

## APPLICATIONS OF THE DUALITY: DETECTABILITY VERSUS STABILIZABILITY IN THE STOCHASTIC FRAMEWORK\*

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*Dedicated to the memory of Professor Mihail Megan*

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### Abstract

The aim of this paper is to exploit the property of duality between detectability and stabilizability from the stochastic framework, to derive some results from the domain of control of systems with random parameters. First, we shall derive a dual Barbashin-Krasovski type criterion for asymptotic stability of the zero solution of an Itô differential equation under the conditions when there exists a Lyapunov function with negative semidefinite derivative. Then, we provide a set of necessary and sufficient conditions that guaranty the existence of the bounded on  $\mathbb{R}$  and stabilizing solution of a forward matrix Riccati differential equation with periodic coefficients arising in concordance with the problem of filtering of a signal generated by a dynamical system modeled by stochastic differential equations of Itô type.

**Keywords:** stochastic detectability, stochastic stabilizability, Barbashin-Krasovski type criterion, Riccati differential equations of filtering.

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# 1 Introduction

Controlability and observability, as well as stabilizability and detectability are structural properties of controlled systems which are involved in the solution of various problems of optimal control in both deterministic framework and stochastic framework (see [20], [2], [23], [4], [5], [6], [12]).

In the deterministic framework it is known a duality relation between controlability and observability properties as well as between stabilizability and detectability properties of a linear system (see [20], [2], [23], [4]). In [8] the duality between stochastic stabilizability and stochastic detectability was analyzed in the case of linear systems modeled by stochastic linear differential equations of Itô type with periodic coefficients possibly subject to random switching modeled by a Markov process.

One of the aim of the present paper is to use the duality between stochastic detectability and stochastic stabilizability of a linear stochastic system modeled by stochastic differential equations of Itô type to derive a dual Barbashin-Krasovski type criterion for exponential stability in mean square of the zero solution of a stochastic linear differential equation under the conditions when only a quadratic Lyapunov function with negative semidefinite derivative is available.

In many applications of the Lyapunov function method for the study of the asymptotic stability of some systems from real life, were found some Lyapunov functions attached in a natural way to the system under consideration, as, for example, the case of the energy function for mechanical systems. In most cases, these functions are positive definite, but their derivative is only negative semidefinite. In those cases, the original method of the Lyapunov functions cannot be used to obtain information about the asymptotic stability of the considered systems. In [3], Barbashin and Krasovski proposed a set of conditions which guaranty the asymptotic stability also in the case when the derivative of the Lyapunov function is only negative semidefinite. In the particular case of a linear time-invariant deterministic system, a sufficient condition which guaranty that a quadratic Lyapunov function with negative semidefinite derivative allows us to conclude if the system under consideration is or not exponentially stable. That condition is expressed by the detectability of a pair of matrices adequately chosen (see [7]). This result was then extended in the case of some linear time-varying systems both in continuous-time and discrete-time in the deterministic framework and stochastic framework. For the reader convenience we refer to [14], [24], [10], [11] and their references. In the present paper, we derive a dual Barbashin-Krasovski type criterion where the detectability

condition involved in the afore mentioned references is replaced by a stabilizability condition.

Another aim of this work is to provide a set of necessary and sufficient conditions that guaranty the existence of a bounded and stabilizing solution of a matrix Riccati differential equation with periodic coefficients arising in connection with the problem of filtering of some signals generated by dynamical systems modeled by stochastic differential equations with state multiplicative white noise perturbations. This kind of Riccati equations are forward matrix Riccati differential equations less studied in the time-varying case. In the existing literature, a greater attention was paid to backward matrix Riccati differential equations [1], [12]. In this work, we shall use the duality between stochastic detectability and stochastic stabilizability to obtain conditions for the existence of a bounded and stabilizing solution of a forward matrix Riccati differential equation. Also, we provide an iterative procedure for the numerical computation of this kind of solutions.

## 2 The problem setting

### 2.1 The model description

We consider the stochastic linear system  $\mathbb{G}$  having the state space representation described by the following stochastic linear differential equations of Itô type:

$$dx(t) = (A_0(t)x(t) + B_0(t)u(t)) + \sum_{k=1}^r (A_k(t)x(t) + B_k(t)u(t))dw_k(t) \quad (1a)$$

$$dy(t) = C_0(t)x(t)dt + \sum_{k=1}^r C_k(t)x(t)dw_k(t) \quad (1b)$$

$t \in \mathbb{R}_+ = [0, \infty)$ , where  $x(t) \in \mathbb{R}^n$  is the vector of the state variables,  $u(t) \in \mathbb{R}^m$  is the vector of inputs of the system and  $y(t) \in \mathbb{R}^q$  is the vector of an output of the system. The vector  $u(t)$  includes control parameters as well as and/or some exogenous disturbances which affect the behavior of the system.

The vectors  $y(t)$  include signals generated by the system  $\mathbb{G}$  which can be measured and used to elaborate a control law with given performances specifications. The output  $y(t)$  may include also some signals generated by system under consideration named quality outputs whose evolution in time can be changed using the adopted control law.

In (1),  $\{w_k(t)\}_{t \geq 0}$  are processes of the Brownian motions defined on a given probability space  $(\Omega, \mathfrak{F}, \mathcal{P})$  and are satisfying the assumption:

**(H<sub>1</sub>)**  $\{w(t)\}_{t \geq 0}$ ,  $(w(t) = (w_1(t) \ w_2(t) \ \dots \ w_n(t))^\top)$  is an  $r$ -dimensional standard Wiener process i.e.  $w(0) = 0$ ,  $\mathbb{E}[w(t)] = 0$ ,  $\mathbb{E}[(w(s) - w(t))(w(s) - w(t))^\top] = I_r(s - t)$ , for all  $s > t \geq 0$ ,  $I_r$  being the identity matrix of size  $r \times r$ .

Here and in the sequel  $\mathbb{E}[\cdot]$  stands for the mathematical expectation.

Regarding the coefficients of the system (1) we make the assumption:

**(H<sub>2</sub>)**  $(A_k(\cdot), B_k(\cdot), C_k(\cdot)) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{q \times n}$ ,  $0 \leq k \leq r$ , are continuous matrix valued functions which are periodic with common period  $\theta > 0$ .

**Remark 1.** *Even if the system (1) evolves for  $t \in \mathbb{R}_+$ , the functions  $A_k(\cdot)$ ,  $B_k(\cdot)$ ,  $C_k(\cdot)$  can be extended in a natural way by periodicity to the whole real axis.*

For each  $t > 0$ ,  $\mathcal{F}_t \subset \mathfrak{F}$  denotes the smallest  $\sigma$ -algebra with respect to which the random vector  $w(s)$ ,  $0 \leq s \leq t$  are measurable. For  $t = 0$ ,  $\mathcal{F}_0$  is the trivial  $\sigma$ -algebra  $(\emptyset, \Omega)$ . For each  $t \geq 0$ , the  $\sigma$ -algebra  $\mathcal{F}_t$  is augmented with all subsets  $\mathfrak{A} \in \mathfrak{F}$  which is satisfying  $\mathcal{P}(\mathfrak{A}) = 0$ .

If  $J \subset \mathbb{R}_+$ ,  $L^2_{\mathcal{F}}\{J, \mathbb{R}^d\}$  stands for the linear space of the measurable stochastic processes  $v : J \times \Omega \rightarrow \mathbb{R}^d$  adapted to the filtration  $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$  and have the property

$$\mathbb{E} \left[ \int_J |v(t)|^2 dt \right] < \infty.$$

Based on Theorem 1.1 Chapter 5 from [13] (see also Theorem 5.2.1 from [22]) we obtain:

**Proposition 1.** *For any pair  $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$  and any  $u(\cdot) \in L^2_{\mathcal{F}}\{[t_0, \infty); \mathbb{R}^m\}$  the stochastic linear differential equation (SLDE) (1a) has a unique solution  $x(\cdot; t_0, x_0, u(\cdot)) : [t_0, \infty) \rightarrow \mathbb{R}^n$  with the properties:*

- (a)  $\{x(t; t_0, x_0, u(\cdot))\}_{t \geq t_0}$  is a continuous stochastic process;
- (b)  $x(\cdot; t_0, x_0, u(\cdot)) \in L^2_{\mathcal{F}}\{[t_0, T]; \mathbb{R}^n\}$ , for all  $T > t_0$ ;
- (c)  $x(t_0; t_0, x_0, u(\cdot)) = x_0$ .

To ease the presentation of the results derived in the next sections we shall say sometimes that the system  $\mathbb{G}$  is associated to the triple  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot), \mathcal{B}(\cdot))$  where

$$\mathcal{A}(\cdot) = (A_0(\cdot), A_1(\cdot), \dots, A_r(\cdot)) \quad (2a)$$

$$\mathcal{B}(\cdot) = (B_0(\cdot), B_1(\cdot), \dots, B_r(\cdot)) \quad (2b)$$

$$\mathcal{C}(\cdot) = (C_0(\cdot), C_1(\cdot), \dots, C_r(\cdot)) \quad (2c)$$

$A_k(\cdot)$ ,  $B_k(\cdot)$ ,  $C_k(\cdot)$ ,  $0 \leq k \leq r$ , are the matrix valued functions which are coefficients of (1) and are satisfying the condition from assumption  $(\mathbf{H}_2)$ .

The system (1) will be named the state-space representation of the triple  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot), \mathcal{B}(\cdot))$ .

## 2.2 Basic definitions

**Definition 1.** (a) We say that the system described by SLDE (1a) is stochastic stabilizable by linear state feedback, or equivalently the pair  $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$  is stabilizable by linear state feedback if there exist continuous and  $\theta$ -periodic matrix valued functions  $F(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$  with the property that the zero solution of the corresponding closed-loop SLDE:

$$dx(t) = (A_0(t) + B_0(t)F(t))x(t)dt + \sum_{k=1}^r (A_k(t) + B_k(t)F(t))x(t)dw_k(t) \quad (3)$$

is exponentially stable in mean square (ESMS);

(b) We say that the linear system

$$dx(t) = A_0(t)x(t)dt + \sum_{k=1}^r A_k(t)x(t)dw_k(t) \quad (4a)$$

$$dy(t) = C_0(t)x(t)dt + \sum_{k=1}^r C_k(t)x(t)dw_k(t) \quad (4b)$$

(obtained from (1) taking  $u(t) = 0$ ) is named stochastic detectable, or equivalently, the pair  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  is detectable if there exist continuous and  $\theta$ -periodic matrix valued functions  $K(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times q}$  with the property that the zero solution of the corresponding resulting system:

$$dx(t) = (A_0(t) + K(t)C_0(t))x(t)dt + \sum_{k=1}^r (A_k(t) + K(t)C_k(t))x(t)dw_k(t) \quad (5)$$

is ESMS.

**Remark 2.** We assume that the definition of the exponential stability in mean square (ESMS) of the zero solution of a stochastic linear differential equation as (3) or (5), respectively, is known. For the reader convenience we refer to [18] or [12]. In Proposition 3.3 in [8] may be found a list of conditions which are equivalent to the ESMS property of the zero solution of a SLDE.

### 2.3 The problem setting

In [8], we have shown that if  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot), \mathcal{B}(\cdot))$  is as in (2) one may construct a new triple  $(\mathcal{C}^\#(\cdot), \mathcal{A}^\#(\cdot), \mathcal{B}^\#(\cdot))$  named dual triple of the given triple, having the properties:

- (a) the pair  $(\mathcal{A}(\cdot), \mathcal{B}(\cdot))$  is stabilizable if and only if the dual pair  $(\mathcal{C}^\#(\cdot), \mathcal{A}^\#(\cdot))$  is detectable;
- (b) the pair  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  is detectable if and only if the dual pair  $(\mathcal{A}^\#(\cdot), \mathcal{B}^\#(\cdot))$  is stabilizable.

In the present work, we shall use this duality in order to derive a dual Barbasin-Krasovski type criterion for exponential stability in mean square of the zero solution of a SLDE.

A more detailed setting of this issue will be given in Section 3. Another application of the duality between the stabilizability property and detectability property in the stochastic context consists of the derivation of sufficient conditions which guaranty the existence of the stabilizing and bounded on  $\mathbb{R}$  solution of the following forward matrix Riccati differential equation (FRDE):

$$\begin{aligned} \dot{Y}(t) = & A_0(t)Y(t) + Y(t)A_0^\top(t) + \sum_{k=1}^r A_k(t)Y(t)A_k^\top(t) \\ & - (Y(t)C_0^\top(t) + \sum_{k=1}^r A_k(t)Y(t)C_k^\top(t) + L(t))(R(t) + \sum_{k=1}^r C_k(t)Y(t)C_k^\top(t))^{-1} \\ & \cdot (C_0(t)Y(t) + \sum_{k=1}^r C_k(t)Y(t)A_k^\top(t) + L^\top(t)) + M(t) \end{aligned} \quad (6)$$

associated to the pair  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  of the system (2a), (2c) and  $(M(\cdot), L(\cdot), R(\cdot)) : \mathbb{R} \rightarrow \mathcal{S}_n \times \mathbb{R}^{n \times q} \times \mathcal{S}_m$  which are continuous and periodic matrix valued functions with period  $\theta$  (the same as the period from  $(\mathbf{H}_2)$ ).

Throughout this work  $\mathcal{S}_d \subset \mathbb{R}^{d \times d}$  denotes the linear space of symmetric matrices of size  $d \times d$ .

**Definition 2.** A globally defined solution  $Y_s(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  of the FMRDE (6) is named stabilizing solution if the zero solution of the following closed-loop SLDE:

$$dx(t) = (A_0(t) + K_s(t)C_0(t))x(t)dt + \sum_{k=1}^r (A_k(t) + K_s(t)C_k(t))x(t)dw_k(t), \quad (7)$$

$t \geq 0$ , is ESMS, where

$$\begin{aligned} K_s(t) := & - (Y_s(t)C_0^\top(t) + \sum_{k=1}^r A_k(t)Y_s(t)C_k^\top(t) + L(t)) \\ & \cdot (R(t) + \sum_{k=1}^r C_k(t)Y_s(t)C_k^\top(t))^{-1}, \quad t \in \mathbb{R}. \end{aligned} \quad (8)$$

Our aim is to provide a set of conditions which guaranty the existence and uniqueness of the stabilizing and bounded on  $\mathbb{R}$  solution of FMRDE (6). We shall provide also an iterative procedure for the numerical computation of the bounded and stabilizing solution of the equation (6).

**Remark 3.** (a) It is worth mentioning that we do not know neither an initial value, nor a boundary condition of the stabilizing solution of the equation (6). The only available information is that the zero solution of the corresponding resulting SLDE (7)-(8) is ESMS and additionally it is the unique globally defined solution having the property that the zero solution of the resulting SLDE is ESMS. That is why the known methods for numerical computation of the solution with given initial value or a bilocal problem are not applicable for the numerical computation of the bounded and stabilizing solution of the FMRDE (6).

(b) Starting with the classic works of Kalman (see [16], [17]) the stabilizing solution of some particular forms of the Riccati equation of the form (6) played a central role in the construction of the filter with applications in various research domains. See for example [9], [21], [19] and [25] for a detailed discussion.

### 3 A dual Barbasin-Krasovski criterion for exponential stability in mean-square

Let us consider the stochastic linear differential equation (SLDE):

$$dx(t) = A_0(t)x(t)dt + \sum_{k=1}^r A_k(t)x(t)dw_k(t), \quad t \geq 0. \quad (9)$$

Our aim is to extend the result stated in Proposition 4.5 b) from [15] to the case of stochastic time-varying system (9).

According to the equivalence (i)  $\Leftrightarrow$  (vi) from Proposition 3.3 in [8] with the updates  $M_k(\cdot) \leftarrow A_k(\cdot)$  we deduce that the zero solution of the SLDE (9) is ESMS if and only if there exists a matrix valued function  $\tilde{H}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  continuous and  $\theta$ -periodic,  $\tilde{H}(t) > 0$ , for all  $t \in \mathbb{R}$  with the property that the non-homogeneous forward Lyapunov-type matrix linear differential equation:

$$\dot{X}(t) = A_0(t)X(t) + X(t)A_0^\top(t) + \sum_{k=1}^r A_k(t)X(t)A_k^\top(t) + \tilde{H}(t), \quad (10)$$

$t \in \mathbb{R}$  has a unique solution  $\tilde{X}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  which is a  $\theta$ -periodic function and satisfies  $\tilde{X}(t) > 0$ ,  $t \in \mathbb{R}$ .

In this section we show that under a suitable stabilizability assumption, the fact that the non-homogeneous Lyapunov type linear differential equation (10) with the free term  $\tilde{H}(t) \geq 0$ ,  $t \in \mathbb{R}$  (instead of  $\tilde{H}(t) > 0$ ,  $t \in \mathbb{R}$ ) has a solution  $\tilde{X}(t) \geq 0$  (instead of  $\tilde{X}(t) > 0$ ,)  $t \in \mathbb{R}$ , guaranties the ESMS property of the zero solution of the SLDE (9).

Let  $B_0(t) \in \mathbb{R}^{m \times m}$  be obtained from the factorization

$$\tilde{H}(t) = B_0(t)B_0^\top(t), \quad t \in \mathbb{R}. \quad (11)$$

If  $\tilde{H}(\cdot)$  is a  $\theta$ -periodic function, then  $B_0(\cdot)$  from (11) may be chosen to be a  $\theta$ -periodic function (with the same period).

We set

$$\check{B}(\cdot) := (B_0(\cdot), 0, \dots, 0). \quad (12)$$

If  $\mathcal{A}(\cdot)$  is that described in (2a) and  $\check{B}(\cdot)$  is introduced in (12), then, Definition 1 (a) becomes:

**Definition 3.** *We say that the pair  $(\mathcal{A}(\cdot), \check{B}(\cdot))$  is stabilizable if there exists a continuous and  $\theta$ -periodic matrix valued function  $F(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$  with the property that the zero solution of the corresponding closed-loop system:*

$$dx(t) = (A_0(t) + B_0(t)F(t))x(t)dt + \sum_{k=1}^r A_k(t)x(t)dw_k(t), \quad t \geq 0 \quad (13)$$

is ESMS.

The main result of this section is:

**Theorem 1.** *Assume:*

- (a) *the assumptions  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$  hold;*
- (b)  *$(\mathcal{A}(\cdot), \tilde{\mathcal{B}}(\cdot))$  is stabilizable.*

*Under these conditions, the following are equivalent:*

- (i) *the zero solution of the SLDE (9) is ESMS;*
- (ii) *the non-homogeneous forward Lyapunov type matrix linear differential equation:*

$$\dot{X}(t) = A_0(t)X(t) + X(t)A_0^\top(t) + \sum_{k=1}^r A_k(t)X(t)A_k^\top(t) + B_0(t)B_0^\top(t), \quad (14)$$

*$t \in \mathbb{R}$  has a unique solution  $\tilde{X}(t) : \mathbb{R} \rightarrow \mathcal{S}_n$  which is  $\theta$ -periodic and satisfying  $\tilde{X}(t) \geq 0$ , for all  $t \in \mathbb{R}$ .*

*Proof.* Let  $\mathbf{T}(t, t_0) : \mathcal{S}_n \rightarrow \mathcal{S}_n$  be the linear evolution operator in  $\mathcal{S}_n$  defined by the linear differential equation:

$$\dot{\mathbf{T}}(t) = A_0(t)\mathbf{T}(t) + \mathbf{T}(t)A_0^\top(t) + \sum_{k=1}^r A_k(t)\mathbf{T}(t)A_k^\top(t). \quad (15)$$

Employing Theorem 3.2.2 from [12] in the particular case  $\mathcal{D} = \{1\}$ , we may infer that the statement (i) holds if and only if

$$\|\mathbf{T}(t, t_0)\| \leq \beta e^{-\alpha(t-t_0)},$$

for all  $t \geq t_0 \in \mathbb{R}$ , where  $\beta \geq 1$ ,  $\alpha > 0$  do not depend upon  $t, t_0$ . We set

$$\tilde{X}(t) := \int_{-\infty}^t \mathbf{T}(t, s)[B_0(s)B_0^\top(s)]ds.$$

By direct calculation, we show that  $\tilde{X}(\cdot)$  defined as before, is the unique solution of the equation (14) which is  $\theta$ -periodic and satisfies  $\tilde{X}(t) \geq 0$ ,  $t \in \mathbb{R}$ . So, the validity of the implication (i)  $\Rightarrow$  (ii) is confirmed.

Let us now show that the implication (ii)  $\Rightarrow$  (i) holds too. If  $\tilde{X}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  is the unique  $\theta$ -periodic and positive semidefinite solution of the equation (14), we define

$$\tilde{Y}(t) := \tilde{X}(-t), \quad t \in \mathbb{R}.$$

By direct calculation one obtains that  $\tilde{Y}(\cdot)$  solves the non-homogeneous backward Lyapunov type matrix linear differential equation:

$$-\dot{Y}(t) = A_0^\top(t)Y(t) + Y(t)\tilde{A}_0(t) + \sum_{k=1}^r \tilde{A}_k^\top(t)Y(t)\tilde{A}_k(t) + \tilde{C}_0^\top(t)\tilde{C}_0(t), \quad (16)$$

$t \in \mathbb{R}$ , where

$$\tilde{A}_k(t) := A_k^\top(-t), \quad 0 \leq k \leq r \quad (17a)$$

$$\tilde{C}_0(t) := B_0^\top(-t), \quad t \in \mathbb{R}. \quad (17b)$$

The non-homogeneous backward Lyapunov type matrix linear differential equation (16) is associated to the system:

$$d\tilde{x}(t) = \tilde{A}_0(t)\tilde{x}(t)dt + \sum_{k=1}^r \tilde{A}_k(t)\tilde{x}(t)dw_k(t) \quad (18a)$$

$$\tilde{y}(t) = \tilde{C}_0(t)\tilde{x}(t), \quad t \in \mathbb{R}_+. \quad (18b)$$

We show that the system (18) is stochastic detectable in the sense of Definition 4.1.1 from [12] if the pair  $(\mathcal{A}(\cdot), \tilde{\mathcal{B}}(\cdot))$  is stabilizable.

According to Definition 3,  $(\mathcal{A}(\cdot), \tilde{\mathcal{B}}(\cdot))$  is stabilizable if and only if there exists a continuous and  $\theta$ -periodic matrix valued function  $F(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$  with the property that the zero solution of the SLDE (13) is ESMS. Employing the equivalence  $(i) \Leftrightarrow (v)$  from Proposition 3.3 [8] with the updates  $M_0(t) \leftarrow A_0(t) + B_0(t)F(t)$ ,  $M_k(t) \leftarrow A_k(t)$ ,  $1 \leq k \leq r$ , we deduce that the zero solution of the SLDE (13) is ESMS if and only if the following non-homogeneous forward Lyapunov type matrix linear differential equation:

$$\begin{aligned} \dot{Y}(t) = & (A_0(t) + B_0(t)F(t))Y(t) + Y(t)(A_0(t) + B_0(t)F(t))^\top \\ & + \sum_{k=1}^r A_k(t)Y(t)A_k^\top(t) + I_n, \quad t \in \mathbb{R} \end{aligned} \quad (19)$$

has a unique solution  $\tilde{Y}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  which is  $\theta$ -periodic and satisfies

$$\tilde{Y}(t) > 0, \quad t \in \mathbb{R}.$$

Let  $\tilde{Z}(t) := \tilde{Y}(-t)$ ,  $t \in \mathbb{R}$ . By direct calculation, involving (19), we obtain that  $\tilde{Z}(\cdot)$  is the unique  $\theta$ -periodic and uniform positive defined solution

of the following non-homogeneous backward Lyapunov type matrix linear differential equation:

$$\begin{aligned} -\dot{\tilde{Z}}(t) = & (\tilde{A}_0(t) + \tilde{K}(t)\tilde{C}_0(t))^\top \tilde{Z}(t) + \tilde{Z}(t)(\tilde{A}_0(t) + \tilde{K}(t)\tilde{C}_0(t)) \\ & + \sum_{k=1}^r \tilde{A}_k^\top(t) \tilde{Z}(t) \tilde{A}_k(t) + I_n, \end{aligned} \quad (20)$$

$t \in \mathbb{R}$ , where  $\tilde{A}_k(t)$ ,  $0 \leq k \leq r$ , and  $\tilde{C}_0(t)$  were introduced by (17) and  $\tilde{K}(t) := F^\top(-t)$ ,  $t \in \mathbb{R}$ . Invoking the implication  $(iv) \Rightarrow (i)$  from Proposition 3.3 in [8] with the updates  $M_0(t) \leftarrow \tilde{A}_0(t) + \tilde{K}(t)\tilde{C}_0(t)$  and  $M_k(t) \leftarrow \tilde{A}_k(t)$ ,  $1 \leq k \leq r$ , we may infer that the existence of a  $\theta$ -periodic and uniform positive definite solution of the non-homogeneous backward Lyapunov type matrix linear differential equation (20) guaranties the ESMS property of the zero solution of the following SLDE:

$$d\tilde{x}(t) = (\tilde{A}_0(t) + \tilde{K}(t)\tilde{C}_0(t))\tilde{x}(t)dt + \sum_{k=1}^r \tilde{A}_k(t)\tilde{x}(t)dw_k(t), \quad t \in \mathbb{R}_+. \quad (21)$$

The exponential stability in mean-square of the zero solution of the SLDE (21) is equivalent to the stochastic detectability of the system (18) in the sense of the Definition 4.1.1 (b) from [12]. Hence, we can apply Theorem 4.1.7 from [12] in the case of the pair formed by the system (18) and the non-homogeneous backward Lyapunov type matrix linear differential equation (16), in order to conclude that the zero solution of the SLDE:

$$d\tilde{x}(t) = \tilde{A}_0(t)\tilde{x}(t)dt + \sum_{k=1}^r \tilde{A}_k(t)\tilde{x}(t)dw_k(t), \quad t \in \mathbb{R}_+ \quad (22)$$

is ESMS. Let  $\tilde{\mathbf{T}}(t, t_0) : \mathcal{S}_n \rightarrow \mathcal{S}_n$ ,  $t, t_0 \in \mathbb{R}$  be the linear evolution operator defined by the linear differential equation on  $\mathcal{S}_n$ :

$$\dot{\mathbb{X}}(t) = \tilde{A}_0(t)\mathbb{X}(t) + \mathbb{X}(t)\tilde{A}_0^\top(t) + \sum_{k=1}^r \tilde{A}_k(t)\mathbb{X}(t)\tilde{A}_k^\top(t), \quad t \in \mathbb{R}. \quad (23)$$

According to Theorem 3.2.2 (i) from [12] in the particular case  $\mathcal{D} = \{1\}$  we may infer that the zero solution of the SLDE (22) is ESMS if and only if

$$\|\tilde{\mathbf{T}}(t, t_0)\| \leq \tilde{\beta}e^{-\tilde{\alpha}(t-t_0)}, \quad (24)$$

for all  $t \geq t_0$ ,  $t, t_0 \in \mathbb{R}$ , where  $\tilde{\beta} \geq 1$ ,  $\tilde{\alpha} > 0$ , do not depend upon  $t$  and  $t_0$ .

Let  $\mathcal{L}(t) : \mathcal{S}_n \rightarrow \mathcal{S}_n$  be the linear operator defined by:

$$\mathcal{L}(t)[X] = A_0(t)X + XA_0^\top(t) + \sum_{k=1}^r A_k(t)XA_k^\top(t),$$

for all  $X \in \mathcal{S}_n$ . With this notation the linear differential equation (15) may be written as:

$$\dot{X}(t) = \mathcal{L}(t)[X(t)]. \quad (25)$$

Bearing in mind (17a) we may rewrite the differential equation (23) as:

$$\dot{\mathbb{X}}(t) = \mathcal{L}^*(-t)[\mathbb{X}(t)], \quad (26)$$

$\mathcal{L}^*(t)$  being the adjoint operator of  $\mathcal{L}(t)$  with respect to the inner product  $\langle X, Y \rangle = \text{Tr}[XY]$ , for all  $X, Y \in \mathcal{S}_n$ ,  $\text{Tr}[X]$  being the trace operator.

Recalling that  $\mathbf{T}(t, s)$  is the linear evolution operator of the linear differential equation (25) and  $\tilde{\mathbf{T}}(t, s)$  is the linear evolution operator of the linear differential equation (26) we obtain that

$$\tilde{\mathbf{T}}(s, t) = \mathbf{T}^*(-t, -s),$$

for all  $s \geq t$ ,  $t, s \in \mathbb{R}$ . Employing (24) we get:

$$\|\mathbf{T}^*(-t, -s)\| \leq \tilde{\beta}e^{-\tilde{\alpha}(s-t)}, \quad (27)$$

for all  $s \geq t$ , or equivalently, for any  $-t \geq -s$ ,  $t, s \in \mathbb{R}$ . This allows us to reply  $t$  by  $-t$ ,  $s$  by  $-s$  in (27) to obtain

$$\|\mathbf{T}^*(t, s)\| \leq \tilde{\beta}e^{-\tilde{\alpha}(t-s)}, \quad (28)$$

for all  $t \geq s$ ,  $t, s \in \mathbb{R}$ . Taking into account that

$$c_1\|\mathbf{T}(t, s)\| \leq \|\mathbf{T}^*(t, s)\| \leq c_2\|\mathbf{T}(t, s)\|,$$

for all  $t, s \in \mathbb{R}$  (where  $c_j > 0$ ,  $j = 1, 2$ , do not depend on  $t, s$ ) we obtain from (28) that

$$\|\mathbf{T}(t, s)\| \leq \tilde{\beta}c_1^{-1}e^{-\tilde{\alpha}(t-s)}, \quad (29)$$

for all  $t \geq s$ ,  $t, s \in \mathbb{R}$ , which means that the zero solution of the linear differential equation (25) is exponentially stable. Invoking again Theorem 3.2.2 from [12] in the particular case  $\mathcal{D} = \{1\}$  we find that (29) is equivalent to the ESMS property of the zero solution of the SLDE (9). Thus, the proof ends.  $\square$

## 4 On the bounded and stabilizing solution of a forward matrix Riccati differential equation

### 4.1 A dual backward matrix Riccati differential equation associated to a forward matrix Riccati differential equation

Let  $\tilde{Y}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  be a bounded on  $\mathbb{R}$  solution of the FMRDE (6). We set

$$\tilde{X}(t) := \tilde{Y}(-t), \quad t \in \mathbb{R}.$$

By direct calculations one obtains that  $\tilde{X}(\cdot)$  is a bounded on  $\mathbb{R}$  solution of the following backward matrix Riccati equation (BMRDE):

$$\begin{aligned} -\dot{\tilde{X}}(t) = & \hat{A}_0^\top(t)\tilde{X}(t) + \tilde{X}(t)\hat{A}_0(t) + \sum_{k=1}^r \hat{A}_k^\top(t)\tilde{X}(t)\hat{A}_k(t) \\ & - \left( \tilde{X}(t)\hat{B}_0(t) + \sum_{k=1}^r \hat{A}_k^\top(t)\tilde{X}(t)\hat{B}_k(t) + \hat{L}(t) \right) \\ & \cdot \left( \hat{R}(t) + \sum_{k=1}^r \hat{B}_k^\top(t)\tilde{X}(t)\hat{B}_k(t) \right)^{-1} \\ & \cdot \left( \hat{B}_0^\top(t)\tilde{X}(t) + \sum_{k=1}^r \hat{B}_k^\top(t)\tilde{X}(t)\hat{A}_k(t) + \hat{L}^\top(t) \right) \end{aligned} \quad (30)$$

where

$$\hat{A}_k(t) := A_k^\top(-t), \quad (31a)$$

$$\hat{B}_k(t) := C_k^\top(-t), \quad 0 \leq k \leq r \quad (31b)$$

$$\hat{M}(t) := M(-t) \quad (31c)$$

$$\hat{L}(t) := L(-t) \quad (31d)$$

$$\hat{R}(t) := R(-t), \quad t \in \mathbb{R}. \quad (31e)$$

The backward matrix Riccati type differential equation (30)-(31) will be named the dual BMRDE associated to the FMRDE (6). The BMRDE (30) is a kind of matrix Riccati differential equation involved in the solution of a problem of stochastic linear quadratic optimal control.

For the reader convenience we recall:

**Definition 4.** A solution  $X_s(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  of BMRDE (30) is named stabilizing solution if the zero solution of the corresponding closed-loop stochastic linear differential equation:

$$dx(t) = (\hat{A}_0(t) + \hat{B}_0(t)F_s(t))x(t)dt + \sum_{k=1}^r (\hat{A}_k(t) + \hat{B}_k(t)F_s(t))x(t)dw_k(t), \quad (32)$$

$t \in \mathbb{R}_+$ , is ESMS, where

$$F_s(t) := - \left( \hat{R}(t) + \sum_{k=1}^r \hat{B}_k^\top(t)X_s(t)\hat{B}_k(t) \right)^{-1} \cdot \left( \hat{B}_0^\top(t)X_s(t) + \sum_{k=1}^r \hat{B}_k^\top(t)X_s(t)\hat{A}_k(t) + \hat{L}^\top(t) \right), \quad t \in \mathbb{R} \quad (33)$$

The problem of the existence of the stabilizing and bounded on  $\mathbb{R}$  solution for a BMRDE of type (30) was studied in Chapter 5 from [12].

## 4.2 An auxiliary result

Let us consider the following two SLDEs:

$$dx(t) = M_0(t)x(t)dt + \sum_{k=1}^r M_k(t)x(t)dw_k(t) \quad (34)$$

and

$$dx(t) = \hat{M}_0(t)x(t)dt + \sum_{k=1}^r \hat{M}_k(t)x(t)dw_k(t) \quad (35)$$

where  $M_k(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ ,  $0 \leq k \leq r$  are continuous and  $\theta$ -periodic matrix valued functions and

$$\hat{M}_k(t) := M_k^\top(-t), \quad 0 \leq k \leq r. \quad (36)$$

We have:

**Lemma 1.** *The following are equivalent:*

- (i) the zero solution of the SLDE (34) is ESMS;
- (ii) the zero solution of the SLDE (35) is ESMS.

*Proof.* If the zero solution of the SLDE (34) is ESMS then, based on the implication (i)  $\Rightarrow$  (v) from Proposition 3.3 from [8] applied in the case of the equation (34), we deduce that the non-homogeneous forward Lyapunov type linear differential equation:

$$\dot{X}(t) = M_0(t)X(t) + X(t)M_0^\top(t) + \sum_{k=1}^r M_k(t)X(t)M_k^\top(t) + I_n \quad (37)$$

has a unique solution  $\tilde{X}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  which is a  $\theta$ -periodic function and is uniform positive on  $\mathbb{R}$ . We set

$$\tilde{Y}(t) \triangleq \tilde{X}(-t), \quad t \in \mathbb{R}.$$

By direct calculations involving (36) and (37) we obtain that  $\tilde{Y}(\cdot)$  is unique uniform positive definite on  $\mathbb{R}$  and  $\theta$ -periodic solution of the following non-homogeneous backward Lyapunov type linear differential equation:

$$-\dot{\tilde{Y}}(t) = \widehat{M}_0^\top(t)\tilde{Y}(t) + \tilde{Y}(t)\widehat{M}_0(t) + \sum_{k=1}^r \widehat{M}_k^\top(t)\tilde{Y}(t)\widehat{M}_k(t) + I_n. \quad (38)$$

Invoking the implication (iv)  $\Rightarrow$  (i) from Proposition 3.3 from [8] in the particular case of the SLDE (35) and of non-homogeneous Lyapunov type differential equation (38) we may conclude that the zero solution of the SLDE (35) is ESMS. Thus we have checked the validity of the implication (i)  $\Rightarrow$  (ii) from the statement.

Conversely, if (ii) holds, then the implication (i)  $\Rightarrow$  (iii) from Proposition 3.3 in [8] applied in the case of the SLDE (35), we deduce that the non-homogeneous backward Lyapunov type matrix linear differential equation (38) has a unique solution  $\tilde{Y}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  which is a  $\theta$ -periodic function and satisfies

$$\tilde{Y}(t) > 0, \quad \forall t \in \mathbb{R}.$$

Setting

$$\tilde{X}(t) = \tilde{Y}(-t), \quad t \in \mathbb{R},$$

we obtain via (36) and (38) that  $\tilde{X}(\cdot)$  is the unique uniform positive and  $\theta$ -periodic solution of the non-homogeneous forward Lyapunov type matrix linear differential equation (37). Based on the implication (vi)  $\Rightarrow$  (i) from Proposition 3.3 in [8] we deduce that the zero solution of the SLDE (34) is ESMS. Thus the proof is complete.  $\square$

The equivalence from Lemma 1 applied in the case of the pair of SLDE (7)-(8) and (32)-(33), respectively, gives:

**Corollary 1.** *Under the assumptions  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$ ,  $Y_s(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  is the bounded on  $\mathbb{R}$  and stabilizing solution of the FMRDE (6) if and only if  $X_s(\cdot)$  defined by*

$$X_s(t) = Y_s(-t), \quad t \in \mathbb{R}$$

*is bounded on  $\mathbb{R}$  and stabilizing solution of its dual BMRDE (30)-(31).*

*Proof.* Taking

$$M_k(t) \triangleq A_k(t) + K_s(t)C_k(t), \quad 0 \leq k \leq r,$$

$K_s(t)$  being defined in (8) and

$$\widehat{M}_k(t) \triangleq \widehat{A}_k(t) + \widehat{B}_k(t)F_s(t), \quad 0 \leq k \leq r,$$

with  $F_s(t)$  defined by (33) we obtain via (31) that

$$\widehat{M}_k(t) = M_k^\top(-t), \quad 0 \leq k \leq r, \quad t \in \mathbb{R}.$$

The conclusion is obtained from Lemma 1 applied to the SLDEs (7) and (32), respectively.  $\square$

So, conditions which guaranty the existence and some useful properties of the bounded and stabilizing solution of the FMRDE (6) can be obtained applying to the dual FMRDE (30)-(31) some results from Chapter 5 in [12].

### 4.3 On the bounded and maximal solution of the FMRDE

Let us remark that a FMRDE of type (6) is defined by the following triple  $\Sigma = \{\mathcal{A}(\cdot), \mathcal{C}(\cdot), \mathcal{Q}(\cdot)\}$  where  $\mathcal{A}(\cdot)$ ,  $\mathcal{C}(\cdot)$  are those from (2a), (2c) and

$$\mathcal{Q}(\cdot) = \begin{pmatrix} M(\cdot) & L(\cdot) \\ L^\top(\cdot) & R(\cdot) \end{pmatrix} = \mathcal{Q}^\top(\cdot). \quad (39)$$

To the triple  $\Sigma$  we associate the set of functions  $\Gamma^\Sigma$  formed by all functions  $Z(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  continuously differentiable,  $\theta$ -periodic satisfying the following conditions:

$$\begin{aligned} & \begin{pmatrix} -\dot{Z}(t) + A_0(t)Z(t) + Z(t)A_0^\top(t) + M(t) & Z(t)C_0^\top(t) + L(t) \\ C_0(t)Z(t) + L^\top(t) & R(t) \end{pmatrix} \\ & + \sum_{k=1}^r \begin{pmatrix} A_k(t) \\ C_k(t) \end{pmatrix} Z(t) \begin{pmatrix} A_k(t) \\ C_k(t) \end{pmatrix}^\top \geq 0 \end{aligned} \quad (40)$$

and

$$R(t) + \sum_{k=1}^r C_k(t)Z(t)C_k^\top(t) > 0, \quad t \in \mathbb{R}. \quad (41)$$

A crucial role in the development from this section will be played by the subset  $\tilde{\Gamma}^\Sigma \subset \Gamma^\Sigma$  formed by all functions  $Z(\cdot) \in \Gamma^\Sigma$  which are satisfying

$$\begin{pmatrix} -\dot{Z}(t) + A_0(t)Z(t) + Z(t)A_0^\top(t) + M(t) & Z(t)C_0^\top(t) + L(t) \\ C_0(t)Z(t) + L^\top(t) & R(t) \end{pmatrix} + \sum_{k=1}^r \begin{pmatrix} A_k(t) \\ C_k(t) \end{pmatrix} Z(t) \begin{pmatrix} A_k(t) \\ C_k(t) \end{pmatrix}^\top > 0, \quad t \in \mathbb{R}. \quad (42)$$

**Remark 4.** (a) The inequalities from (41) and (42) are uniform with respect to  $t \in \mathbb{R}$  due to the periodicity property of the matrix valued functions arising in these inequalities.

(b) Employing the Schur complement technique, one may shown that the set  $\Gamma^\Sigma$  contains all  $\theta$ -periodic solutions  $Y(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  of the FMRDE (6) which are satisfying the condition:

$$R(t) + \sum_{k=1}^r C_k(t)Y(t)C_k^\top(t) > 0, \quad t \in \mathbb{R}. \quad (43)$$

(c) From (40), (41) we obtain via the Schur complement technique that  $Z(t) \equiv 0 \in \Gamma^\Sigma$  if and only if

$$R(t) > 0, \quad (44)$$

$$M(t) - L(t)R^{-1}(t)L^\top(t) \geq 0, \quad t \in \mathbb{R}. \quad (45)$$

The dual BMRDE (30) is associated to the dual triple

$$\Sigma^\# = (\mathcal{A}^\#(\cdot), \mathcal{B}^\#(\cdot), \mathcal{Q}^\#(\cdot)), \quad (46)$$

where

$$\begin{aligned} \mathcal{A}^\#(\cdot) &= (\hat{A}_0(\cdot), \hat{A}_1(\cdot), \dots, \hat{A}_r(\cdot)), \\ \mathcal{B}^\#(\cdot) &= (\hat{B}_0(\cdot), \widehat{AB}_1(\cdot), \dots, \hat{B}_r(\cdot)) \\ \mathcal{Q}^\#(\cdot) &= \begin{pmatrix} \widehat{M}(\cdot) & \widehat{L}(\cdot) \\ \widehat{L}^\top(\cdot) & \widehat{R}(\cdot) \end{pmatrix} \end{aligned} \quad (47)$$

where  $\hat{A}_k(\cdot)$ ,  $\hat{B}_k(\cdot)$ ,  $0 \leq k \leq r$ , and  $\hat{M}(\cdot)$ ,  $\hat{L}(\cdot)$ ,  $\hat{R}(\cdot)$  were defined in (31). To the triple  $\Sigma^\#$  we associate the set  $\Gamma^{\Sigma^\#}$  formed by all continuously differentiable and  $\theta$ -periodic matrix valued functions  $\hat{Z}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  which are satisfying:

$$\begin{pmatrix} \dot{\hat{Z}}(\cdot) + \hat{A}_0^\top(t)\hat{Z}(t) + \hat{Z}(t) + \hat{A}_0(t) + \hat{M}(t) & \hat{Z}(t)\hat{B}_0(t) + \hat{L}(t) \\ \hat{B}_0^\top(t)\hat{Z}(t) + \hat{L}^\top(t) & \hat{R}(t) \end{pmatrix} + \sum_{k=1}^r (\hat{A}_k(t) \hat{B}_k(t))^\top \hat{Z}(t) (\hat{A}_k(t) \hat{B}_k(t)) \geq 0 \quad (48)$$

$$\hat{R}(t) + \sum_{k=1}^r \hat{B}_k^\top(t)\hat{Z}(t)\hat{B}_k(t) > 0, \quad (49)$$

$t \in \mathbb{R}$ . The set of functions  $\Gamma^{\Sigma^\#}$  is the analogous of the set  $\Gamma^\Sigma$  introduced in Section 5.1 in [12] being associated to the particular case of the BMRDE (30).

**Remark 5.** From (40), (41) on one hand and (48), (49) on the other hand, one obtains via (31) that a function  $Z(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  lies in  $\Gamma^\Sigma$  if and only if the function  $\hat{Z}(\cdot)$  defined by  $\hat{Z}(t) = Z(-t)$ ,  $t \in \mathbb{R}$  lies in  $\Gamma^{\Sigma^\#}$ .

**Definition 5.** (a) A solution  $Y_{\max}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  of the FMRDE (6) is named maximal solution with respect to the set  $\Gamma^\Sigma$  or maximal solution (for shortness) of (6) if

$$Y_{\max}(t) \geq Z(t),$$

$t \in \mathbb{R}$ , for any  $Z(\cdot) \in \Gamma^\Sigma$ .

(b) A solution  $X_{\max}(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  of the BMRDE (30) is named maximal solution with respect to the set  $\Gamma^{\Sigma^\#}$  or maximal solution (for shortness) of (30) if

$$X_{\max}(t) \geq \hat{Z}(t),$$

$t \in \mathbb{R}$ , for any  $\hat{Z}(\cdot) \in \Gamma^{\Sigma^\#}$ .

**Remark 6.** According to Remark 5 and Definition 5 it follows that  $Y_{\max}(\cdot)$  is the maximal solution of the FMRDE (35) if and only if  $X_{\max}(\cdot)$  defined by

$$X_{\max}(t) = Y_{\max}(-t),$$

$t \in \mathbb{R}$ , is the maximal solution of its dual BMRDE (30)-(31). Moreover, if it exists, the maximal solution  $Y_{\max}(\cdot)$  of (6) is unique.

**Theorem 2.** *Assume:*

- (a) *the assumptions  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$  hold;*
- (b) *the pair  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  is detectable.*

*Under these conditions, the following are equivalent:*

- (i) *the set  $\Gamma^\Sigma$  is not empty;*
- (ii) *the FMRDE (6) has a bounded and maximal solution  $Y_{max} : \mathbb{R} \rightarrow \mathcal{S}_n$  which is satisfying the sign condition:*

$$R(t) + \sum_{k=1}^r C_k(t) Y_{max}(t) C_k^\top(t) > 0, \quad t \in \mathbb{R}. \quad (50)$$

*Moreover  $Y_{max}(\cdot)$  is a periodic function with period  $\theta$ .*

*Proof.* The implication  $(ii) \Rightarrow (i)$  is obvious because according to Remark 4 (d), the maximal solution  $Y_{max}(\cdot)$ , if any, lies in  $\Gamma^\Sigma$ . To check the validity of the implication  $(i) \Rightarrow (ii)$  we invoke the implication  $(i) \Rightarrow (ii)$  from Theorem 5.3.5 in [12] applied in the case of the dual BMRDE (30)-(31).

First, let us remark that according to Proposition 4.2 from [8], the pair  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  is detectable if and only if the dual pair  $(\mathcal{A}^\#(\cdot), \mathcal{B}^\#(\cdot))$  is stabilizable. On the other hand based on Remark 5 we may infer that the set  $\Gamma^{\Sigma^\#}$  is nonempty if the set  $\Gamma^\Sigma$  is not empty. So, we can apply  $(i) \Rightarrow (ii)$  of Theorem 5.3.5 [12] in the case of BMRDE (30)-(31), to deduce that this equation has a maximal solution  $X_{max} : \mathbb{R} \rightarrow \mathcal{S}_n$  which is a  $\theta$ -periodic function and satisfies the sign condition:

$$\hat{R}(t) + \sum_{k=1}^r \hat{B}_k^\top(t) X_{max}(t) \hat{B}_k(t) > 0, \quad t \in \mathbb{R}. \quad (51)$$

Then, invoking Remark 6, we may conclude that  $Y_{max}(\cdot)$  defined by

$$Y_{max}(t) \triangleq X_{max}(-t), \quad t \in \mathbb{R}$$

is the maximal solution of the FMRDE (6). Replacing  $t$  by  $-t$  in (51) we obtain that  $Y_{max}(\cdot)$  is satisfying (50) too. Thus, the proof ends.  $\square$

#### 4.4 The bounded and stabilizing solution of the FMRDE (6)

The next result establishes the relation between the maximal solution and the stabilizing solution of a FMRDE of type (6).

**Proposition 2.** *Assume:*

- (a) *the assumptions  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$  hold;*
- (b) *the set  $\Gamma^\Sigma$  is not empty.*

*Under these conditions, the following are true:*

- (i) *the stabilizing solution  $Y_s(\cdot)$  of the FMRDE (6) if any coincides to its maximal solution  $Y_{max}(\cdot)$ ;*
- (ii) *the bounded on  $\mathbb{R}$  and stabilizing solution of the FMRDE (6) if any is unique and it is  $\theta$ -periodic function.*

*Proof.* Let  $Y_s(\cdot)$  be the bounded and stabilizing solution of the FMRDE (6). According to Corollary 1,  $X_s(\cdot)$  defined by  $X_s(t) \triangleq Y_s(-t)$ ,  $t \in \mathbb{R}$  is the bounded and stabilizing solution of the dual BMRDE (30)-(31). Let  $Z(\cdot) \in \Gamma^\Sigma$ . Based on Remark 5, the function  $\hat{Z}(\cdot)$  defined by  $\hat{Z}(t) \triangleq Z(-t)$ ,  $t \in \mathbb{R}$ , lies in  $\Gamma^\Sigma$ . Employing Theorem 5.4.1 from [12] in the case of the pair formed by the BMRDE (30)-(31) and the set  $\Gamma^\Sigma$ , we obtain that  $X_s(\cdot)$  coincides to the maximal solution  $X_{max}(\cdot)$  of (30)-(31). Finally, employing Corollary 1 and Remark 6, we conclude that

$$Y_s(t) = Y_{max}(t), \quad t \in \mathbb{R},$$

which confirms (i) from the statement. The validity of the assertion (ii) is obtained invoking (ii) from Theorem 2 together with the previous equality.  $\square$

The next result provides a set of necessary and sufficient conditions which are guarantying the existence of the bounded on  $\mathbb{R}$  and stabilizing solution of the FMRDE (6).

**Theorem 3.** *Under the assumptions  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$  the following are equivalent:*

- (i) *the pair  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  is detectable and the set  $\tilde{\Gamma}^\Sigma$  is not empty;*

(ii) the FMRDE (6) has a bounded on  $\mathbb{R}$  and stabilizing solution  $Y_s(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  which is satisfying the sign condition:

$$R(t) + \sum_{k=1}^r C_k(t) Y_s(t) C_k^\top(t) > 0, \quad t \in \mathbb{R}. \quad (52)$$

*Proof.* To show that the implication (i)  $\Rightarrow$  (ii) holds, we apply the implication (i)  $\Rightarrow$  (ii) from Theorem 5.4.6 in [12], in the particular case of the BMRDE (30)-(31). First, let us remark that if  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  is detectable, then the dual pair  $(\mathcal{A}^\#(\cdot), \mathcal{B}^\#(\cdot))$  is stabilizable,  $\mathcal{A}^\#(\cdot)$ ,  $\mathcal{B}^\#(\cdot)$  being introduced in (46). On the other hand, if  $Z(\cdot) \in \tilde{\Gamma}^\Sigma$  one obtains directly from (31) and (42) that  $\hat{Z}(\cdot)$  defined by  $\hat{Z}(t) \triangleq Z(-t)$ ,  $t \in \mathbb{R}$ , lies in  $\tilde{\Gamma}^{\Sigma^\#}$ . So, the conditions from the statement (i) from Theorem 5.4.6 [12] particularized to the case of the BMRDE (30)-(31) are fulfilled. Hence, the implication (i)  $\Rightarrow$  (ii) from Theorem 5.4.6 from the afore mentioned reference allows us to deduce that under the considered assumption the BMRDE (30)-(31) has a unique bounded and stabilizing solution  $X_s(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  which is satisfying the sign condition

$$\hat{R}(t) + \sum_{k=1}^r \hat{B}_k^\top(t) X_s(t) \hat{B}_k(t) > 0, \quad t \in \mathbb{R}. \quad (53)$$

Employing Corollary 1 we may infer that  $Y_s(\cdot)$  defined by

$$Y_s(\cdot) \triangleq X_s(-t), \quad t \in \mathbb{R}$$

is just the bounded and stabilizing solution of the FMRDE (6). Writing (53) for  $t$  replaced by  $-t$ , we obtain that  $Y_s(\cdot)$  is satisfying the sign condition (52) too. Thus, the validity of the implication (i)  $\Rightarrow$  (ii) is confirmed.

Let us show that the implication (ii)  $\Rightarrow$  (i) holds too. If  $Y_s(\cdot)$  is the bounded and stabilizing solution of the FMRDE (6) which is satisfying the sign condition (52), it follows that  $X_s(\cdot)$  defined by

$$X_s(t) = Y_s(-t), \quad t \in \mathbb{R},$$

is the bounded and stabilizing solution of the BMRDE (30)-(31) which is satisfying the sign condition (53). Based on the implication (ii)  $\Rightarrow$  (i) from Theorem 5.4.6 in [12], applied in the case of the BMRDE (30)-(31) we deduce that the pair  $(\mathcal{A}^\#(\cdot), \mathcal{B}^\#(\cdot))$  is stabilizable and the set  $\tilde{\Gamma}^{\Sigma^\#}$  is not empty. Based on Proposition 4.2 in [8] we may infer that the pair  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  is detectable if  $(\mathcal{A}^\#(\cdot), \mathcal{B}^\#(\cdot))$  is stabilizable. On the other hand, if  $\hat{Z}(\cdot) \in \tilde{\Gamma}^{\Sigma^\#}$  one obtains that the function  $Z(\cdot)$  defined by  $Z(t) := \hat{Z}(-t)$ ,  $t \in \mathbb{R}$  lies in  $\tilde{\Gamma}^\Sigma$ . Thus we have shown that the statement (i) holds if (ii) is true. The proof ends.  $\square$

#### 4.5 An iterative procedure for the numerical computation of the maximal solution and of the stabilizing solution of the FMRDE (6).

As it can be seen from the previous developments, we do not know apriori neither its initial value nor a boundary value of the maximal solution and/or the stabilizing solution of a FMRDE. That is why, the existing procedures for the numerical computation of the solution of Cauchy problem or a solution of a boundary value problem cannot be used to compute the maximal solution or the stabilizing solution of a FMRDE.

In this subsection, we propose an iterative procedure for numerical computation of the maximal solution and of the stabilizing solution, respectively, of the FMRDE (6). The first step in the construction of the iterative procedure is described in the following lemma:

**Lemma 2.** *Assume that  $(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  is detectable. Let  $\tilde{K}_0(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times q}$  be a continuous and  $\theta$ -periodic matrix valued function with the property that the zero solution of the corresponding resulting system:*

$$dx(t) = (A_0(t) + \tilde{K}_0(t)C_0(t))x(t)dt + \sum_{k=1}^r (A_k(t) + \tilde{K}_0(t)C_k(t))x(t)dw_k(t) \quad (54)$$

is ESMS. Let  $Y_0(\cdot) : \mathbb{R} \rightarrow \mathcal{S}_n$  be a continuous differential matrix valued function which is  $\theta$ -periodic and solves the following linear matrix differential inequality (LMDI):

$$\begin{aligned} & -\dot{Y}_0(t) + (A_0(t) + \tilde{K}_0(t)C_0(t))Y_0(t) + Y_0(t)(A_0(t) + \tilde{K}_0(t)C_0(t))^\top \\ & + \sum_{k=1}^r (A_k(t) + \tilde{K}_0(t)C_k(t))Y_0(t)(A_k(t) + \tilde{K}_0(t)C_0(t))^\top + M_0(t) \leq 0 \end{aligned} \quad (55)$$

where

$$M_0(t) := M(t) + \varepsilon I_n + L(t)\tilde{K}_0^\top(t) + \tilde{K}_0(t)L^\top(t) + \tilde{K}_0(t)R(t)\tilde{K}_0^\top(t), \quad t \in \mathbb{R}.$$

Under these conditions we have:

$$Y_0(t) - Z(t) \geq \gamma I_n \quad (56)$$

for all  $t \in \mathbb{R}$ , and any  $Z(\cdot) \in \Gamma^\Sigma$ .

*Proof.* Let  $\tilde{F}_0(t) := \tilde{K}_0^\top(-t)$ ,  $t \in \mathbb{R}$ . Applying Lemma 1 with the updates  $M_k(t) \leftarrow A_k(t) + \tilde{K}_0(t)C_k(t)$ ,  $0 \leq k \leq r$ , we obtain that the zero solution of the SLDE

$$dx(t) = (\hat{A}_0(t) + \hat{B}_0(t)\tilde{F}_0(t))x(t)dt + \sum_{k=1}^r (\hat{A}_k(t) + \hat{B}_k(t)\tilde{F}_0(t))x(t)dw_k(t),$$

is ESMS if and only if the zero solution of (54) is ESMS,  $\hat{A}_k(\cdot)$ ,  $\hat{B}_k(\cdot)$  being defined in (31a), (31b), respectively.

If we set  $X_0(t) := Y_0(-t)$ , we obtain that  $X(\cdot)$  satisfies

$$\begin{aligned} \dot{X}_0(t) + (\hat{A}_0(t) + \hat{B}_0(t)\tilde{F}_0(t))^\top X_0(t) + X_0(t)(\hat{A}_0(t) + \hat{B}_0(t)\tilde{F}_0(t)) \\ + \sum_{k=1}^r (\hat{A}_k(t) + \hat{B}_k(t)\tilde{F}_0(t))^\top X_0(t)(\hat{A}_k(t) + \hat{B}_k(t)\tilde{F}_0(t)) + \hat{M}_0(t) \leq 0, \quad t \in \mathbb{R}, \end{aligned} \quad (57)$$

where

$$\hat{M}_0(t) := \hat{M}(t) + \varepsilon I_n + \hat{L}(t)\tilde{F}_0(t) + \tilde{F}_0^\top(t)\hat{L}^\top(t) + \tilde{F}_0^\top(t)\hat{R}(t)\tilde{F}_0(t),$$

$\hat{M}(\cdot)$ ,  $\hat{L}(\cdot)$ ,  $\hat{R}(\cdot)$  being defined in (31c)-(31e). Applying Lemma 5.8.1 from [12] in the case of (57), we obtain via Remark 5 that

$$X_0(t) - Z(-t) \geq \gamma I_n,$$

for all  $t \in \mathbb{R}$  and for any  $Z(\cdot) \in \Gamma^\Sigma$ . The conclusion is obtained setting  $-t$  instead of  $t$  in the last inequality.  $\square$

From (56) we obtain via (41) that

$$R(t) + \sum_{k=1}^r C_k(t)Y_0(t)C_k^\top(t) > 0, \quad t \in \mathbb{R}.$$

Hence, we may define:

$$\begin{aligned} K_0(t) := - \left( Y_0(t)C_0^\top(t) + \sum_{k=1}^r A_k(t)Y_0(t)C_k^\top(t) + L(t) \right) \\ \cdot \left( R(t) + \sum_{k=1}^r C_k(t)Y_0(t)C_k^\top(t) \right)^{-1}, \quad t \in \mathbb{R}. \end{aligned} \quad (58)$$

Employing again (56) together with (40), (41), (55), (58), we obtain:

**Corollary 2.** *If  $K_0(t)$  is computed via (58), based on a  $\theta$ -periodic solution of the LMDI (55), then the zero solution of the SLDE*

$$dx(t) = (A_0(t) + K_0(t)C_0(t))x(t)dt + \sum_{k=1}^r (A_k(t) + K_0(t)C_k(t))x(t)dw_k(t)$$

*is ESMS. Particularly, the zero solution of the linear differential equation on  $\mathbb{R}^n$*

$$\dot{x}(t) = (A_0(t) + K_0(t)C_0(t))x(t),$$

*is exponentially stable.*

Taking  $Y_0(\cdot)$ ,  $K_0(\cdot)$  as first terms, we construct the sequences  $\{Y_p(t)\}_{p \geq 1}$ ,  $\{K_p(t)\}_{p \geq 1}$  as:

- (a)  $Y_p(\cdot)$  is the unique  $\theta$ -periodic solution of the following Lyapunov differential equation:

$$\begin{aligned} \dot{Y}_p(t) = & (A_0(t) + K_{p-1}(t)C_0(t))Y_p(t) \\ & + Y_p(t)(A_0(t) + K_{p-1}(t)C_0(t))^\top + M_p(t) \end{aligned} \quad (59)$$

where

$$\begin{aligned} M_p(t) = & M(t) + \frac{\varepsilon}{p+1}I_n + L(t)K_{p-1}^\top(t) \\ & + K_{p-1}(t)L^\top(t) + K_{p-1}(t)R(t)K_{p-1}^\top(t) \\ & + \sum_{k=1}^r (A_k(t) + K_{p-1}(t)C_k(t))Y_{p-1}(t)(A_k(t) + K_{p-1}(t)C_k(t))^\top. \end{aligned} \quad (60)$$

- (b) the gain matrices  $K_p(t)$  are defined by:

$$\begin{aligned} K_p(t) := & - \left( Y_p(t)C_0^\top(t) + \sum_{k=1}^r A_k(t)Y_{p-1}(t)C_k^\top(t) + L(t) \right) \\ & \cdot \left( R(t) + \sum_{k=1}^r C_k(t)Y_{p-1}(t)C_k^\top(t) \right)^{-1}, \quad t \in \mathbb{R}, \quad p \geq 1. \end{aligned} \quad (61)$$

**Theorem 4.** *Assume:*

- (a) *the assumptions  $(\mathbf{H}_1)$ ,  $(\mathbf{H}_2)$  hold;*  
 (b)  *$(\mathcal{C}(\cdot), \mathcal{A}(\cdot))$  is detectable;*

(c) the set  $\Gamma^\Sigma$  is not empty.

Under these conditions, for any continuous and  $\theta$ -periodic matrix valued functions  $K_0(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times q}$  with the property that the zero solution of the corresponding resulting SLDE (54), is ESMS, the sequences  $\{Y_p(t)\}_{p \geq 1}$ ,  $\{K_p(t)\}_{p \geq 1}$ ,  $t \in \mathbb{R}$  constructed via (59)-(60) and (61), respectively, starting from an arbitrary  $\theta$ -periodic solution of the LMDI (55), are convergent. If

$$\tilde{Y}(t) := \lim_{t \rightarrow \infty} Y_p(t), \quad t \in \mathbb{R}, \quad (62)$$

then  $\tilde{Y}(\cdot)$  is the maximal and bounded on  $\mathbb{R}$  solution of the FMRDE (6) which is satisfying the sign condition (50).

If the set  $\tilde{\Gamma}^\Sigma$  is not empty, then  $\tilde{Y}(\cdot)$  defined in (62) is just the bounded on  $\mathbb{R}$  and stabilizing solution of the FMRDE (6) which is satisfying the sign condition (52).

*Proof. Hint.* One shows inductively, with respect to  $p$ , that the terms of the sequence  $\{Y_p(t)\}_{p \geq 1}$  have the properties:

(a<sub>p</sub>)

$$Y_p(t) - Z(t) \geq \gamma I_n, \quad t \in \mathbb{R}, \quad (63)$$

for any  $Z(\cdot) \in \Gamma^\Sigma$ , where  $\gamma > 0$  does not depend upon  $p$ , but it may depends upon the function  $Z(\cdot)$ ;

(b<sub>p</sub>) the zero solution of the linear differential equation on  $\mathbb{R}^n$  :

$$\dot{x}(t) = (A_0(t) + K_0(t)C_0(t))x(t) \quad (64)$$

is exponentially stable;

(c<sub>p</sub>)

$$Y_p(t) \geq Y_{p+1}(t), \quad t \in \mathbb{R}. \quad (65)$$

The exponential stability of the zero solution of the equation (64) guaranties the existence of the unique  $\theta$ -periodic solution  $Y_p(\cdot)$  of the Lyapunov differential equation (59)-(60). The condition (63) together with (41) guaranty that  $K_p(\cdot)$  are well defined via (61). Bearing in mind that (63) and (65) hold for any  $p \geq 1$  and  $t \in \mathbb{R}$ , we may infer that  $\{Y_p(t)\}_{p \geq 1}$  is point-wise convergent. Employing dominant convergence theorem one shows that  $\tilde{Y}(\cdot)$  defined by (62) is a solution of the FMRDE (6). Letting  $p \rightarrow \infty$  in (63) we deduce that

$$\tilde{Y}(t) \geq Z(t),$$

for all  $t \in \mathbb{R}$ , and any  $Z(\cdot) \in \Gamma^\Sigma$ . This means that  $\tilde{Y}(\cdot)$  defined by (62) coincides to the maximal solution of the FMRDE (6). If  $\tilde{\Gamma}^\Sigma$  is not empty, then, from Proposition 2 and Theorem 3, we deduce that  $\tilde{Y}(\cdot)$  is just the stabilizing solution of (6). Thus the proof is complete.  $\square$

**Remark 7.** Since all functions  $Y_p(\cdot)$ ,  $K_p(\cdot)$ ,  $p \geq 0$  are  $\theta$ -periodic functions, it is sufficient to compute them on an interval  $[t_0, t_0 + \theta]$ .

## References

- [1] H. Abou-Kandil, G. Freiling, V. Ionescu and G. Jank, *Matrix Riccati Equations in Control Systems Theory*, Birkhauser, Basel, 2003.
- [2] B.D.O. Anderson and J.B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice-Hall, Englewood Cliffs, 1989.
- [3] E.M. Barbashin and N.N. Krasovskii, Stability of motion in the large, *Dokl. Akad. Nauk.* 86 (1952), 453-456.
- [4] S. Bittanti and P. Colaneri, *Periodic Systems, Filtering and Control*, Springer-Verlag, London, 2009.
- [5] O.L.V. Costa, M.D. Fragoso and R.P. Marques, *Discrete-Time Markov Jump Linear Systems*, Springer, 2005.
- [6] O.L.V. Costa, M.D. Fragoso and M.G. Todorov, *Continuous-Time Markov Jump Linear Systems*, Springer-Verlag Berlin Heidelberg, 2013.
- [7] J.C. Doyle, K. Glover, P.P. Khargonekar and B.A. Francis, State space solutions to standard  $H_2$  and  $H_\infty$  control problems, *IEEE Trans. Automatic Control* 34 (1989), 831-847.
- [8] V. Drăgan, Duality: detectability versus stabilizability in the stochastic context, *Ann. Acad. Rom. Sci., Serie Math. Appl.* 16 (2024), 287-310.
- [9] V. Drăgan, Optimal filtering for discrete-time linear systems with multiplicative white noise perturbations and periodic coefficients, *IEEE Trans. Autom. Control.* 58 (2012), 1029-1034.
- [10] V. Drăgan and T. Morozan, Global solutions to a game theoretic Riccati equation of stochastic control, *J. Differ. Equations* 138 (1997), 328-350.

- [11] V. Drăgan, G. Freiling, A. Hochhaus and T. Morozan, A class of nonlinear differential equations on the space of symmetric matrices, *Electron. J. Differ. Equations* 96 (2004), 1-48.
- [12] V. Drăgan, T. Morozan and A.M. Stoica, *Mathematical Methods in Robust Control of Linear Stochastic Systems*, Second edition, Springer, New York, 2013.
- [13] A. Friedman, *Stochastic Differential Equations and Applications*, vol. I, Academic Press, New York, 1975.
- [14] A. Halanay and V. Ionescu, *Time Varying Discrete Linear Systems*, Berlin, Birkhauser, 1994.
- [15] A. Ichikawa and H. Katayama, *Linear Time-Varying Systems and Sampled-Data*, Springer, 2001.
- [16] R.E. Kalman, Contributions to the theory of optimal control, *Bol. Soc. Aht. Mexicana* 5 (1960), 102-119.
- [17] R.E. Kalman and R.S. Bucy, New results in linear filtering and prediction theory, *ASME Trans. Part D, J. Basic Eng*, 83 (1961), 95-108.
- [18] R. Khasminskii, *Stochastic Stability of Differential Equations*, Sythoff and Noordhoff, Alpen aan den Ryn, 1980.
- [19] A. Kumar Roy and S. Kannan, Event-based optimal filter for a networked system with multiplicative and auto/cross-correlated process and measurement noise, *Int. J. Adapt. Control Signal Process.* 36 (2022), 2453-2478.
- [20] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.
- [21] W. Liu, H. Zhang, K. Yu and X. Tan, Optimal linear filtering for networked systems with communication constraints, fading measurements, and multiplicative noises, *Int. J. Adapt. Control Signal Process.* 31 (2017), 1019-1039.
- [22] R. Oxendal, *Stochastic Differential Equations*, Springer, Berlin, 1998.
- [23] V.M. Popov, *Hyperstability of Control Systems*, Editura Academiei-Springer Verlag, Berlin, 1973.

- [24] R. Ravi, K.M. Nagpal and P.P. Khargonekar,  $H_\infty$  control for linear time-varying systems: a state space approach, *SIAM J. Control Optim.* 29 (1991), 1394-1413.
- [25] J. Zhang, X. He and D. Zhou, Robust optimal filtering for linear time-varying systems with stochastic uncertainties, *IET Control Theory Appl.* 11 (2017), 3297-3304.