

EXPLORING SOME APPROXIMATION PROCESSES IN QUANTUM CALCULUS*

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Dedicated to the memory of Professor Mihail Megan

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Abstract

The paper is a synthesis on a topic that concerns Approximation Theory. We refer to the construction of some approximation processes using q -Calculus. At first we recall the fundamental elements of q -Calculus and then we focus on the presentation of special discrete-type operators sequences using this theory. The main characteristics of the new constructions are highlighted.

Keywords: positive linear operators, q -Calculus, rate of convergence, approximation process.

MSC: 41A36, 41A25, 26D15.

1 Introduction

Quantum Calculus appeared as a connection between mathematics and physics having the characteristic of describing nature more accurately. It is known that the universe is quantized and if we are at a point x , then the points that we can reach are in geometric progression $q^i x$, in accordance with the astronomer Edwin Hubble.

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There is no general definition of a " q -analogue". A q -analogue, also called q -extension of a mathematical object X , is a family of objects $X(q)$, $q > 0$, in general $q \in (0, 1)$ such that $\lim_{q \rightarrow 1} X(q) = X$.

In the last decades, Quantum Calculus began to be widely used in the construction of linear positive approximation processes.

This work being a synthesis, after presenting the fundamental elements regarding q -Calculus in order to make the material essentially self-contained, the authors pursued two distinct aspects: indicating the representative results in this field established by various specialists along with highlighting the authors' achievements published in time or newly developed. The architecture of the work was created by the second author.

Certainly, a work of this size does not include an exhaustive approach. Our goal is only to offer readers a brief presentation of the application of q -Calculus in approximating the signals belonging to different spaces of functions with the help of linear and positive operators. We mention that we will refer only to a few mathematicians who have made contributions in this field, the selection being subjective therefore random. Among the pioneers of this research direction we mention A. Lupas, G.M. Phillips and V. Gupta. The article provides the reader with a mathematical background and a wide variety of examples for key results.

2 Elements of q -Calculus

For the reader's convenience, we recall basic definitions, notations and results regarding Quantum Calculus. In this direction, the book of Kac and Cheung can be consulted [11]. Let $q > 0$. For any $n \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, the q -integer $[n]_q$ and the q -factorial $[n]_q!$ are respectively defined by

$$[n]_q = \sum_{j=0}^{n-1} q^j, \quad [n]_q! = \prod_{j=1}^n [j]_q, \quad n \in \mathbb{N},$$

and $[0]_q = 0$, $[0]_q! = 1$. The q -binomial coefficients are denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}, \quad k = 0, 1, \dots, n.$$

Clearly, for $q = 1$ one has $[n]_1 = n$, $[n]_1! = n!$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

the ordinary binomial coefficients.

In the sequel, we always will assume that $q \in (0, 1)$. Each q -binomial coefficient is a polynomial in q of degree $k(n - k)$ with 1 as the leading coefficient. Also, the q -binomial coefficients satisfy the following recurrence relations (q-Pascal rule):

$$\begin{cases} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n \\ k \end{bmatrix}_q, \\ \begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n \\ k \end{bmatrix}_q, \end{cases} \quad (1)$$

where $1 \leq k \leq n$. These identities can be easily obtained using an elementary calculus, see for example [11, pp. 17–18].

Definition 1. Let $q > 0$, $q \neq 1$, be a real number. The q -derivative of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \quad D_q f(0) = \lim_{x \rightarrow 0} D_q f(x).$$

Definition 2. The function F is a q -antiderivative of f if $D_q F = f$. It is denoted by

$$F(x) = \int f(x) d_q(x).$$

In the ordinary calculus, the uniqueness is up to adding a constant since the derivative of a function vanishes if and only if it is constant. The situation in Quantum Calculus is more subtle. $D_q \varphi(x) = 0$ if and only if $\varphi(qx) = \varphi(x)$, which does not necessarily imply φ a constant.

Definition 3. Let $b > 0$, $q \in (0, 1)$ and $f : \mathbb{R} \rightarrow \mathbb{R}$. The real number denoted by $I(f; q; b)$ and defined by

$$I(f; q; b) := \int_0^b f(x) d_q x = (1-q) \sum_{k=0}^{\infty} b q^k f(b q^k) \quad (2)$$

is called the finite q -integral of the function f on $[0, b]$.

On a general interval $[a, b]$, $0 < a < b$, one defines

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

If f is Riemann integrable on $[0, b]$, then

$$\int_0^b f(x) dx = \lim_{q \rightarrow 1^-} \int_0^b f(x) d_q x.$$

Moreover, the q -integral defined by (2) satisfies the relation

$$D_q \int_0^x f(t) d_q t = f(x), \quad x \neq 0.$$

Recalling that for any $r \in \mathbb{N}$, the r^{th} order difference operator of a function f is recursively defined by

$$\Delta_q^r f_k = \Delta_q^{r-1} f_{k+1} - q^{r-1} \Delta_q^{r-1} f_k,$$

where $\Delta_q^0 f_k = f_k$ and $f_k := f([k]_q/[n]_q)$, $k, n \in \mathbb{N}$, we present the connection between q -difference $\Delta_q^r f_0$ and r^{th} derivative $f^{(r)}$

$$\frac{\Delta_q^r f_0}{q^{r(r-1)}[r]_q!} = \frac{f^{(r)}(\xi)}{r!}, \quad \xi \in [0, [r]_q),$$

see [15, p.268]. This proves the existence of a bridge between q -Calculus and classical analysis.

3 An extension of q -Stirling numbers of the second kind

In this section, we present a new generalization of Stirling numbers based on Quantum Calculus.

Following [3], for each $(m, r) \in \mathbb{N}_0 \times \mathbb{N}_0$, the number $\sigma_q(m, r)$ defined by

$$\sigma_q(m, r) = \frac{1}{[r]_q!} \sum_{j=0}^r (-1)^j q^{(r-j)(r-j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}}, \quad (3)$$

represents a new q -analogue of Stirling number of the second kind.

Theorem 1. *The numbers $\sigma_q(m, r)$, $(m, r) \in \mathbb{N}_0 \times \mathbb{N}_0$, given by (3) enjoy the following properties.*

$$\sigma_q(m, 0) = 0 \quad (m \in \mathbb{N}) \quad \text{and} \quad \sigma_q(0, 0) = 1, \quad (4)$$

$$q^r \sigma_q(m+1, r) = [r]_q \sigma_q(m, r) + \sigma_q(m, r-1), \quad m \in \mathbb{N}_0, \quad r \in \mathbb{N}, \quad (5)$$

$$\sigma_q(m, r) = 0, \quad r > m. \quad (6)$$

Proof. The identity (4) follows immediately from (3). By using the same definition, for each $m \in \mathbb{N}_0$, $r \in \mathbb{N}$, we can write

$$\sigma_q(m, r-1) = \frac{1}{[r-1]_q!} \sum_{j=1}^r (-1)^{j-1} q^{(r-j)(r-j-1)/2} \begin{bmatrix} r-1 \\ j-1 \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}}. \quad (7)$$

On the other hand,

$$\begin{aligned} & q^r \sigma_q(m+1, r) - [r]_q \sigma_q(m, r) \\ &= \frac{[r]_q}{[r]_q!} q^{r-1} \sum_{j=0}^r (-1)^j q^{(r-j-1)(r-j-2)/2} \begin{bmatrix} r-1 \\ j \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}} \\ & \quad - \frac{[r]_q}{[r]_q!} \sum_{j=0}^r (-1)^j q^{(r-j)(r-j-1)/2} \begin{bmatrix} r \\ j \end{bmatrix}_q \frac{[r-j]_q^m}{q^{(r-j)m}} \\ &= \frac{1}{[r-1]_q!} \sum_{j=0}^r (-1)^j q^{(r-j-1)(r-j-2)/2} \frac{[r-j]_q^m}{q^{(r-j)m}} \left(q^{r-1} \begin{bmatrix} r-1 \\ j \end{bmatrix}_q - q^{r-j-1} \begin{bmatrix} r \\ j \end{bmatrix}_q \right) \\ &= \frac{1}{[r-1]_q!} \sum_{j=0}^r (-1)^j q^{(r-j-1)(r-j-2)/2} \frac{[r-j]_q^m}{q^{(r-j)m}} q^{r-1} \left(\begin{bmatrix} r-1 \\ j \end{bmatrix}_q - \frac{1}{q^j} \begin{bmatrix} r \\ j \end{bmatrix}_q \right). \end{aligned}$$

Taking into account (1) with $n := r-1$, $k := j$ and (7) the proof is complete. For establishing relation (6), on the basis of (5), it is enough to prove $\sigma_q(m, m+1) = 0$ for any $m \in \mathbb{N}_0$. Since

$$[m+1-j]_q^m = \frac{1}{(1-q)^m} \sum_{k=0}^m (-1)^k \binom{m}{k} q^{k(m+1)-kj}, \quad \text{for } 0 \leq j \leq m+1,$$

and

$$\sum_{j=0}^{m+1} q^{j(j-1)/2} \begin{bmatrix} m+1 \\ j \end{bmatrix}_q \left(-\frac{1}{q^k} \right)^j = \left(1 - \frac{1}{q^k} \right)_q^{m+1} = 0, \quad \text{for } 0 \leq k < m+1,$$

we can write

$$\begin{aligned}\sigma_q(m, m+1) &= \frac{1}{[m+1]_q! q^{m(m+1)/2}} \sum_{j=0}^{m+1} (-1)^j j^{j(j-1)/2} \begin{bmatrix} m+1 \\ j \end{bmatrix}_q [m+1-j]_q^m \\ &= \frac{(1-q)^{-m}}{[m+1]_q! q^{m(m+1)/2}} \sum_{k=0}^m (-1)^k \binom{m}{k} q^{k(m+1)} \left(1 - \frac{1}{q^k}\right)_q^{m+1} = 0.\end{aligned}$$

□

Easily it can be proved

$$\lim_{q \rightarrow 1^-} \sigma_q(m, r) = \frac{1}{r!} \sum_{j=0}^r (-1)^{r-j} \binom{r}{j} j^m, \quad m \in \mathbb{N}_0, \quad r \in \mathbb{N}_0,$$

the limit representing $S(m, r)$, the classical Stirling numbers of the second kind, see [1, p.824].

It is known that a q -analogue of Stirling numbers is not unique. For example Aral [5] introduced the numbers $S_q(m, r)$ in studying a q -generalization of Szász–Mirakjan operators.

Further, we focus on using q -Calculus to obtain sequences of linear approximation operators. The first researches have been achieved by Lupaş [12] and Phillips [16] who proposed q -variants of the original Bernstein operators. Also, along the way, other classes of operators have been extended, for example: Meyer–König and Zeller operators [19], Bleimann, Butzer and Hahn operators [6], Szász–Mirakjan operators [5], Balázs–Szabados operators [8].

4 A class of Stancu operators in q -Calculus

For $f \in C([0, 1])$, $\alpha \geq 0$ and each $n \in \mathbb{N}$, Nowak [14] have been defined the operators

$$(B_n^{q,\alpha} f)(x) = \sum_{k=0}^n p_{n,k}^{q,\alpha}(x) f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0, 1], \quad (8)$$

where

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1} (x + \alpha[i]_q) \prod_{s=0}^{n-1-k} (1 - q^s x + \alpha[s]_q)}{\prod_{i=0}^{n-1} (1 + \alpha[i]_q)}.$$

We mention, an empty product is taken to be equal to 1.

This class contains as special cases to following well-known sequences of linear and positive operators:

(i) For $\alpha = 0$, $B_n^{q,0} \equiv B_n^q$ represents q -Bernstein operator introduced by Phillips.

(ii) For $q = 1$ and $\alpha = 0$, $B_n^{1,0} \equiv B_n$ is the classical Bernstein polynomial.

(iii) For $q = 1$, $B_n^{1,\alpha} \equiv B_n^{(\alpha)}$ turns into Stancu operator defined as follows

$$(B_n^{(\alpha)} f)(x) = \frac{1}{1[n, -\alpha]} \sum_{k=0}^n \binom{n}{k} x^{[k, -\alpha]} (1-x)^{[n-k, -\alpha]} f\left(\frac{k}{n}\right), \quad x \in [0, 1].$$

Here,

$$t^{[m, \alpha]} = \prod_{j=0}^{m-1} (t - j\alpha)$$

represents the generalized factorial power with the step $\alpha \in \mathbb{R}$, $m \in \mathbb{N}$. In [2] the author proved the following identities

$$(B_n^{q, \alpha} e_0)(x) = 1, \quad (B_n^{q, \alpha} e_1)(x) = x,$$

$$(B_n^{q, \alpha} e_2)(x) = \frac{1}{1 + \alpha} \left(x(x + \alpha) + \frac{x(1-x)}{[n]_q} \right), \quad x \in [0, 1], \quad (9)$$

where $e_j, j \in \mathbb{N}$, stands for the monomial of j -th degree and $e_0(x) = 1$.

On the basis of Bohman–Korovkin theorem, the conclusion

$$\lim_{n \rightarrow \infty} (B_n^{q_n, \alpha_n} f)(x) = f(x) \quad \text{uniformly in } x \in [0, 1], \quad (10)$$

takes place with additional hypotheses

$$0 < q_n < 1, \quad \lim_{n \rightarrow \infty} q_n = 1 \quad \text{and} \quad \alpha_n \geq 0, \quad \lim_{n \rightarrow \infty} \alpha_n = 0. \quad (11)$$

The operator $B_n^{q, \alpha}$ can be reintroduced by using probabilistic tools, see [2]. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $Z_q : \mathbb{N} \times [0, 1] \rightarrow \mathcal{M}_2(\Omega)$ a random scheme on $[0, 1]$, where $\mathcal{M}_2(\Omega)$ stands for the space of all real square-integrable random variables on Ω . We consider

$$P \left(Z_q(n, x) = \frac{[k]_q}{[n]_q} \right) = p_{n,k}^{q, \alpha}(x), \quad 0 \leq k \leq n, \quad n \in \mathbb{N}, \quad x \in [0, 1].$$

As usual, the mathematical expectation and the variance of $Z_q(n, x)$ are denoted by $E(Z_q(n, x))$ and $\text{Var}(Z_q(n, x))$, respectively. Setting $P_{Z_q(n, x)}$ the distribution of $Z_q(n, x)$ with respect to P , one has

$$(B_n^{q, \alpha} f)(x) = \int_{\Omega} f \circ Z_q(n, x) dP = \int_0^1 f dP_{Z_q(n, x)} = E(f(Z_q(n, x))), \quad (12)$$

$x \in [0, 1]$ and, in harmony with (9), we obtain

$$E(Z_q(n, x)) = x, \quad \text{Var}(Z_q(n, x)) = \frac{1 + \alpha[n]_q}{(1 + \alpha)[n]_q} x(1 - x). \quad (13)$$

Theorem 2. *Under the hypotheses (11), identity (10) holds.*

Proof. (probabilistic approach). For proving the statement, it is enough to show the following. For any $\varepsilon > 0$, there exist $\eta_\varepsilon > 0$ and $n_\varepsilon \in \mathbb{N}$ such that

$$|(B_n^{q_n, \alpha_n} f)(x) - f(x)| < \varepsilon, \quad (14)$$

for all $x \in [0, 1]$, $q_n \in (1 - \eta_\varepsilon, 1)$, $\alpha_n \in (0, \eta_\varepsilon)$ and $n \geq n_\varepsilon$.

Let $\varepsilon > 0$ be arbitrarily fixed. Since $f \in C([0, 1])$, a constant M exists such that $|f(x)| \leq M$ for all $x \in [0, 1]$. Let $\delta \in (0, 1]$ be chosen such that $|f(t) - f(x)| < \varepsilon/2$, whenever $|t - x| < \delta$, $t \in [0, 1]$, $x \in [0, 1]$. For all t and x belonging to $[0, 1]$ such that $|t - x| \geq \delta$, we can write

$$|f(t) - f(x)| \leq 2M \leq 2M \frac{(t - x)^2}{\delta^2}.$$

Let $\eta_\varepsilon := \varepsilon \delta^2 / M$, where $\delta \leq 1$. The constant M can be taken as large as we want, consequently $\eta_\varepsilon < 1$.

On the other hand, since $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} q_n = 1$, we get $\lim_{n \rightarrow \infty} [n]_{q_n} = \infty$ and, for the above chosen η_ε , $n_\varepsilon \in \mathbb{N}$ exists such that

$$\alpha_n + \frac{1}{[n]_{q_n}} < \eta_\varepsilon \quad \text{for each } n \geq n_\varepsilon. \quad (15)$$

Taking in view both (12), (13) and (15) we can write successively

$$\begin{aligned} |(B_n^{q_n, \alpha_n} f)(x) - f(x)| &\leq \int_0^1 |f(t) - f(x)| P_{Z_{q_n}(n, x)}(dt) \\ &= \int_{\substack{t \in [0, 1] \\ |t-x| < \delta}} + \int_{\substack{t \in [0, 1] \\ |t-x| \geq \delta}} \leq \frac{\varepsilon}{2} \int_0^1 dP_{Z_{q_n}(n, x)} + \frac{2M}{\delta^2} \int_0^1 (t - x)^2 dP_{Z_{q_n}(n, x)} \\ &= \frac{\varepsilon}{2} + \frac{2M}{\delta^2} \text{Var}(Z_{q_n}(n, x)) \leq \frac{\varepsilon}{2} + \frac{2M}{\delta^2} \left(\alpha_n + \frac{1}{[n]_{q_n}} \right) \max_{x \in [0, 1]} x(1 - x) < \varepsilon. \end{aligned}$$

Therefore, (14) is true and the proof is ended. $\square$

Further we explore the rate of convergence of $B_n^{q,\alpha}$, $0 < q < 1$, $\alpha > 0$, $n \in \mathbb{N}$, in terms of the modulus of continuity $\omega_1(f; \cdot)$, where

$$\omega_1(f; \delta) = \sup_{\substack{x, y \in [0, 1] \\ |x - y| \leq \delta}} |f(x) - f(y)|, \quad \delta \geq 0.$$

Theorem 3. Let $B_n^{q,\alpha}$, $n \in \mathbb{N}$, be defined by (8).

(i) If $f \in C([0, 1])$, then

$$|(B_n^{q,\alpha} f)(x) - f(x)| \leq \frac{3}{2} \omega_1 \left(f; \sqrt{\frac{1 + \alpha[n]_q}{(1 + \alpha)[n]_q}} \right), \quad x \in [0, 1].$$

(ii) If f is differentiable and $f' \in C([0, 1])$, then

$$|(B_n^{q,\alpha} f)(x) - f(x)| \leq \frac{3}{4} \sqrt{\frac{1 + \alpha[n]_q}{(1 + \alpha)[n]_q}} \omega_1 \left(f'; \sqrt{\frac{1 + \alpha[n]_q}{(1 + \alpha)[n]_q}} \right), \quad x \in [0, 1].$$

Proof. (i) Based on a result due to Shisha and Mond [18], for any linear positive operator $L : C([0, 1]) \rightarrow B([0, 1])$ the following inequality involving the modulus of continuity, takes place

$$\begin{aligned} |(Lf)(x) - f(x)| &\leq |f(x)| |(Le_0)(x) - 1| \\ &+ \left((Le_0)(x) + \frac{1}{\delta} \sqrt{(L\varphi_n^2)(x) (Le_0)(x)} \right) \omega_1(f; \delta), \end{aligned} \quad (16)$$

$x \in [0, 1]$, $\delta > 0$, where $\varphi_x(t) = t - x$, $t \in [0, 1]$. In view of (9) we obtain

$$(B_n^{q,\alpha} \varphi_x^2)(x) = \frac{1 + \alpha[n]_q}{(1 + \alpha)[n]_q} x(1 - x).$$

Since $x(1 - x) \leq \frac{1}{4}$, $x \in [0, 1]$, choosing

$$\delta = \sqrt{\frac{1 + \alpha[n]_q}{(1 + \alpha)[n]_q}},$$

the conclusion follows.

(ii) Under the assumption $f' \in C([0, 1])$ one has [18]

$$\begin{aligned} |(Lf)(x) - f(x)| &\leq |f(x)| |(Le_0)(x) - 1| + |f'(x)| |(L\varphi_x)(x)| \\ &+ \sqrt{(L\varphi_x^2)(x)} \left(\sqrt{(Le_0)(x)} + \frac{1}{\delta} \sqrt{(L\varphi_x^2)(x)} \right) \omega_1(f'; \delta). \end{aligned}$$

Since $(B_n^{q,\alpha} \varphi_x)(x) = 0$ with the same choice of δ , the conclusion follows. $\square$

By definition the m -th iterate of $B_n^{q,\alpha}$ is

$${}^1B_n^{q,\alpha} := B_n^{q,\alpha}, \quad {}^mB_n^{q,\alpha} := B_n^{q,\alpha}({}^{(m-1)}B_n^{q,\alpha}), \quad m = 2, 3, \dots$$

Our next aim is to study the convergence of the iterates ${}^mB_n^{q,\alpha}$ as m tends to infinity. To achieve this, we recall a result obtained by using the contraction principle.

Theorem 4. ([4]) Let L_n , $n \in \mathbb{N}$, be defined as follows

$$L_n : C([a, b]) \rightarrow C([a, b]), \quad (L_n f)(x) = \sum_{k=0}^n \psi_{n,k}(x) f(x_{n,k}),$$

where $0 = x_{n,0} < x_{n,1} < \dots < x_{n,n} = b$ and for each $0 \leq k \leq n$, $\psi_{n,k} \in C([a, b])$. We assume that $L_n e_j = e_j$, $j \in \{0, 1\}$. Let us denote

$$u_n = \min_{x \in [a, b]} (\psi_{n,0}(x) + \psi_{n,n}(x)).$$

If $u_n > 0$, then

$$\lim_{m \rightarrow \infty} ({}^mL_n f)(x) = f(a) + \frac{f(b) - f(a)}{b - a} (x - a), \quad f \in C([a, b]),$$

uniformly on $[a, b]$.

Theorem 5. Let $B_n^{q,\alpha}$, $n \in \mathbb{N}$, be defined by (8). For any fixed $n \in \mathbb{N}$, one has

$$\lim_{m \rightarrow \infty} ({}^mB_n^{q,\alpha})(x) = f(0) + (f(1) - f(0))x, \quad f \in C([0, 1]),$$

uniformly on $[0, 1]$.

Proof. Choosing in Theorem 4.3 $a = 0$, $b = 1$, $x_{n,k} = [k]_q/[n]_q$, $\psi_{n,k} = p_{n,k}^{q,\alpha}$, $0 \leq k \leq n$, and knowing the identities (9), all is left to be proved is the relation

$$\min_{x \in [0, 1]} (p_{n,0}^{q,\alpha}(x) + p_{n,n}^{q,\alpha}(x)) > 0.$$

We get

$$\begin{aligned}
p_{n,0}^{q,\alpha}(x) + p_{n,n}^{q,\alpha}(x) &= \left(\prod_{i=0}^{n-1} (1 + \alpha[i]_q) \right)^{-1} \left(\prod_{s=0}^{n-1} (x + \alpha[s]_q) + \prod_{s=0}^{n-1} (1 - q^s x + \alpha[s]_q) \right) \\
&\geq \left(\prod_{i=0}^{n-1} (1 + \alpha[i]_q) \right)^{-1} (x^n + (1-x)^n) \\
&\geq \frac{1}{2^{n-1}} \left(\prod_{i=0}^{n-1} (1 + \alpha[i]_q) \right)^{-1} \\
&\geq \frac{1}{2^{n-1}} \left(1 + \frac{\alpha}{1-q} \right)^{-n},
\end{aligned}$$

and, consequently, Theorem 4.3 can be applied and the proof is complete. $\square$

5 q-Operators on unbounded intervals

We propose to present two families of operators that act on spaces of functions defined on $\mathbb{R}_+ = [0, \infty)$.

5.1 q-Baskakov operators

Aral and Gupta [7] defined a q -analogue of Baskakov operators, which for $q \in (0, 1)$, $f \in C(\mathbb{R}_+)$, $x \in \mathbb{R}_+$, $n \in \mathbb{N}$, is defined by

$$(V_{n,q}f)(x) = \sum_{k=0}^{\infty} v_{n,k}(q; x) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right), \quad (17)$$

where

$$v_{n,k}(q; x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}}.$$

Taking into account q -Taylor Theorem and the identity

$$(-x)_q^k = (-x)^k q^{\frac{k(k-1)}{2}},$$

we get

$$\sum_{k=0}^{\infty} v_{n,k}(q; x) = 1.$$

It is clear that the operators given by (17) are positive and linear operators. For $q = 1$, they reduce to the classical Baskakov operators.

In [10], the authors proved

$$\begin{cases} V_{n,q}(e_0; x) = 1, & V_{n,q}(e_1; x) = x, \\ V_{n,q}(e_2; x) = \frac{[n+1]_q}{q[n]_q} x^2 + \frac{1}{[n]_q} x. \end{cases} \quad (18)$$

Based on q -integration, Gupta and Radu [10] also introduced the Kantorovich variant of the above operators as follows

$$VK_{n,q}(f; x) = [n]_q \sum_{k=0}^{\infty} v_{n,k}(q; x) \int_{q \frac{[k]_q}{[n]_q}}^{q \frac{[k+1]_q}{[n]_q}} f(q^{-k+1}t) d_q t,$$

$x \in \mathbb{R}_+$, $n \in \mathbb{N}$.

From Definition 2.3, by a simple computation arising from (18), the following identities

$$\begin{aligned} VK_{n,q}(e_0; x) &= 1, \\ VK_{n,q}(e_1; x) &= x + \frac{q}{[2]_q[n]_q}, \\ VK_{n,q}(e_2; x) &= \frac{[n+1]_q}{q[n]_q} x^2 + \frac{q(1 + [2]_q) + [3]_q}{[3]_q[n]_q} x + \frac{q^2}{[3]_q[n]_q^2} \end{aligned}$$

were obtained. Examining the last identity, it is clear that the sequence of the operators $(VK_{n,q})_{n \geq 1}$ does not satisfy the conditions of Bohman-Korovkin theorem. The transformation of this sequence into an approximation process is achieved by replacing the parameter q with a certain sequence, as can be read in the following result.

Theorem 6. ([10]) *Let $(q_n)_{n \in \mathbb{N}}$ be a sequence satisfying $0 < q_n < 1$, $\lim_{n \rightarrow \infty} q_n = 1$. Then, for any compact $J \subset \mathbb{R}_+$ and for each $f \in C(\mathbb{R}_+)$, we have*

$$\lim_{n \rightarrow \infty} VK_{n,q_n}(f; x) = f(x),$$

uniformly in $x \in J$.

5.2 q -Baskakov–Mastroianni operators

Let $(\phi_n)_{n \geq 1}$ be a sequence of real valued functions defined on $\mathbb{R}_+$, continuously infinitely q -differentiable on $\mathbb{R}_+$, $q \in (0, 1)$, and satisfying the following conditions:

$$\begin{aligned} (P_1) \quad & \phi_n(0) = 1, \quad n \in \mathbb{N}, \\ (P_2) \quad & (-1)^k D_q^k \phi_n(x) \geq 0, \quad n \in \mathbb{N}, k \in \mathbb{N}_0, x \geq 0. \end{aligned}$$

For all $(n, k) \in \mathbb{N} \times \mathbb{N}_0$ there exists a positive integer i_k , $0 \leq i_k \leq k$, such that

$$(P_3) \quad D_q^{k+1} \phi_n(x) = (-1)^{i_k+1} D_q^{k-i_k} \phi_n(q^{i_k+1}x) \beta_{n,k,i_k,q}(x),$$

where

$$\lim_{n \rightarrow \infty} \frac{\beta_{n,k,i_k,q}(0)}{[n]_q^{i_k+1} q^{k-i_k}} = 1.$$

Inspired by the general class introduced in [13], Radu [17] defined the following operators

$$T_{n,q}(f; x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{\frac{k(k-1)}{2}} D_q^k \phi_n(x) f\left(\frac{[k]_q}{[n]_q q^{k-1}}\right), \quad x \geq 0, \quad (19)$$

$$f \in \mathcal{F}(\mathbb{R}_+) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R}, \text{ the series in (19) is convergent}\},$$

see also [3, Eq. (16)].

For each $n \in \mathbb{N}$, $T_{n,q}$ are linear positive operators satisfying the interpolating property $T_{n,q}(f; 0) = f(0)$. With the help of q -Stirling numbers presented at Section 3, the next result indicates the moments of the operators $T_{n,q}$, see [3, Lemma 4].

Theorem 7. *Let $T_{n,q}$, $n \in \mathbb{N}$, be defined by (19). One has*

$$T_{n,q}(e_m; x) = \sum_{r=0}^m \frac{(-x)^r}{[n]_q^m} q^m D_q^r \phi_n(0) \sigma_q(m, r), \quad x \geq 0. \quad (20)$$

Proof. Let $f \in \mathcal{F}(\mathbb{R}_+)$ be arbitrarily fixed. By using $\Delta_q^0 f_{k,s} = f_{k,s}$, $f_{k,s} = f\left(\frac{[k]_q}{q^s [n]_q}\right)$, $k \in \mathbb{N}_0$, $s \in \mathbb{Z}$, the operator $T_{n,q}$ can be expressed as follows

$$T_{n,q}(f; x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{\frac{k(k-1)}{2}} D_q^k \phi_n(x) \Delta_q^0 f_{k,k-1}.$$

On the basis of the product rule in q -Calculus, the q -derivative of $T_{n,q}f$ is given by

$$\begin{aligned}
D_q(T_{n,q}(f; x)) &= - \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{\frac{k(k+1)}{2}} D_q^{k+1} \phi_n(x) \Delta_q^0 f_{k+1,k} \\
&\quad + \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^k q^{\frac{k(k-1)}{2}} D_q^{k+1} \phi_n(x) \Delta_q^0 f_{k,k-1} \\
&= - \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{\frac{k(k-1)}{2}} q^k D_q^{k+1} \phi_n(x) (\Delta_q^0 f_{k+1,k} - \Delta_q^0 f_{k,k-1}) \\
&= - \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{\frac{k(k-1)}{2}} q^k D_q^{k+1} \phi_n(x) \Delta_q^1 f_{k,k}.
\end{aligned}$$

For $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$, by induction with respect to $r \in \mathbb{N}$ we can prove

$$D_q^r T_{n,q}(f; x) = (-1)^r \sum_{k=0}^{\infty} \frac{(-x)^k}{[k]_q!} q^{\frac{k(k-1)}{2}} q^{rk} D_q^{k+r} \phi_n(x) \Delta_q^r f_{k,k+r-1}.$$

Choosing $x = 0$, we deduce

$$D_q^r T_{n,q}(f; 0) = (-1)^r D_q^r \phi_n(0) \sigma_q(m, r).$$

Consequently, taking into account the identity [3, Eq. (11)], we obtain

$$\Delta_q^r f_{0,r-1} = \frac{q^m [r]_q!}{[n]_q^m} \sigma_q(m, r), \quad r \in \mathbb{N}_0, \quad f := e_m,$$

involving

$$D_q^r T_{n,q}(e_m; 0) = (-1)^r q^m \frac{[r]_q!}{[n]_q^m} D_q^r \phi_n(0) \sigma_q(m, r).$$

Choosing $a = 0$ in the q -Taylor theorem (see [9, p. 103]), we obtain

$$\begin{aligned}
T_{n,q}(e_m; x) &= \sum_{r=0}^{\infty} \frac{x^r}{[r]_q!} D_q^r T_{n,q}(e_m; 0) \\
&= \sum_{r=0}^{\infty} \frac{(-x)^r}{[n]_q^m} q^m D_q^r \phi_n(0) \sigma_q(m, r).
\end{aligned}$$

Taking into account (6), the proof is complete. $\square$

Formula (20) gives the explicit form of the first three moments of $T_{n,q}$, $n \in \mathbb{N}$, operators.

$$\begin{aligned} T_{n,q}(e_0; x) &= 1, \\ T_{n,q}(e_1; x) &= -x \frac{D_q \phi_n(0)}{[n]_q}, \\ T_{n,q}(e_2; x) &= x^2 \frac{D_q^2 \phi_n(0)}{q [n]_q^2} - x \frac{D_q \phi_n(0)}{[n]_q^2}. \end{aligned}$$

Consequently, the second central moment of $T_{n,q}$, $n \in \mathbb{N}$, is given as follows

$$T_{n,q}(\varphi_x^2; x) = a_{n,q} x^2 + b_{n,q} x, \quad x \geq 0,$$

where where

$$a_{n,q} = 1 + 2 \frac{D_q \phi_n(0)}{[n]_q} + \frac{D_q^2 \phi_n(0)}{q [n]_q^2}, \quad b_{n,q} = -\frac{D_q \phi_n(0)}{[n]_q^2}. \quad (21)$$

Clearly, $\lim_{n \rightarrow \infty} T_{n,q}(e_2; x) \neq x^2$. To transform these operators into an approximation process, the constant q will be replaced by a $q_n \in (0, 1)$, such that $\lim_{n \rightarrow \infty} q_n = 1$.

We are able to present both the convergence of the operators and their rate of convergence. The results appear in [3], see Theorems 1 and 2.

Theorem 8. *Let $(q_n)_{n \geq 1}$, $0 < q_n < 1$, be a sequence and let T_{n,q_n} , $n \in \mathbb{N}$, be defined as in (19). If $\lim_{n \rightarrow \infty} q_n = 1$, for any compact $K \subset \mathbb{R}_+$ and for each $f \in \mathcal{F}(\mathbb{R}_+) \cap C(\mathbb{R}_+)$ one has*

$$\lim_{n \rightarrow \infty} T_{n,q_n}(f; x) = f(x) \quad \text{uniformly in } x \in K.$$

Moreover, for every $f \in C_B(\mathbb{R}_+)$ one has

$$|T_{n,q_n}(f; x) - f(x)| \leq \left(1 + \sqrt{\max\{x, x^2\}}\right) \omega_1(f; \sqrt{c_{n,q_n}}), \quad x \geq 0,$$

where

$$c_{n,q_n} = |a_{n,q_n}| + b_{n,q_n},$$

these sequences being defined by (21).

Proof. The first statement is based on the Bohman–Korovkin theorem and the expression of the first three calculated moments.

For determine the convergence speed, we used relation (16) written to an arbitrary interval $[0, x]$ and the inequalities

$$0 \leq T_{n,q_n}(\varphi_x^2; x) \leq \max\{x, x^2\} c_{n,q_n}.$$

□

References

- [1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series 55, Issued June, 1964.
- [2] O. Agratini, On a q -analogue of Stancu operators, *Cent. Eur. J. Math.* 8 (2010), 191-198.
- [3] O. Agratini and C. Radu, On q -Baskakov–Mastroianni operators, *Rocky Mt. J. Math.* 42 (2012), 773-790.
- [4] O. Agratini and I.A. Rus, Iterates of a class of discrete operators via contraction principle, *Comment. Math. Univ. Carolinae* 44 (2003), 555-563.
- [5] A. Aral, A generalization of Szász–Mirakyan operators based on q -integers, *Math. Comput. Model.* 47 (2008), 1052-1062.
- [6] A. Aral and O. Dođru, Bleimann, Butzer and Hahn operators based on q -integers, *J. Inequal. Appl.* 2007 (2007), 79410.
- [7] A. Aral and V. Gupta, On the Durrmeyer type modification of the q -Baskakov type operators, *Nonlinear Anal.* 72 (2010), 1171-1180.
- [8] O. Dođru, On statistical approximation properties of Stancu type bivariate generalization of q -Balázs–Szabados operators, In: O. Agratini, P. Blaga (Eds.), *Proc. Int. Conference on Numerical Analysis and Approximation Theory*, Cluj-Napoca, Romania, July 5–8, 2006, 179–194, Casa Cartii de Stiinta, Cluj-Napoca, 2006.
- [9] T. Ernst, The history of q -Calculus and a new method, *U.U.D.M. Report* 2000, Vol. 16, Uppsala, Department of Mathematics, Uppsala University, 2000.
- [10] V. Gupta and C. Radu, Statistical approximation properties of q -Baskakov–Kantorovich operators, *Cent. Eur. J. Math.* 7 (2009), 809-818.
- [11] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer-Verlag, New York, 2002.
- [12] A. Lupaş, A q -analogue of the Bernstein operators, University of Cluj-Napoca, *Seminar on Numerical and Statistical Calculus*, Preprint, 1987, 85-92.

- [13] G. Mastroianni, Sur un operatore lineare e positivo, *Rend. Acc. Sc. Fis. Mat.* 46 (1979), 161-176.
- [14] G. Nowak, Approximation properties for generalized q -Bernstein polynomials, *J. Math. Anal. Appl.* 350 (2009), 50-55.
- [15] G.M. Phillips, *Interpolation and Approximation by Polynomials*, Springer-Verlag, 2003.
- [16] G.M. Phillips, Bernstein polynomials based on the q -integers, *Ann. Numer. Math.*, 4 (1997), 511-518.
- [17] C. Radu, On statistical approximation of a general class of positive linear operators extended in q -calculus, *Appl. Math. Comput.* 215 (2009), 2317-2325.
- [18] O. Shisha and B. Mond, The degree of convergence of sequences of linear positive operators, *Proc. Nat. Acad. Sci. USA* 60 (1968), 1196-1200.
- [19] T. Trif, Meyer-König and Zeller operators based on the q -integers, *Rev. Anal. Numer. Theory Approx.* 29 (2000), 221-229.

