) - g'(x_m)(x_{m+1} - x_m))$.

Thus, x_{m+1} is obtained from the Newton method (2) given x_m. Furthermore, we have

$$\|x_{m+1} - x_m\| \leq \bar{p} \leq s_{m+1} - s_m.$$

and

$$\begin{aligned} \|x_{m+1} - x_0\| &\leq \sum_{i=1}^{m+1} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{m+1} (s_i - s_{i-1}) \\ &= s_{m+1} - s_0 = s_{m+1} < s_* \leq \alpha. \end{aligned}$$

The induction for the assertion (8) is terminated. It follows by the last two inequalities that the sequence $\{x_m\}$ is Cauchy in a complete normed space and as such it is convergent to some $x_* \in T(x_0, s_*)$. By letting $n \to +\infty$ in the Newton method (2), we conclude that x_* solves the problem (1). The assertion (8) follows from the estimate $\|x_{m+j} - x_m\| \leq s_{m+j} - s_m$ for $j \to \infty$.

We still need to show the uniqueness of the sequence x_m in $T(x_0, s_*)$.

Assume that there is another sequence $\{u_m\}$ in $T(0,\beta)$ with $u_{m+1} \neq x_{m+1}$. Then, it follows that $x_{m+1} = \psi_m(x_{m+1})$ and $u_{m+1} = \psi_m(u_{m+1})$. By applying the definition of the excess

$$\begin{aligned} & \|u_{m+1-x_{m+1}}\| = e(H_m(u_{m+1}) \cap [x_*, s_m], H_m(x_{m+1})) \\ \leq \; & (1 - qv_0(\|x_m - x_0\|)\|u_{m+1} - x_{m+1}\| < \|u_{m+1} - x_{m+1}\|. \end{aligned}$$

□

4 Special cases and comparisons to earlier works

A direct comparison cannot be given with similar results such as in [12, 15] for $G \neq 0$ or in [14] for $G = 0$. However, such a comparison can be given if we specialize the majorant functions v_0, v and v_1 (see also (3) and (4)).

Case $G \neq 0$.

We compare our results to the ones in [12], (see also [14], [15]) by specializing the functions v_0 and v to

$$v_0(t) = q\frac{l_0}{2}t^2 - t + \eta, s_1 = \eta = q\|y_0\|,$$

$$v(t) = q\frac{l}{2}t^2 - t + \eta$$

and

$$v_1(t) = q\frac{l_1}{2}t^2 - t + \eta.$$

Clearly, we have $l_0 \leq l$ and $l \leq l_1$.

These inequalities can be strict (see the numerical example). Then, the majorizing sequence $\{t_n\}$ in [12],(see also [14], [15]) is

$$t_0 = 0, t_1 = s_1, t_{n+2} = t_{n+1} - \frac{q_1 l_1 (t_{n+1} - t_n)^2}{2(1 - ql_1(t_{n+1}))}, n = 0, 1, 2, \ldots \tag{11}$$

Moreover, the error bounds are available

$$\|x_n - x^*\| \leq \epsilon_n, \tag{12}$$

where for

$$\begin{aligned} \lambda &= \frac{1 - \sqrt{1-2h}}{1 + \sqrt{1-2h}} \in [0, 1) \\ \epsilon_n &= \frac{2\sqrt{1-2h}\lambda^{2^n}}{ql_1(1 - \lambda^{2^n})}, n = 1, 2, \ldots \end{aligned} \tag{13}$$

and provided that the famous for its simplicity and clarity Newton- Kantorovich semi-local condition [1–7,16–18,22,23] for the convergence of Newton's method is fulfilled, which is given by

$$h_1 = L_1\eta < \frac{1}{2}, L_1 = ql_1.. \tag{14}$$

In our case the majorizing sequence $\{s_n\}$ reduces to

$$s_0 = 0, s_1 = \eta, s_2 = s_1 + \frac{ql_0(s_1 - s_0)^2}{2(1 - l_0q)} \tag{15}$$

and

$$s_{n+2} = s_{n+1} + \frac{ql(s_{n+1} - s_n)^2}{2(1 - l_0qs_{n+1})}, n = 1, 2, \ldots$$

Set for simplicity $L_0 = ql_0$ and $L = ql$.

The convergence of the sequence $\{s_n\}$ has been studied extensively by us in [4–7]. In particular, the corresponding to (14) semi-local convergence criteria in [[5], see Lemma 2.1] are

$$h = \bar{L}\eta \leq \frac{1}{2}, \tag{16}$$

where

$$\bar{L} = \frac{1}{8}(4L_0 + \sqrt{L_2L_0 + 8L_0^2} + \sqrt{L_0L_2}), L_2 = L_1 \text{ or } \ L_2 = L.$$

Then, under (14) and (16) a simple inductive argument shows

$$0 \leq s_n \leq t_n, \tag{17}$$

$$0 \leq s_{n+1} - s_n \leq t_{n+1} - t_n, \tag{18}$$

so

$$\lim_{n\to+\infty} s_n = s_* \leq t_* = \lim_{n\to+\infty} t_n = \frac{1}{L_1}(1 - \sqrt{1 - 2h_1}) \tag{19}$$

and

$$h_1 < \frac{1}{2} \implies h \leq \frac{1}{2}, \tag{20}$$

but not necessarily vice versa unless $l_0 = l = l + 1$. (see also the numerical example where (14) is violated but (16) is valid). It is worth noting also that as $\frac{l_1}{l_1} \to 0$, then

$$\frac{h}{h_1} \to 0 \tag{21}$$

Therefore, we justify the items (1)'-(3)' as stated in the introduction of this paper. It is also worth stating (21) indicates by how many times (infinitely) our approach extends the applicability of Newton method (2).

Case $G = 0$.

We compare our results to the ones in [14,15] by specializing the functions v_0, v and v_1 to be

$$\begin{aligned} v_0(t) &= \varphi_0'(t) - \varphi_0'(0), \\ v((1-\theta)t) &= \varphi'(t) - \varphi'(\theta t) \end{aligned}$$

and

$$v_1((1-\theta)t) = \varphi_1'(t) - \varphi_1'(\theta t),$$

where the functions φ_0, φ and φ_1 are twice continuously differentiable, convex, strictly increasing and satisfy

$$\varphi_0(0) = \varphi(0) = \varphi_1(0) = 0$$

and

$$\varphi_0'(0) = \varphi'(0) = \varphi_1'(0) = -1.$$

Notice that (see also (3) and (4))

$$\varphi_0'(t) \leq \varphi_1'(t) < 0 \tag{22}$$

and

$$\varphi'(t) \leq \varphi_1'(t) < 0 \tag{23}$$

on some subinterval of $[0,\infty)$. These inequalities can be strict (see the example).

The majorizing sequence in [14, 15] is

$$t_{n+1} = \left| t_n - \frac{\varphi_1(t_n)}{\varphi_1'(t_n)} \right|, \tag{24}$$

but ours is

$$s_{n+1} = \left| \frac{s_n \varphi_0'(s_n) - \psi(s_n)}{\varphi_0'(s_n)} \right|, \psi = \varphi \text{ or } \psi = \varphi_1 \tag{25}$$

Notice that for $\psi = \varphi_1$ the sequence (25) can be written as

$$s_{n+1} = \left| s_n - \frac{\varphi_1(s_n)}{\varphi_1'(s_n)} \right| \left| \frac{\varphi_1'(s_n)}{\varphi_0'(s_n)} \right|.$$

But by (22) and (23)

$$|\varphi_1'(s_n)| \leq |\varphi_0'(s_n)|.$$

Consequently, the estimates (17)-(19) are valid in this more general setting. The same is true for the convergence criteria and uniqueness of the solution results. Notice also that in [14] Newton's method is shown to be of order two and the sequence $\{t_n\}$ is its majorant sequence which is of order two (see also (13)).

Let us complete this study with a simple example that shows even for scalar equations the condition (14) is not satisfied but the new condition (16) is valid.

Example 1. *Let* $S_1 = S_2 = (-\infty, +\infty), T = T(1, 1-\gamma), \gamma \in [0, \frac{1}{2})$ *and consider the real function* $f : T \to (-\infty, +\infty)$ *defined by*

$$f(t) = t^3 - \gamma.$$

Then, we have $\eta = \frac{1}{2}(1-\mu), L_0 = 3-\mu, L = 2(1+\frac{1}{1-L_0})$ *and* $L_1 = 2(2-\mu)$. *But, the Kantorovich condition (14) is not satisfied, since* $h_1 = \frac{2}{3}(2-\mu)(1-\mu) > \frac{1}{2}$ *for each* $\mu \in [0, \frac{1}{2})$.

Thus, the approach in [12, 14, 15] cannot assure the convergence of Newton method to the solution $x_* = \sqrt[3]{\gamma}$ *although it converges. But the new condition (16) is valid if we set* $L_2 - L_1$ *for* $\mu \in [0.463, \frac{1}{2})$ *and even better if* $L_2 = L$ *for* $\mu \in [0.447, \frac{1}{2})$. *Other examples, where* $v_0 < v, v_0 < v_1, v_0 < v, l_0 < l_1, l < l_1, l_0 < l, \varphi_0 < \varphi_1, \varphi < \varphi_1$ *and* $\varphi_0 < \varphi$ *can be found in [4–9].*

The applicability of the rest of the methods for solving the generalized equation (1) given in [12, 14, 15] can be extended along the same lines. We leave the details to the motivational reader.

5 Semi-local convergence

It is worth noticing that the Aubin property is related to the metric regularity [13]. Consequently, the results are provided in terms of metric regularity. But first, we need a relationship between different types of majorant conditions.

Assume $R > 0$.

Definition 5. *A function $h_0 : [0, R) \longrightarrow (-\infty, +\infty)$ which is twice continuously differentiable and non-decreasing is said to be a center-majorant function for g on $T(x_0, R)$ with modulus $c > 0$ if for each $y \in T[x_0, R]$*
(A1) $c\|g'(y) - g'(x_0)\| \leq h_0'(\|y - x_0\|) - h_0'(0)$.
(A2) The function $h_0'(t)$ has a smallest positive zero denoted by ρ.

Definition 6. *A function $h = h(h_0) : [0, \rho) \longrightarrow (-\infty, +\infty)$ which is twice continuously differentiable and non-decreasing is said to be a restricted-majorant function for g on $T(x_0, \rho)$ with modulus $c > 0$ if for each x, $y \in T(x_0, \rho)$*
(A3) $c\|g'(y) - g'(x)\| \leq h'(\|y - x\| + \|x - x_0\|) - h'(\|x - x_0\|)$.

Notice that the function h_0 depends on x_0 and $\bar{\rho}$, where as the function h relies on x_0, ρ and h_0.
(A4) Assume:

$$h_0'(s) \leq h'(s) \text{ for each } s \in [0, \rho). \tag{26}$$

Definition 7. *A function $h_1 : [0, R) \longrightarrow (-\infty, +\infty)$ which is twice continuously differentiable and non-decreasing is said to be a majorant function for g on $T(x^*, \rho)$ with modulus $a > 0$ if for each x, $y \in T(x^*, \bar{\rho})$.*
(A3)' $a\|g'(y) - g'(x)\| \leq h_1(\|y - x\| + \|x - x_0\|) - h_1'(\|x - x_0\|)$.

It follows by these definitions that

$$h_0'(s) \leq h_1'(s) \text{ and } h'(s) \leq h_1'(s) \text{ for each } s \in [0, \rho). \tag{27}$$

Thus, the results in the literature using only h_1 (see e.g. [10, 14, 15] for $G = 0$) can be replaced by the pair (h_0, h) resulting to a finer error distances with advantages as already given in the introduction. These advantages are obtained under the same computational cost, since the new majorant functions h_0, h are specializations of the function h_1.

The additional conditions are:

(A4) $h_0(0) > 0$, $h_0'(0) = -1$;

(A5) h_0' is convex and strictly increasing;

(A6) $h_0(t) = 0$ for some $t \in (0, R)$. Take $t_0^* = \min\{t \in [0, R) : h_0(t) = 0\}$;

(A7) $h(0) > 0, h'(0) = -1$;

(A8) h' is convex and strictly increasing;

(A9) $h(t) = 0$ for some $t \in [0, t_0^*)$. Take $t^* = \min\{t \in [0, t_0^*) : h(t) = 0\}$.

The conditions in [15] are:

(A4)' $h_1(0) > 0$, $h_1'(0) = -1$;

(A5)' h_1' is convex and strictly increasing;

(A6)' $h_1(t) = 0$ for some $t \in (0, R)$. Take $t_1^* = \min\{t \in [0, R) : h_1(t) = 0\}$;

Define the majorant sequences corresponding to these conditions to be for

$$\bar{h} = \begin{cases} h_0, & n = 0 \\ h, & n = 1, 2, \ldots \end{cases}$$

$$r_0 = 0, \qquad r_{n+2} = r_{n+1} + \frac{\int_0^1 \bar{h}'(r_n + \theta(r_{n=1} - r_n))d\theta(r_{n+1} - r_n)}{1 - h_0'(r_{n+1})}, \tag{28}$$

$$s_0 = 0, \qquad s_{n+1} = s_n - \frac{\bar{h}(s_n)}{h_0'(s_n)}, \tag{29}$$

and

$$t_0 = 0,\ t_{n+1} = t_n - \frac{h_1(t_n)}{h_1'(t_n)}. \tag{30}$$

The sequence $\{t_n\}$ is studied in [15] under the conditions (A4)'-(A6)' and in [13] if the function h_1 is constant. The sequences $\{r_n\}$ and $\{s_n\}$ correspond to the conditions (A1)-(A9). Clearly, a simple induction argument shows that

$$0 \le s_n \le r_n \le t_n, \tag{31}$$

$$0 \le s_{n+1} - s_n \le r_{n+1} - r_n \le t_{n+1} - t_n, \tag{32}$$

and

$$\lim_{n \longrightarrow +\infty} r_n \le \lim_{n \longrightarrow +\infty} s_n = t^* \le \lim_{n \longrightarrow +\infty} t_n = t_1^*. \tag{33}$$

Thus, the new error analysis is tighter.

Let us consider the conditions:
(A10) The set valued mapping $u \to L^{-1}_{g+G,x_0}(u)$ has the Aubin property at 0 for x_1, with modulus $c > 0$, and associated constants $a > 0$ and $b > 0$. Finally, we define the set-valued mapping $\Psi_x : \Omega \rightrightarrows S_1$ by

$$\Psi_x(v) := L^{-1}_{g+G,x_0}(E_g(x, v) - E_g(x_0, v)). \tag{34}$$

Then, all the results in [1,3,15] can be rewritten with (h_0, h) replacing the function h_1.

Next, we only mention the main ones. The details for the rest are left to the motivated reader.

Theorem 3. *Assume that the conditions (A1)-(A10) hold. Moreover, assume that the iterate x_1 is generated from the initial point $x_0 \in \Omega$ by (2)*

$$g(x_n) + g'(x_n)(x_{n+1} - x_n) + G(x_{n+1}) \ni 0. \tag{35}$$

Furthermore, assume

$$T[x_0, t^*] \subset \Omega.$$

Then, there exists a sequence $\{x_n\}$ generated by (34) such that the iterate $x_1 \in T(x_0, t^*) \cap T(x_1, a)$ such that

$$\|x_{n+1} - x_n\| \leq s_{n+1} - s_n,$$

$$\|x_{n+1} - x_n\| \leq \frac{s_{n+1} - s_n}{(s_n - s_{n-1})^2}\|x_n - x_{n-1}\|^2,$$

$$\|x_{n+1} - x_n\| \leq \frac{1}{2}\|x_n - x_{n-1}\|.$$

Moreover, $\{x_n\} \subset u(x_0, t^*) \cap T(x_1, a)$ is convergent to some point $x^* \in T[x_0, t^*] \cap T[x_1, a]$ satisfying $g(x^*) + G(x^*) \ni 0$ such that

$$\|x^* - x_n\| \leq s^* - s_n.$$

Additionally, if the condition $h_0'(s^*) < 0$ holds, then

$$\|x_{n+1} - x_n\| \leq \frac{h''(s^*)}{2|h_0'(s^*)|}\|x_n - x_{n-1}\|^2.$$

Hence, the sequence $\{x_n\}$ is convergent $Q-$ quadratically to x^*.

Proof. Simply replace h_1 by the pair (h_0, h) as well as the sequence $\{t_n\}$ by $\{s_n\}$ in the proof of Theorem 6 in [15]. Clearly, these conclusions can be rewritten provided that tighter sequence $\{r_n\}$ replaces $\{s_n\}$. □

Theorem 4. *Assume that the conditions (A1)-(A10) hold and the mapping in (34) is single-valued in $T(0,b)$.*

Then, there exists a sequence $\{x_n\}$ generated by (35) belonging to $T(x_0,t^*)\cap T(x_1,a)$ which converges to $x^*\in T[x_0,t^*]\cap T[x_1,a]$ solving $g(x^*)+G(x^*)\ni 0$. Moreover, the sequence $\{x_n\}$ converges to x^* such that

$$\|x^*-x_{n+1}\|\leq \frac{s^*-s_{n+1}}{(s^*-s_n)^2}\|x^*-x_n\|^2,$$

and

$$\|x^*-x_{n+1}\|\leq \frac{1}{2}\|x^*-x_n\|^2.$$

Additionally, if $h_0'(t^*)<0$, then the sequence $\{x_n\}$ is convergent $Q-$quadratically to x^* and satisfies

$$\|x^*-x_{n+1}\|\leq \frac{h''(s^*)}{2|h_0'(s^*)|}\|x^*-x_n\|^2.$$

Moreover, x^* is the unique solution of equation (2) in $T[x_0,s^*]\cap T[x_1,a]$.

Proof. As in the proof of Theorem 3. □

Remark 2. *If $h_0=h=h_1$, $t_n=s_n$, then the results of Theorem 3 and Theorem 4 reduce to Theorem 5 and Theorem 7, respectively, in [15]. Otherwise, they are better with advantages, as already stated in the introduction. Clearly, the results in [15] imply those in this article but not necessarily vise versa.*

Let us specialize the majorant functions h_0,h,h_1 by

$$h_0(t)=q_0t,\ h(t)=qt,\ \text{ and } h_1(t)=q_1t.$$

Clearly, we have $0\leq q_0\leq q\leq q_1$. Then the corresponding sequences $\{r_n\},\{s_n\}$ and $\{t_n\}$ are

$$\begin{aligned} r_0=0,\ r_1 &= \|x_n-x_0\|,\ r_2=r_1+\frac{q_0((r_1-r_0)^2}{2(1-q_0r_1)} \qquad (36)\\ r_{n+2} &= r_{n+1}+\frac{q((r_{n+1}-r_n)^2}{2(1-q_0r_{n+1})}\\ s_0=0,\ s_1 &= \|x_1-x_0\|,\ s_2=s_1+\frac{q_0((s_1-s_0)^2}{2(1-q_0s_1)}\\ s_{n+2} &= s_{n+1}+\frac{q((s_{n+1}-s_n)^2}{2(1-q_0s_{n+1})} \qquad (37)\\ t_0=0,\ t_1 &= \|x_n-x_0\|,\ t_{n+1}=t_n+\frac{q_1((t_n-t_{n-1})^2}{2(1-q_1t_n)},\ n=1,2,\ldots \qquad (38) \end{aligned}$$

The convergence of these sequences is studied in [4–7, 16]. In particular, Kantorovich used the sequence $\{t_n\}$ to solve equation (1) if $G = 0$. The convergence condition is

$$q_1\|x_1 - x_0\| \leq \frac{1}{2}. \tag{39}$$

But the convergence conditions in [4–7, 16] are always weaker. Let us just provide only one for both sequences $\{r_n\}$ and $\{s_n\}$ given by

$$\bar{q}\|x_1 - x_0\| \leq \frac{1}{2}, \tag{40}$$

where

$$\bar{q} = \frac{q_0 + q}{2}.$$

Then, (39) implies (40) but not necessarily vise versa unless $q_0 = q = q_1$. Examples where $q_0 \leq q \leq q_1$ are given in [3–7, 16]. Clearly, the specialized cases given in [15] are also extended along the lines of the above.

6 Conclusion

A finer semi-local convergence analysis for Newton's and related methods is developed for solving the generalized equation (1) with advantages as already stated in the Introduction even if the majorant functions used are specialized to the ones given in earlier papers [11, 14, 15]. The advantages are obtained by using more precise majorizing sequences. The new methodology can be used on other methods in order to extend their applicability in a similar way [1–3, 10, 11, 13–27].Clearly, the new approach can be used to extend results on the local convergence of these methods.

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