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DIFFERENCE SEQUENCES OF BI-COMPLEX UNCERTAIN SEQUENCES*

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Abstract

In this paper, we apply the concept of difference operators to sequence spaces in the framework of uncertainty theory. Using these concepts, we introduce new classes of convergent difference sequences spaces involving bi-complex uncertain variables, and we examine their algebraic and topological properties, including linearity and normed space structures. Furthermore, we establish inclusion relations among these spaces, particularly strict inclusion $\Delta^{n-1} \subset \Delta^n$, and examine important structural characteristics such as solidity, monotonicity, convergence-free, sequence algebra and symmetry, showing that these properties generally do not hold. We also study convergence relations and implications associated with the newly defined sequence spaces. Several results and examples are provided to support the developed theory.

Keywords: bi-complex uncertain variable, convergence, difference sequence space, solid space, symmetric space.

MSC: 40A05, 40A30, 40C15, 40F05, 60B10, 60D05, 60E15, 60F15.

1 Introduction

In functional analysis, especially in the study of sequence spaces, the concept of sequence convergence plays an important and significant role. This

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idea has been discussed in earlier works by Rath and Tripathy [13]. Uncertainty theory was introduced by Liu [10] in 2007 using the idea of an uncertain measure, which satisfies properties such as normality, duality, subadditivity, and product axioms. Later, Liu [11] presented a revised version of this theory in 2016. You [21] developed the notion of uniform almost sure convergence and examined its connections with various other modes of convergence. Zhang [22] studied several important results concerning the convergence behavior of uncertain sequences were established. Furthermore, Guo and Xu [4] investigated the conditions for mean square convergence of uncertain sequences.

The idea of bi-complex numbers was first proposed by Segre and later explored by several researchers such as Wagh [20], Srivastava and Srivastava [18], Sager and Sager [15], Rochon and Shapiro [14], Beg et al. [1] and Singh [17]. However, the most detailed and foundational work in this area was done by Price [12] and later, in 1991, he published a comprehensive book covering bi-complex numbers, multi-complex spaces, and their function theory. Recent studies have explored special bi-complex sequences and related number systems. Kizilates et al. [5] studied bi-complex generalized Tribonacci quaternions. Further developments on hybrid and hypercomplex sequences were reported by Kizilates [6, 7] and Kizilates and Kone [8]. This study motivates further investigation of sequence spaces involving bi-complex structures.

The concept of difference sequences was first introduced by Kizmaz. [9] After that, Esi et al. [2] developed generalized difference sequence spaces as follows:
Let $m, n \geq 0$ be fixed integers,

$$Z(\Delta_m^n) = \{y = (y_k) \in \omega : \Delta_m^n y = (\Delta_m^n y_k) \in Z\},$$

for $Z = \ell_\infty, c$ and c_0; where

$$\Delta_m^n y_k = (\Delta_m^{n-1} y_k - \Delta_m^{n-1} y_{k+m})$$

and $\Delta_m^0 y_k = y_k$ for all $n \in \mathbb{N}$. This notion of generalized difference gave the binomial representation as follows:

$$\Delta_m^n y_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} y_{k+mv}$$

for all $k \in \mathbb{N}$. For $m = 1$ and $n = 1$, we have the spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz. [9]. For $m = 1$, we have the spaces

$\ell_\infty(\Delta^n), c(\Delta^n)$ and $c_0(\Delta^n)$ introduced and studied by Et and Colak [3]. For $n = 1$, we have the spaces $\ell_\infty(\Delta_m), c(\Delta_m)$ and $c_0(\Delta_m)$ introduced and studied by Tripathy and Esi [19].
The sequence spaces $Z(\Delta_m^n)$ for $Z = \ell_\infty, c$ and c_0 are Banach spaces, by the norm

$$\|y\|_{\Delta_m^n} = \sum_{i=1}^{p} |y_i| + \sup_k |\Delta_m^n y_k|,$$

where $p = mn$ for $m \geq 1, n \geq 1$.

2 Definitions and preliminaries

We review basic definitions and prior results before the main contribution. These are essential for the study.

Definition 1. *[10] Suppose, $\mathcal{A}$ is a σ-algebra on a nonempty set $\mathcal{S}$. A set-valued function $\mathcal{N}$ is called an uncertain measure if it satisfies the following four axioms:*

1. Normality Axiom: $\mathcal{N}(\mathcal{S}) = 1$,
2. Duality axiom: For each event $\mathcal{H} \in \mathcal{A}, \mathcal{N}(\mathcal{H}) + \mathcal{N}(\mathcal{H}^c) = 1$,
3. Subadditivity Axiom: If $\{d_m\}$ is a countable collection of events in $\mathcal{A}$, then

$$\mathcal{N}\Big(\bigcup_{m=1}^{\infty} d_m\Big) \leq \sum_{m=1}^{\infty} \mathcal{N}(d_m).$$

4. Product Axiom: Suppose $(\mathcal{S}_i, \mathcal{A}_i, \mathcal{N}_i)$ is the uncertainty space for $i =$ 1,2,3,... . So the product uncertain measure $\mathcal{N}$ fulfill

$$\mathcal{N}\Big(\prod_{i=1}^{\infty} d_i\Big) = \bigwedge_{i=1}^{\infty} \mathcal{N}_i(d_i)$$

where each $d_i \in \mathcal{A}_i$.

A triplet $(\mathcal{S}, \mathcal{A}, \mathcal{N})$ is known as an uncertainty space, and every $\mathcal{H}$ in $\mathcal{A}$ is known as an event.

Definition 2. *[10] An uncertain variable is a measurable function $\vartheta : (\mathcal{S}, \mathcal{A}, \mathcal{N}) \to \mathbb{R}$ such that for any Borel set $B \subset \mathbb{R}$, the set*

$$\{\vartheta \in B\} = \{\omega \in \mathcal{S} : \vartheta(\omega) \in B\}$$

is an event in $\mathcal{A}$.

Definition 3. *A sequence space X is said to be solid (or normal) if $(\alpha_n u_n) \in X$, whenever $(u_n) \in X$, for all sequences (α_n) of scalars such that $|\alpha_n| \leq 1$ for all $n \in \mathbb{N}$.*

Definition 4. *A sequence space X is said to be monotone if X contains the canonical pre-images of all its step spaces.*

Definition 5. *A sequence space X is said to be convergence free if $(v_n) \in X$ whenever $(u_n) \in X$ and $v_n = 0$ whenever $u_n = 0$.*

Definition 6. *A sequence space X is said to be a sequence algebra if $(u_n v_n) \in X$ whenever $(u_n) \in X$ and $(v_n) \in X$.*

Definition 7. *A sequence space X is said to be symmetric if*

$$(u_k) \in X \to (u_\pi(k)) \in X,$$

where π is a permutation of $\mathbb{N}$.

3 Bi-complex difference sequence

Let $(\vartheta_r)_{\mathbb{BC}}$ be a sequence of bi-complex uncertain variables in $(\mathcal{S}, \mathcal{A}, \mathcal{N})$ where $\vartheta_r = \xi_r + i_1\eta_r + i_2\mu_r + i_1 i_2 \varphi_r$.

$$(U, \Delta_m^n) = \{\vartheta = (\vartheta_r(\omega))_{\mathbb{BC}} : \text{ there exists } \vartheta(\omega) \text{ such that}$$

$$\frac{1}{p}\sum_{r=1}^{p} \|\Delta_m^n \vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} \to 0, \text{ as } p \to \infty\}.$$

Using the difference operator, we have defined some new classes of sequences of bi-complex uncertain variables as follows:

$$c_0(U, \Delta_m^n)_{\mathbb{BC}} = \big\{\vartheta : \lim_{r\to\infty} \|\Delta_m^n \vartheta_r(\omega)\|_{\mathbb{BC}} = 0\big\}.$$

$$c(U, \Delta_m^n)_{\mathbb{BC}} = \big\{\vartheta : \lim_{r\to\infty} \|\Delta_m^n \vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} = 0$$

$$\text{for some bi-complex uncertain variable } \vartheta(\omega)\big\}.$$

$$\ell_\infty(U, \Delta_m^n)_{\mathbb{BC}} = \big\{\vartheta : \sup_r \|\Delta_m^n \vartheta_r(\omega)\|_{\mathbb{BC}} < \infty\big\}.$$

$$(U, \Delta_m^n)_{\mathbb{BC}} = \big\{\vartheta : \lim_{p\to\infty} \frac{1}{p}\sum_{j=1}^{p} \|\Delta_m^n \vartheta_{k+j}(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} = 0, \text{ uniformly in } k\big\}.$$

Theorem 1. *The sets* $c_0(U, \Delta_m^n)_{\mathbb{BC}},\ c(U, \Delta_m^n)_{\mathbb{BC}}$ *and* $\ell_\infty(U, \Delta_m^n)_{\mathbb{BC}}$ *are linear spaces.*

Proof. We examine the result for the class of bi-complex uncertain sequences $c_0(U, \Delta_m^n)_{\mathbb{BC}}$. The other cases can be treated in a similar way.

Let

$$(\vartheta_r), (\vartheta_r') \in c_0(U, \Delta_m^n)_{\mathbb{BC}}.$$

Then,

$$\lim_{r\to\infty} \|\Delta_m^n \vartheta_r(\omega)\|_{\mathbb{BC}} = 0$$

$$\lim_{r\to\infty} \|\Delta_m^n \vartheta_r{}'(\omega)\|_{\mathbb{BC}} = 0$$

Now, for $\alpha,\ \beta \in \mathbb{BC}$,

$$\lim_{r\to\infty} \|\Delta_m^n(\alpha\vartheta_r(\omega) + \beta\vartheta_r^{'}(\omega))\|_{\mathbb{BC}} = \lim_{r\to\infty} \|\alpha\Delta_m^n\vartheta_r(\omega) + \beta\Delta_m^n\vartheta_r^{'}(\omega)\|_{\mathbb{BC}}$$

$$\leq |\alpha| \lim_{r\to\infty} \|\Delta_m^n\vartheta_r(\omega)\|_{\mathbb{BC}} + |\beta| \lim_{r\to\infty} \|\Delta_m^n\vartheta_r^{'}(\omega)\|_{\mathbb{BC}},\ \text{as } r \to \infty.$$

Hence,

$$(\alpha\vartheta_r(\omega) + \beta\vartheta_r^{'}(\omega)) \in c_0(U, \Delta_m^n)_{\mathbb{BC}}.$$

Consequently, $c_0(U, \Delta_m^n)_{\mathbb{BC}}$ forms a linear space. The same argument proves that $c(U, \Delta_m^n)_{\mathbb{BC}}$ and $\ell_\infty(U, \Delta_m^n)_{\mathbb{BC}}$ are also linear spaces. □

Theorem 2. *The sets* $c_0(U, \Delta_m^n)_{\mathbb{BC}},\ c(U, \Delta_m^n)_{\mathbb{BC}}$ *and* $\ell_\infty(U, \Delta_m^n)_{\mathbb{BC}}$ *are normed linear spaces, normed by*

$$\|\vartheta_r(\omega)\|_{(\Delta_m^n)_{\mathbb{BC}}} = \sum_{q=1}^{j} \|\vartheta_q(\omega)\| + \sup_r \|\Delta_m^n\vartheta_r(\omega)\|$$

where $j = mn$ *for* $m \geq 1, n \geq 1$.

Proof. Let

$$\vartheta_r \in \ell_\infty(U, \Delta_m^n)_{\mathbb{BC}}$$

and suppose that

$$\|\vartheta(\omega)\|_{(\Delta_m^n)_{\mathbb{BC}}} = 0.$$

Then

$$\sum_{q=1}^{j} \|\vartheta_q(\omega)\| + \sup_r \|\Delta_m^n\vartheta_r(\omega)\| = 0.$$

Since every term is non-negative, it follows that

$$\vartheta_q(\omega) = 0, \text{ for } q = 1, 2, ..., mn \text{ and } \|\Delta_m^n \vartheta_r(\omega)\| = 0.$$

For

$$r = 1, \|\Delta_m^n \vartheta_1(\omega)\| = 0$$

which implies

$$\|\Delta_m^{n-1} \vartheta_1(\omega)\| - \|\Delta_m^{n-1} \vartheta_{m+1}(\omega)\| = 0.$$

Since,

$$\vartheta_q(\omega) = 0, \text{ for } q = 1, 2, ..., mn,$$

we obtain $\vartheta_{m+1} = 0$. Proceeding inductively in the usual Kizmaz-type manner, we conclude that $\vartheta_r(\omega) = 0, r \in \mathbb{N}$.

Hence,

$$\|\vartheta(\omega)\|_{(\Delta_m^n)_{\mathbb{BC}}} = 0 \implies \vartheta(\omega) = 0.$$

Now,

$$\|\vartheta(\omega) + \vartheta'(\omega)\|_{(\Delta_m^n)_{\mathbb{BC}}}$$

$$= \sum_{q=1}^{j} \|\vartheta_q(\omega) + \vartheta'_q(\omega)\| + \sup_r \|\Delta_m^n(\vartheta_r(\omega) + \vartheta'_r(\omega))\|$$

$$\leq \sum_{q=1}^{j} \|\vartheta_q(\omega)\| + \sup_r \|\Delta_m^n \vartheta_r(\omega)\| + \sum_{q=1}^{j} \|\vartheta'_q(\omega)\| + \sup_r \|\Delta_m^n \vartheta'_r(\omega)\|$$

$$= \|\vartheta(\omega)\|_{(\Delta_m^n)_{\mathbb{BC}}} + \|\vartheta'(\omega)\|_{(\Delta_m^n)_{\mathbb{BC}}}$$

Finally, we have for any $\lambda^\circ \in \mathbb{BC}$,

$$\|\lambda^\circ \vartheta(\omega)\|_{(\Delta_m^n)_{\mathbb{BC}}} = \sum_{q=1}^{j} \|\lambda^\circ \vartheta_q(\omega)\| + \sup_r \|\Delta_m^n(\lambda^\circ \vartheta_r(\omega))\|$$

$$= |\lambda^\circ| \left(\sum_{q=1}^{j} \|\vartheta_q(\omega)\| + \sup_r \|\Delta_m^n \vartheta_r(\omega)\| \right)$$

$$= |\lambda^\circ| \|\vartheta(\omega)\|_{(\Delta_m^n)_{\mathbb{BC}}}.$$

Hence, $\|.\|_{(\Delta_m^n)_{\mathbb{BC}}}$ is a norm on sets $c_0(U, \Delta_m^n)_{\mathbb{BC}}$, $c(U, \Delta_m^n)_{\mathbb{BC}}$, $\ell_\infty(U, \Delta_m^n)_{\mathbb{BC}}$. This establishes the theorem. □

Theorem 3. *Let $m \geq 1$ and $n \geq 1$. Then the bi-complex sequence spaces satisfy $Z(U, \Delta_m^p)_{\mathbb{BC}} \subset Z(U, \Delta_m^n)_{\mathbb{BC}}$, for $Z = c, c_0, \ell_\infty$. More generally, for $p = 0, 1, ..., n-1$, we have $Z(U, \Delta_m^p)_{\mathbb{BC}} \subset Z(U, \Delta_m^n)_{\mathbb{BC}}$ for $Z = c, c_0, \ell_\infty$.*

Proof. Let

$$\vartheta_r \in Z(U, \Delta_m^{n-1})_{\mathbb{BC}},$$

where $Z = c, c_0, \ell_\infty$. We show that $\vartheta_r \in Z(U, \Delta_m^n)_{\mathbb{BC}}$.

For $Z = c$, let $\vartheta_r \in c(U, \Delta_m^{n-1})_{\mathbb{BC}}$ we have,

$$\lim_{r\to\infty} \|\Delta_m^{n-1}\vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} = 0.$$

Now, using the relation

$$\Delta_m^n \vartheta_r(\omega) = \Delta_m^{n-1}\vartheta_r(\omega) - \Delta_m^{n-1}\vartheta_{r+m}(\omega),$$

we have

$$\lim_{r\to\infty} \|\Delta_m^n \vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} = \lim_{r\to\infty} \|\Delta_m^{n-1}\vartheta_r(\omega) - \Delta_m^{n-1}\vartheta_{r+m}(\omega) - \vartheta(\omega)\|_{\mathbb{BC}}$$

$$\leq \left(\lim_{r\to\infty} \|\Delta_m^{n-1}\vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} + \lim_{r\to\infty} \|\Delta_m^{n-1}\vartheta_{r+m}(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} \right)$$

Both limits are zero. Hence,

$$\lim_{r\to\infty} \|\Delta_m^n \vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} = 0,$$

which implies $(\vartheta_r) \in c(U, \Delta_m^n)_{\mathbb{BC}}$. Similarly we can apply the same argument for $Z = c_0, \ell_\infty$, giving $Z(U, \Delta_m^p)_{\mathbb{BC}} \subset Z(U, \Delta_m^n)_{\mathbb{BC}}$. Using principle of induction, we obtain $Z(U, \Delta_m^p)_{\mathbb{BC}} \subset Z(U, \Delta_m^n)_{\mathbb{BC}}$ for $p = 0, 1, ..., n-1$. □

Result 1. The bi-complex sequence spaces $Z(U, \Delta_m^n)_{\mathbb{BC}}$, for $Z = c, c_0, \ell_\infty$ are neither solid and nor monotone.

The above result follows from the following examples.

Example 1. *Let the bi-complex uncertainty space $(\mathcal{S}, \mathcal{A}, \mathcal{N})$ be $\{\omega_1, \omega_2, \omega_3, \omega_4, ...\}$ equipped with the power set and let $\mathcal{H} \in \mathcal{A}$ be an arbitrary event such that*

$$\mathcal{N}(\mathcal{H}) = \begin{cases} \sup\limits_{\omega_r \in \mathcal{H}} \{\frac{1}{r^2+1}\}, & if \quad \sup\limits_{\omega_r \in \mathcal{H}} \{\frac{1}{r^2+1}\} < \frac{1}{2}, \\ 1 - \sup\limits_{\omega_r \in \mathcal{H}^c} \{\frac{1}{r^2+1}\}, & if \quad \sup\limits_{\omega_r \in \mathcal{H}^c} \{\frac{1}{r^2+1}\} < \frac{1}{2}, \\ \frac{1}{2}, & otherwise. \end{cases}$$

Consider the bi-complex uncertain variables defined by

$$(\vartheta_r(\omega))_{\mathbb{BC}} = \begin{cases} i_1 + i_2 + (r^2+1)i_1 i_2, & if\ \omega = \omega_r, \\ 0, & otherwise, \end{cases}$$

for $r = 1, 2, ...$ *and* $\vartheta \equiv \theta$, *where* θ *is a zero element. Then* $\vartheta_r \in c(U, \Delta_m^n)_{\mathbb{BC}} \subset \ell_\infty(U, \Delta_m^n)_{\mathbb{BC}}$, *but* $\Big(\vartheta_r\, \mathcal{N}_r(\mathcal{H})\Big) \notin \ell_\infty(U, \Delta_m^n)_{\mathbb{BC}}$. *Hence, the spaces* $Z(U, \Delta_m^n)_{\mathbb{BC}}$, *for* $Z = c\ ,\ \ell_\infty$ *are not solid.*

Example 2. *Now, for proving the case for not monotone we take the class of sequence* $c_0(U, \Delta_m^n)_{\mathbb{BC}}$, *we consider the sequence with*

$$\mathcal{N}(\mathcal{H}) = \sum_{\omega_r \in \omega} \frac{1}{4^r}.$$

and consider the bi-complex uncertain variables defined as

$$(\vartheta_r(\omega))_{\mathbb{BC}} = \begin{cases} (i_1 + i_2)4^r, & if\ \omega = \omega_r, \\ 0, & otherwise, \end{cases}$$

Then $\vartheta_r \in c_0(U, \Delta_m^n)_{\mathbb{BC}}$, *but* $(\vartheta_r\, \mathcal{N}_r(\mathcal{H})) \notin c_0(U, \Delta_m^n)_{\mathbb{BC}}$. *Hence* $c_0(U, \Delta_m^n)_{\mathbb{BC}}$ *is not monotone. Consider the sequence* ϑ_r' *in its pre-image space defined by*

$$(\vartheta_r'(\omega))_{\mathbb{BC}} = \begin{cases} \vartheta_r, & if\ \omega = \omega_r, \\ 0, & otherwise, \end{cases}$$

Then $\vartheta_r' \notin c_0(U, \Delta_m^n)_{\mathbb{BC}}$. *Hence, the space is not monotone.*

Result 2. The difference bi-complex uncertain sequence space $Z(U, \Delta_m^n)_{\mathbb{BC}}$, for $Z = c, c_0, \ell_\infty$ are neither convergence free nor sequence algebra.

The above result follows from the following examples.

Example 3. *Consider the two uncertain bi-complex sequences* (ϑ_r) *and* (ϑ_r') *defined by*

$$(\vartheta_r(\omega))_{\mathbb{BC}} = \begin{cases} \frac{1}{\sqrt{r}}(1 + i_1 + i_2 + i_1 i_2), & if\ \omega = \omega_1, \\ 0, & otherwise, \end{cases}$$

Hence, $(\vartheta_r) \in c(U, \Delta_m^n)_{\mathbb{BC}}$. *Now, define another sequence* (ϑ_r') *by*

$$(\vartheta_r'(\omega))_{\mathbb{BC}} = \begin{cases} \sqrt{r}(1 + i_1 + i_2 + i_1 i_2), & if\ \omega = \omega_1, \\ 0, & otherwise, \end{cases}$$

Thus, $(\vartheta') \notin c(U, \Delta_m^n)_{\mathbb{BC}}$. *This shows that* $c(U, \Delta_m^n)_{\mathbb{BC}}$ *is not convergence free. Similarly the other cases can be proved.*

Example 4. *Next for sequence algebra, consider the above two sequences. Their product is*

$$(\vartheta_r(\omega))_{\mathbb{BC}}(\vartheta'_r(\omega))_{\mathbb{BC}} = \begin{cases} (1+i_1+i_2+i_1i_2)^2, & if\ \omega=\omega_1, \\ 0, & otherwise. \end{cases}$$

Since $(\vartheta'_r) \notin Z(U,\Delta_m^n)_{\mathbb{BC}}$*, their product* $(\vartheta_r\vartheta'_r) \notin Z(U,\Delta_m^n)_{\mathbb{BC}}$*. Hence,* $Z(U,\Delta_m^n)_{\mathbb{BC}}$ *for* $Z = c, c_0, \ell_\infty$ *is not a sequence algebra.*

Result 3. The bi-complex sequence spaces $Z(U,\Delta_m^n)_{\mathbb{BC}}$, for $Z = c, c_0, \ell_\infty$ are not symmetric.

The above result follows from the following example.

Example 5. *We prove the result for* $Z(U,\Delta_m^n)_{\mathbb{BC}}$ *for* $Z = c_0$*, the other spaces will be proved accordingly. Consider a bi-complex uncertainty space* $(\mathcal{S},\mathcal{A},\mathcal{N})$ *as* $\omega_1,\omega_2,\omega_3,\omega_4,...$ *with the power set and for any event* $\mathcal{H}\in\mathcal{A}$ *such that*

$$\mathcal{N}(\mathcal{H}) = \begin{cases} \sup_{\omega_r\in\mathcal{H}}\{\frac{r}{2r+1}\}, & if\ \sup_{\omega_r\in\mathcal{H}}\{\frac{r}{2r+1}\} < \frac{1}{2}, \\ 1-\sup_{\omega_r\in\mathcal{H}^c}\{\frac{r}{2r+1}\}, & if\ \sup_{\omega_r\in\mathcal{H}^c}\{\frac{r}{2r+1}\} < \frac{1}{2}, \\ \frac{1}{2}, & otherwise. \end{cases}$$

Define the bi-complex uncertain variables by

$$(\vartheta_r(\omega_t))_{\mathbb{BC}} = \begin{cases} r, & if\ t=r, \\ 0, & otherwise. \end{cases}$$

Then $(\vartheta_r)_{\mathbb{BC}} \in c_0(U,\Delta_m^n)_{\mathbb{BC}}$*. Now consider the rearrangement of the sequence* $(\vartheta_r)_{\mathbb{BC}}$ *by a permutation* π *of* $\mathbb{N}$*, defined by*

$$(\vartheta_{\pi(r)})_{\mathbb{BC}} = \{\vartheta_1,\vartheta_4,\vartheta_9,\vartheta_2...\} \notin c_0(U,\Delta_m^n)_{\mathbb{BC}}.$$

Therefore, $c_0(U,\Delta_m^n)_{\mathbb{BC}}$ *is not symmetric.*

4 Convergence of bi-complex uncertain difference sequence

This section discusses the convergence concepts within the framework of bi-complex uncertain difference sequences.

Definition 8. *The bi-complex uncertain sequence $(\vartheta_r)_{\mathbb{BC}}$ is said to be convergent almost surely (a.s.) to ϑ with respect to difference sequence if there exists an event $\mathcal{H}$ with $\mathcal{N}(\mathcal{H}) = 1$ such that*

$$\lim_{r\to\infty}\{\|\Delta_m^n\vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}}\} = 0$$

for every $\omega \in \mathcal{H}$.

Definition 9. *The bi-complex uncertain sequence $(\vartheta_r)_{\mathbb{BC}}$ is said to be convergent in measure to ϑ with respect to difference sequence if*

$$\lim_{r\to\infty}\mathcal{N}\{\omega \in S : \|\Delta_m^n\vartheta_r - \vartheta\|_{\mathbb{BC}} \geq \varepsilon\} = 0$$

Definition 10. *Let $\vartheta, \vartheta_1, \vartheta_2, ...$ be bi-complex uncertain variables with finite expected values. Then the bi-complex uncertain sequence $(\vartheta_r)_{\mathbb{BC}}$ is said to converge in mean to ϑ if*

$$\lim_{r\to\infty} E[\omega \in \mathcal{S} : \|\Delta_m^n\vartheta_r - \vartheta\|_{\mathbb{BC}}] = 0.$$

Definition 11. *Let $D, D_1, D_2...$ be the Bi-complex uncertainty distributions of Bi-complex uncertain variables $\vartheta, \vartheta_1, \vartheta_2, ...$, respectively. We say that the bi-complex uncertain sequence $(\vartheta_r)_{\mathbb{BC}}$ converges in distribution to ϑ with respect to the difference sequence if*

$$\lim_{r\to\infty}\|\Delta_m^n D_r(\mathbb{BC}) - D(\mathbb{BC})\| = 0$$

for all real numbers $\alpha, \beta, \gamma, \theta$ at which $D(\mathbb{BC})$ is continuous.

Definition 12. *The sequence $(\vartheta_r)_{\mathbb{BC}}$ is said to converge uniformly almost surely to ϑ if there exists $(\mathcal{X}'_k), \mathcal{N}(\mathcal{X}'_k) \to 0$ such that $(\vartheta_r)_{\mathbb{BC}}$ converges uniformly to ϑ in $\mathcal{S} - \mathcal{X}'_k$, for any fixed $k \in \mathbb{N}$.*

The relations among the convergence concepts of bi-complex uncertain sequences are discussed below.

Theorem 4. *If the bi-complex uncertain sequence converges in mean to ϑ with respect to the difference sequence then converges in measure to ϑ but not necessarily converse.*

Proof. By extending markov inequality complex to bi-complex for any given number $\varepsilon > 0$, we have

$$\mathcal{N}\{\omega \in \mathcal{S} : \|\Delta_m^n\vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} \geq \varepsilon) \leq \frac{E[\|\Delta_m^n\vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}}]}{\varepsilon} \to 0,$$

as $n \to \infty$. Thus $(\vartheta_r)_{\mathbb{BC}}$ converges in measure to ϑ but converse of the above theorem does not satisfy and this can be proved by considering a bi-complex uncertainty space $(\mathcal{S}, \mathcal{A}, \mathcal{N})$ to be $\{\omega_1, \omega_2, ...\}$ with power set and

$$\mathcal{N}(\mathcal{H}) = \begin{cases} \sup\limits_{\omega_l \in \mathcal{H}} \{\frac{1}{l^2+1}\}, & if \ \sup\limits_{\omega_l \in \mathcal{H}} \{\frac{1}{l^2+1}\} < \frac{1}{2}, \\ 1 - \sup\limits_{\omega_l \in \mathcal{H}^c} \{\frac{1}{l^2+1}\}, & if \ \sup\limits_{\omega_l \in \mathcal{H}^c} \{\frac{1}{l^2+1}\} < \frac{1}{2}, \\ \frac{1}{2}, & otherwise. \end{cases}$$

Consider the bi-complex uncertain variables defined by

$$(\vartheta_l(\omega))_{\mathbb{BC}} = \begin{cases} i_1 + i_2 + (l^2+1) i_1 i_2, & if \ \omega = \omega_l, \\ 0, & otherwise, \end{cases}$$

for l=1,2,... and $\vartheta \equiv 0$. For some small number, $\varepsilon > 0$ and $l \geq 1$, we have

$$\mathcal{N}\bigg(\bigg\{\omega \in \mathcal{S} : \|\Delta_m^n \vartheta_l(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} \geq \varepsilon\bigg\}\bigg) = \mathcal{N}\bigg(\bigg\{\omega \in \mathcal{S} : \|\Delta_m^n \vartheta_l(\omega)\|_{\mathbb{BC}} \geq \varepsilon\bigg\}\bigg)$$

$$= \mathcal{N}(\omega_l)$$

$$= \frac{1}{l^2+1} \to 0,$$

as $l \to \infty$. So, the sequence $(\vartheta_l)_{\mathbb{BC}}$ converges in measure to ϑ. However, for each $l \in \mathbb{N}$, the uncertainty distribution of the uncertain variable $\|\vartheta_l - \vartheta\| = \|\vartheta_l\|$ is

$$D_l(u) = \begin{cases} 0, & if \ u < 0, \\ 1 - \frac{1}{l^2+1}, & if \ 0 \leq \ u < l^2 + 1, \\ 1, & if \ u \geq l^2 + 1. \end{cases}$$

So for each $l \geq 1$, we have

$$E\Big[\sum_{l=1}^{p} \|\Delta_m^n \vartheta_l(\omega) - \vartheta(\omega)\|_{\mathbb{BC}}\Big] = \int_o^{l^2+1} 1 - (1 - \frac{1}{l^2+1}) du = 1,$$

That is, the sequence $(\vartheta_r)_{\mathbb{BC}}$ does not converge in mean to ϑ with respect to the difference sequence, which proves the theorem. □

Remark 1. *Almost sure convergence with respect to difference operator does not imply convergence in mean or in measure.*

The following example shows this.

Example 6. *Consider the bi-complex uncertainty space* $(\mathcal{S}, \mathcal{A}, \mathcal{N})$ *to be* $\omega_1, \omega_2, \omega_3, \omega_4, ...$ *with*

$$\mathcal{N}(\mathcal{H}) = \sum_{\omega_r \in \omega} \frac{1}{2^r},$$

and the bi-complex uncertain variables are defined by

$$(\vartheta_r(\omega))_{\mathbb{BC}} = \begin{cases} (i_1 i_2) 2^r, & if\ \omega = \omega_r, \\ 0, & otherwise, \end{cases}$$

for $r = 1, 2, ...$ *and* $\vartheta \equiv 0$. *shows that* $(\vartheta_r)_{\mathbb{BC}}$ *converges a.s. to* ϑ *and further the uncertainty distribution of* $\|\vartheta_r\|_{\mathbb{BC}}$ *are defined by*

$$D_r(u) = \begin{cases} 0, & if\ u < 0, \\ 1 - \frac{1}{2^r}, & if\ 0 \leq u < 2^r, \\ 1, & if\ u \geq 2^r, \end{cases}$$

for $r = 1, 2, ...$, *respectively. Therefore, we get*

$$E[\|\Delta_m^n \vartheta_r - \vartheta\|_{\mathbb{BC}}] = \int_0^{2^r} 1 - (1 - \frac{1}{2^r}) du = 1,$$

imply that the sequence $(\vartheta_n)_{\mathbb{BC}}$ *does not converge in mean to* ϑ.

Example 7. *Now, for proving almost sure convergence a.s. does not imply convergence in measure we consider an uncertainty space* $(\mathcal{S}, \mathcal{A}, \mathcal{N})$ *to be* $\omega_1, \omega_2, \omega_3, \omega_4, ...$ *with*

$$\mathcal{N}(\mathcal{H}) = \begin{cases} \sup\limits_{\omega_r \in \mathcal{H}} \{\frac{r}{2r+1}\}, & if\ \sup\limits_{\omega_r \in \mathcal{H}} \{\frac{r}{2r+1}\} < \frac{1}{2}, \\ 1 - \sup\limits_{\omega_r \in \mathcal{H}^c} \{\frac{r}{2r+1} < \frac{1}{2}\}, & if\ \sup\limits_{\omega_r \in \mathcal{H}^c} \{\frac{r}{2r+1}\} < \frac{1}{2}, \\ \frac{1}{2}, & otherwise, \end{cases}$$

and the bi-complex uncertain variables are given by

$$(\vartheta_r(\omega))_{\mathbb{BC}} = \begin{cases} (i_1 i_2) r, & if\ \omega = \omega_r, \\ 0, & otherwise, \end{cases}$$

for $r = 1, 2, ...$ and $\vartheta \equiv \theta$, where θ is the zero element. Then we get

$$\begin{aligned}\mathcal{N}\{\|\Delta_m^n \vartheta_r(\omega) - \vartheta(\omega)\|_{\mathbb{BC}} \geq \varepsilon\} &= \mathcal{N}\{\omega \in S : \|\Delta_m^n \vartheta_r(\omega)\|_{\mathbb{BC}} \geq \varepsilon\} \\ &= \mathcal{N}\{\omega_r\} \\ &= \frac{r}{2r+1} \to \frac{1}{2},\end{aligned}$$

as $r \to \infty$ implies that the sequence $(\vartheta_r)_{\mathbb{BC}}$ does not converge in measure to ϑ and also the uncertainty distribution of $(\vartheta_r)_{\mathbb{BC}}$ are

$$\begin{aligned}D_r(\mathbb{BC}) = \; & D_r(\alpha + i_1\beta + i_2\gamma + i_1 i_2\theta) \\ &= \begin{cases} 0, & if\ \alpha < 0,\ \beta < 0,\ \gamma < 0,\ \theta < +\infty, \\ 0, & if\ \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ \theta < 0, \\ 1 - \frac{r}{2r+1}, & if\ \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ 0 \leq \theta < r, \\ 1, & if\ \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ \theta \geq r, \end{cases}\end{aligned}$$

.

$$D(\mathbb{BC}) = \begin{cases} 0, & if\ \alpha < 0,\ \beta < 0,\ \gamma < 0,\ \theta < +\infty, \\ 0, & if\ \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ \theta < 0, \\ 1, & if\ \alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ \theta \geq 0. \end{cases}$$

Therefore, showing that $D_r(\mathbb{BC})$ does not converge to $D(\mathbb{BC})$ at $\alpha \geq 0,\ \beta \geq 0,\ \gamma \geq 0,\ \theta \geq 0$. That is, the sequence $(\vartheta_r)_{\mathbb{BC}}$ does not converge to ϑ in distribution.

So, we can say that almost surely *a.s.* convergence with respect to the difference operator does not implies neither convergence in mean nor convergence in measure by the above examples 6 and 7.

Theorem 5. *If the bi-complex uncertain sequence $(\vartheta_r)_{\mathbb{BC}}$ converges uniformly almost surely (u.a.s.) to ϑ with respect to difference operator, then $(\vartheta_r)_{\mathbb{BC}}$ converges almost surely (a.s.) to ϑ with respect to difference operator.*

Proof. We know that if $(\vartheta_r)_{\mathbb{BC}}$ converges uniformly almost surely (u.a.s.) to ϑ, then

$$\lim_{r\to\infty} \mathcal{N}\left(\bigcup_{r=k}^{\infty} \{\|\Delta_m^n \vartheta_r - \vartheta\|_{\mathbb{BC}} \geq \varepsilon\} \right) = 0.$$

Since,

$$\mathcal{N}\left(\bigcap_{k=1}^{\infty} \bigcup_{r=k}^{\infty} \{\|\Delta_m^n \vartheta_r - \vartheta\|_{\mathbb{BC}} \geq \varepsilon\} \right) \leq \inf_k \mathcal{N}\left(\bigcup_{r=k}^{\infty} \{\|\Delta_m^n \vartheta_r - \vartheta\|_{\mathbb{BC}} \geq \varepsilon\} \right).$$

By applying the limit above $r \to \infty$, we get

$$\mathcal{N}\left(\bigcap_{k=1}^{\infty}\bigcup_{r=k}^{\infty}\{\|\Delta_m^n\vartheta_r - \vartheta\|_{\mathbb{BC}} \geq \varepsilon\}\right) = 0.$$

Hence, $(\vartheta_r)_{\mathbb{BC}}$ converges almost surely $(a.s.)$ to ϑ which proves the theorem. □

Theorem 6. *Let the bi-complex uncertain sequence $(\vartheta_r)_{\mathbb{BC}}$ has real part (ξ_r) and imaginary parts $(\eta_r), (\mu_r)$ and (φ_r), respectively, for all $n \in \mathbb{N}$. If uncertain sequences $\xi_r \to \xi, \eta_r \to \eta, \mu_r \to \mu$ and $\varphi_r \to \varphi$ are convergent in measure with respect to difference operator on sequence, the bi-complex uncertain sequence (ϑ_r) converge uniformly with respect to difference operator in distribution to $\vartheta = \xi + i_1\eta + i_2\mu + i_1i_2\varphi$.*

Proof. Let $\mathbb{BC} = \alpha + i_1\beta + i_2\gamma + i_1i_2\theta$ be a continuity point of the distribution D. For any $w > \alpha,\ x > \beta,\ y > \gamma,\ z > \theta$, we have

$$\begin{aligned}
&\{\xi_r \leq\ \alpha,\ \eta_r \leq\ \beta,\ \mu_r \leq\ \gamma,\ \varphi_r \leq\ \theta\}\\
&\quad= \{\xi_r \leq\ \alpha,\ \eta_r \leq\ \beta,\ \mu_r \leq\ \gamma,\ \varphi_r \leq\ \theta,\ \xi \leq w,\ \eta \leq x,\ \mu \leq y,\ \varphi \leq z\}\\
&\quad\cup \{\xi_r \leq\ \alpha,\ \eta_r \leq\ \beta,\ \mu_r \leq\ \gamma,\ \varphi_r \leq\ \theta,\ \xi > w,\ \eta > x,\ \mu > y,\ \varphi > z\}\\
&\quad\cup \{\xi_r \leq\ \alpha,\ \eta_r \leq\ \beta,\ \mu_r \leq\ \gamma,\ \varphi_r \leq\ \theta,\ \xi > w,\ \eta \leq x,\ \mu \leq y,\ \varphi \leq z\}\\
&\quad\subset \{\xi \leq w,\ \eta \leq\ x,\ \mu \leq\ y,\ \varphi \leq\ z\} \cup \{\|\Delta_m^n\xi_r(\omega) - \xi(\omega)\|_{\mathbb{BC}} \geq w - \alpha\}\\
&\quad\cup \{\|\Delta_m^n\eta_r(\omega) - \eta(\omega)\|_{\mathbb{BC}} \geq x - \beta\} \cup\ \{\|\Delta_m^n\mu_r(\omega) - \mu(\omega)\|_{\mathbb{BC}} \geq y - \gamma\}\\
&\qquad\cup\{\|\Delta_m^n\varphi_r(\omega) - \varphi(\omega)\|_{\mathbb{BC}} \geq z - \theta\}.
\end{aligned}$$

By sub-additivity,

$$\begin{aligned}
D_r(\mathbb{BC}) &= D_r(\alpha + i_1\beta + i_2\gamma + i_1i_2\theta)\\
&\leq D(w + i_1x + i_2y + i_1i_2z) + \mathcal{N}(\|\Delta_m^n\xi_r(\omega) - \xi(\omega)\|_{\mathbb{BC}} \geq w - \alpha)\\
&+\mathcal{N}(\|\Delta_m^n\eta_r(\omega) - \eta(\omega)\|_{\mathbb{BC}} \geq x - \beta) + \mathcal{N}(\|\Delta_m^n\mu_r(\omega) - \mu(\omega)\|_{\mathbb{BC}} \geq y - \gamma)\\
&+\mathcal{N}(\|\Delta_m^n\varphi_r(\omega) - \varphi(\omega)\|_{\mathbb{BC}} \geq z - \theta).
\end{aligned}$$

Given each $(\xi_r) \to \xi,\ (\eta_r) \to \eta,\ (\mu_r) \to \mu$ *and* $(\varphi_r) \to \varphi$ converge in measure, so

$$\lim_{r\to\infty}\mathcal{N}\{\omega \in \mathcal{S} : \|\Delta_m^n\xi_r(\omega) - \xi(\omega)\|_{\mathbb{BC}} \geq w - \alpha = 0\},$$

$$\lim_{n\to\infty} \mathcal{N}\{\omega \in \mathcal{S} : \|\Delta_m^n \eta_r(\omega) - \eta(\omega)\|_{\mathbb{BC}} \geq x - \beta = 0\},$$
$$\lim_{n\to\infty} \mathcal{N}\{\omega \in \mathcal{S} : \|\Delta_m^n \mu_r(\omega) - \mu(\omega)\|_{\mathbb{BC}} \geq y - \gamma = 0\},$$

and

$$\lim_{r\to\infty} \mathcal{N}\{\omega \in \mathcal{S} : \|\Delta_m^n \varphi_r(\omega) - \varphi(\omega)\|_{\mathbb{BC}} \geq z - \theta = 0\}.$$

Also

$$\lim_{r\to\infty} \sup D_r(\mathbb{BC}) \leq D(w + i_1 x + i_2 y + i_1 i_2 z),$$

for any $w > \alpha,\ x > \beta,\ y > \gamma$ *and* $z > \theta$ and $w + i_1 x + i_2 y + i_1 i_2 z \to \alpha + i_1\beta + i_2\gamma + i_1 i_2\theta$, we obtain

$$\lim_{r\to\infty} \sup D_r(\mathbb{BC}) \leq D(\mathbb{BC}). \tag{1}$$

And for $p < \alpha,\ \lambda < \beta,\ \kappa < \gamma,\ q < \theta$,

$\{\xi \leq\ p,\ \eta \leq\ \lambda,\ \mu \leq\ \kappa,\ \varphi \leq\ q\}$

$= \{\xi_r \leq\ \alpha,\ \eta_r \leq\ \beta,\ \mu_r \leq\ \gamma,\ \varphi_r \leq\ \theta,\ \xi \leq\ p,\ \eta \leq\ \lambda,\ \mu \leq\ \kappa,\ \varphi \leq\ q\}$

$\cup \{\xi_r \leq\ \alpha,\ \eta_r >\ \beta,\ \mu_r\ \ \gamma,\ \varphi_r >\ \theta,\ \xi \leq\ p,\ \eta \leq\ \lambda,\ \mu \leq\ \kappa,\ \varphi \leq\ q\}$

$\cup \{\xi_r >\ \alpha,\ \eta_r \leq\ \beta,\ \mu_r \leq\ \gamma,\ \varphi_r \leq\ \theta,\ \xi \leq\ p,\ \eta \leq\ \lambda,\ \mu \leq\ \kappa,\ \varphi \leq\ q\}$

$\cup \{\xi_r \geq\ \alpha,\ \eta_r \geq\ \beta,\ \mu_r \geq\ \gamma,\ \varphi_r \geq\ \theta,\ \xi \leq\ p,\ \eta \leq\ \lambda,\ \mu \leq\ \kappa,\ \varphi \leq\ q\}$

$\subset \{\xi_r \leq\ \alpha,\ \eta_r \leq\ \beta,\ \mu_r \leq\ \gamma,\ \varphi_r \leq\ \theta\} \cup |\xi_r - \xi| \geq\ \alpha - p \cup \{|\eta_r - \eta| \geq\ \beta - \lambda$

$\cup\ \ |\mu_r - \mu| \geq\ \gamma - \kappa|\ \cup\ \ |\varphi_r - \varphi| \geq\ \theta - q\}$

implies

$$D(p + i_1\lambda + i_2\kappa + i_1 i_2 q) \leq D_r(\alpha + i_1\beta + i_2\gamma + i_1 i_2\theta) + \mathcal{N}(|\xi_r - \xi| \geq\ \alpha - p)$$
$$+\mathcal{N}(|\eta_r - \eta| \geq\ \beta - \lambda) + \mathcal{N}(|\mu_r - \mu| \geq\ \gamma - \kappa\} + \mathcal{N}(|\varphi_r - \varphi| \geq\ \theta - q).$$

Thus

$$\lim_{r\to\infty} \mathcal{N}(|\xi_r - \xi| \geq \alpha - p) = 0,$$
$$\lim_{r\to\infty} \mathcal{N}(|\eta_r - \eta| \geq \beta - \lambda) = 0,$$
$$\lim_{r\to\infty} \mathcal{N}(|\mu_r - \mu| \geq \gamma - \kappa) = 0,$$

and

$$\lim_{r\to\infty} \mathcal{N}(|\varphi_r - \varphi| \geq \theta - q) = 0,$$

we get

$$D(p + i_1\lambda + i_2\kappa + i_1i_2q) \leq \lim_{r\to\infty} \inf D_r(\alpha + i_1\beta + i_2\gamma + i_1i_2\theta),$$

for any $p < \alpha, \lambda < \beta, \kappa < \gamma, q < \theta$ and

$$p + i_1\lambda + i_2\kappa + i_1i_2q \to \alpha + i_1\beta + i_2\gamma + i_1i_2\theta,$$

we get

$$D(\mathbb{BC}) \leq \lim_{r\to\infty} \inf D_r(\mathbb{BC}). \tag{2}$$

From (1) and (2) , $D_r(\mathbb{BC}) \to D(\mathbb{BC})$ as $r \to \infty$. Thus,

$$(\vartheta_r)_{\mathbb{BC}} \to \vartheta = \xi + i_1\eta + i_2\mu + i_1i_2\varphi$$

converges in distribution, which proves the theorem. □

5 Conclusion

This study develops a new framework of convergent difference sequence spaces involving bi-complex uncertain variables by combining difference operators with uncertainty theory. We established their fundamental algebraic and topological properties, including linearity and normed space structures, and investigated the relationships among these newly defined spaces. In particular, we proved the strict inclusion $\Delta^{n-1} \subset \Delta^n$. Furthermore, we examine their structural properties and show that, in general, these spaces are neither solid, monotone, nor symmetric. We also investigated the convergence behavior of sequences in these spaces and established the implications among different modes of convergence.

The results presented in this paper provide a foundation for further research on sequence spaces involving uncertain and bi-complex-valued structures. Future work may focus on study of the bi-complex uncertain sequences defined by Orlicz functions and extending the present framework to statistical convergence and related summability methods in the bi-complex uncertain setting.

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