Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 18, No. 4/2026

AN EVOLUTION EQUATION WITH COMPETING p-LAPLACIAN OPERATORS*

Dumitru Motreanu†

Dedicated to Professor Constantin Fetecau on the occasion of his 80th anniversary

DOI 10.56082/annalsarscimath.2026.4.289

Abstract

The paper investigates an initial-boundary value problem involving the competing (p,q)-Laplacian $-\Delta_p + \Delta_q$. Under some conditions, which are different from the usual ellipticity and monotonicity conditions, the existence of a generalized solution is proven through a special Galerkin approach. An example illustrates the applicability of our result.

Keywords: initial-boundary value problem, time depending competing operators, p-Laplacian, generalized solution.
MSC: 35K92, 47J35, 35K20.

1 Introduction and main result

Our aim is to study the following initial-boundary value problem

$$\begin{cases} u_t - \Delta_p u + \Delta_q u = f(x,t,u) & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{P}$$

*Accepted for publication on October 07, 2026
†motreanu@univ-perp.fr, Département de Mathématiques, Université de Perpignan Via Domitia, Perpignan, 66860, France

on a bounded domain Ω in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and with fixed real numbers $T > 0$, $p \geq 2$, $q \in (1, p)$, and a function $u_0 \in L^2(\Omega)$. The unknown in problem (P) is a function $u(x,t)$ that will be subsequently viewed as $u(t)(x)$, with $(x,t) \in \Omega \times (0,T)$. The notation u_t (or $u'(t)$) stands for the time derivative, i.e., the partial derivative of u with respect to the time variable t.

The main difficulty in dealing with problem (P) comes from the operator $-\Delta_p + \Delta_q$ that we call the competing (p,q)-Laplacian. This operator is not monotone and no ellipticity condition can be matched. We note that if $u \in W_0^{1,p}(\Omega)$ is a nonzero function and λ is a positive number, the expression

$$\langle -\Delta_p(\lambda u) + \Delta_q(\lambda u), \lambda u\rangle_{W_0^{1,p}(\Omega)} = \lambda^p \langle -\Delta_p u, u\rangle_{W_0^{1,p}(\Omega)} - \lambda^q \langle -\Delta_q u, u\rangle_{W_0^{1,q}(\Omega)}$$

is positive for λ sufficiently large and negative for λ small enough. Until now, problems involving competing operators like $-\Delta_p + \Delta_q$ have been examined only for stationary problems. Their study was initiated in [12] and after that developed in [6, 8, 11, 13, 15, 17, 20] (refer also to the recent book [10]).

Here, we consider for the first time evolution problems with competing operators. Due to the competing character, the monotonicity method of Browder and Minty as well as the theory of semigroups are not applicable (see, e.g., [9]).

Another major difficulty in treating problem (P) is the presence of the operator Δ_q which is not pseudomonotone. Indeed, if Δ_q were pseudomonotone on $W_0^{1,q}(\Omega)$, then for each sequence $\{u_n\}$ in $W_0^{1,q}(\Omega)$ with $u_n \rightharpoonup u$ in $W_0^{1,q}(\Omega)$ and

$$\limsup_{n\to\infty} \langle \Delta_q u_n, u_n - u\rangle_{W_0^{1,q}(\Omega)} \leq 0$$

it would hold $\lim_{n\to\infty} \langle \Delta_q u_n, u_n\rangle_{W_0^{1,q}(\Omega)} = \langle \Delta_q u, u\rangle_{W_0^{1,q}(\Omega)}$ or equivalently $u_n \to u$ in $W_0^{1,q}(\Omega)$ because the space $W_0^{1,q}(\Omega)$ is uniformly convex. Nevertheless, we always have

$$\limsup_{n\to\infty} \langle \Delta_q u_n, u_n - u\rangle_{W_0^{1,q}(\Omega)} = \limsup_{n\to\infty} \langle \Delta_q u_n - \Delta_q u, u_n - u\rangle_{W_0^{1,q}(\Omega)} \leq 0$$

due to the monotonicity of $-\Delta_q$. Therefore the pseudomonotonicity of Δ_q would lead to the contradiction that the weak convergence and strong convergence in $W_0^{1,q}(\Omega)$ are equivalent. We are thus prevented to use the surjectivity theorem for pseudomonotone operators relative to time-derivative (see [3, Theorem 2.152]) as well as the corresponding sub-supersolution approach (refer to [2, 4, 5, 14]).

The lower-order term in the equation of (P) is described by a Carathéodory function $f : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ (i.e., $f(\cdot,\cdot,s)$ is measurable for all

$s \in \mathbb{R}$ and $f(x,t,\cdot)$ is continuous for a.e. $(x,t) \in \Omega \times (0,T)$). Due to the lack of needed convergence of the gradient ∇u of the solution (see [1]), we don't consider the gradient dependence in the lower-order term.

We require the following growth condition:

(H) There exist a function $\sigma \in L^{\frac{p}{p-1}}(\Omega \times (0,T))$ and a constant $a \in (0, \lambda_{1,p})$ such that

$$|f(x,t,s)| \leq \sigma(x,t) + a|s|^{p-1}$$

for a.e. $(x,t) \in \Omega \times (0,T)$ and all $s \in \mathbb{R}$,

where $\lambda_{1,p}$ denotes the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$, that is

$$\lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}}. \tag{1}$$

In order to overcome the lack of ellipticity and monotonicity in problem (P) we weaken the notion of solution as follows.

Definition 1. *By a generalized solution to problem (P) we mean a function $u \in L^p(0,T;W_0^{1,p}(\Omega))$ for which $u(0) = u_0$ such that there exists the generalized derivative $u' \in L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))$ and there exists a sequence $\{u_n\}$ in $L^p(0,T;W_0^{1,p}(\Omega))$ with the properties:*

(i) *$u_n \rightharpoonup u$ in $L^p(0,T;W_0^{1,p}(\Omega))$ and $u_n(0) \rightharpoonup u_0$ and $u_n(T) \rightharpoonup u(T)$ in $L^2(\Omega)$;*

(ii) *there exists the generalized derivative $u_n' \in L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))$ for every n and $u_n' \rightharpoonup u'$ in $L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))$ as $n \to \infty$;*

(iii) *$-\Delta_p u_n + \Delta_q u_n \rightharpoonup f(\cdot,\cdot,u) - u'$ in $L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))$;*

(iv)

$$\lim_{n\to\infty} \langle u_n' - \Delta_p u_n + \Delta_q u_n, u_n - u \rangle_{L^p(0,T;W_0^{1,p}(\Omega))} = 0.$$

We state our existence result.

Theorem 1. *Under hypothesis (H), problem (P) possesses a generalized solution in the sense of Definition 1.*

The proof of Theorem 1 proceeds along a special Galerkin-type approach without any use of monotonicity, ellipticity and coerciveness, relying on a

finite dimensional differential system associated with problem (P) that is resolved through the Carathéodory theorem for ordinary differential equations. In Section 5 we illustrate the applicability of Theorem 1 by an example.

The rest of the paper consists of the following material. Section 2 comprises preliminary results. Section 3 focuses on resolving the finite dimensional differential system associated with problem (P). Section 4 sets forth the analysis of the sequence required in Definition 1. Section 5 contains the proof of Theorem 1 as well as an example.

2 Preliminary results

For any normed space E we denote by $\|\cdot\|_E$ the norm of E, by E^* its dual and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between E and E^*. The weak convergence is denoted by $\rightharpoonup$ and the convergence in norm by $\to$.

Throughout the paper, the space $W_0^{1,p}(\Omega)$ is endowed with the norm $\|u\|_{W_0^{1,p}} = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}$. The gradient of u with respect to $x \in \Omega$ is denoted ∇u, while $|\nabla u(x,t))|$ represents the Euclidean norm of the vector $\nabla u(x,t) \in \mathbb{R}^N$. The dual of $W_0^{1,p}(\Omega)$ is the space $W^{-1,\frac{p}{p-1}}(\Omega)$. Noting that $p \geq 2$, we have the evolution triple

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,\frac{p}{p-1}}(\Omega) \tag{2}$$

with each embedding dense and compact. Recall that $L^2(\Omega)$ is a Hilbert space with the scalar product $(u,v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx$.

The space $X := L^p(0,T;W_0^{1,p}(\Omega))$ is equipped with the norm $\|u\|_X := \big(\int_0^T \|u(t)\|^p_{W_0^{1,p}} dt\big)^{\frac{1}{p}}$ making it a separable and uniformly convex Banach space. The dual of X is $X^* = L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))$. There is a dense and continuous embedding

$$C([0,T],W_0^{1,p}(\Omega)) \subset X. \tag{3}$$

Moreover, the set P of polynomials $\sum_{i=0}^m t^i v_i$ with $v_i \in W_0^{1,p}(\Omega)$ and $m \geq 0$, is dense in X, that is,

$$\overline{P} = X. \tag{4}$$

In the statement of problem (P) there appears the time-dependent neg-

ative p-Laplacian $-\Delta_p : X \to X^*$ given by

$$\begin{aligned} \langle -\Delta_p u, v\rangle_X &= \int_0^T \langle -\Delta_p u(t), v(t)\rangle_{W_0^{1,p}(\Omega)} dt \\ &= \int_0^T \int_\Omega |\nabla u(x,t)|^{p-2}\nabla u(x,t)\nabla v(x,t) dxdt \end{aligned} \tag{5}$$

for all $u, v \in X$. The operator $-\Delta_p$ in (5) is the time realization on the space X of the ordinary negative p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,\frac{p}{p-1}}(\Omega)$. The fact that the operator in (5) is well defined is seen by

$$\int_0^T \| -\Delta_p u(t)\|^{\frac{p}{p-1}}_{W^{-1,\frac{p}{p-1}}(\Omega)} dt = \int_0^T \|u(t)\|^p_{W_0^{1,p}(\Omega)} dt \quad \text{for all } u \in X.$$

Likewise, the time-dependent negative q-Laplacian

$$-\Delta_q : L^q(0,T; W_0^{1,q}(\Omega)) \to L^{\frac{q}{q-1}}(0,T; W^{-1,\frac{q}{q-1}}(\Omega))$$

is defined by

$$\begin{aligned} \langle -\Delta_q u, v\rangle_{L^q(0,T;W_0^{1,q}(\Omega))} &= \int_0^T \langle -\Delta_q u(t), v(t)\rangle_{W_0^{1,q}(\Omega)} dt \\ &= \int_0^T \int_\Omega |\nabla u(x,t)|^{q-2}\nabla u(x,t)\nabla v(x,t) dxdt \end{aligned} \tag{6}$$

for all $u, v \in L^q(0,T; W_0^{1,p}(\Omega))$. Since $q < p$, Hölder's inequality produces a constant $C > 0$ independent of t such that

$$\left(\int_0^t \int_\Omega |\nabla u(x,s)|^q dxds\right)^{\frac{1}{q}} \le C \left(\int_0^t \int_\Omega |\nabla u(x,s)|^p dxds\right)^{\frac{1}{p}} \tag{7}$$

for all $u \in X$ and $t \in [0,T]$. Taking $t = T$, we obtain the continuous embedding $X \subset L^q(0,T; W_0^{1,q}(\Omega))$. Therefore the operator $-\Delta_p + \Delta_q$ is well defined on the space X, so the formulation of problem (P) is meaningful.

The following lemma provides important properties of the operator $-\Delta_p$ in (5). Similarly, we can proceed with the operator $-\Delta_q$ in (6).

Lemma 1. *The time-dependent negative p-Laplacian $-\Delta_p : X \to X^*$ is well defined, demicontinuous, strictly monotone, bounded (i.e., bounded on bounded sets), and satisfies*

$$\| -\Delta_p u\|_{X^*} = \|u\|_X^{p-1} \quad \textit{for all } u \in X. \tag{8}$$

Proof. From (5) and Hölder's inequality we infer that

$$\begin{aligned}|\langle -\Delta_p u, v\rangle_X| &\le \int_0^T \int_\Omega |\nabla u(x,t)|^{p-1} |\nabla v(x,t)| dxdt \\ &\le \left(\int_0^T \int_\Omega |\nabla u(x,t)|^p dxdt \right)^{\frac{p-1}{p}} \left(\int_0^T \int_\Omega |\nabla v(x,t)|^p dxdt \right)^{\frac{1}{p}} \\ &= \|u\|_X^{p-1} \|v\|_X\end{aligned}$$

for all $u, v \in X$. It turns out that the operator $-\Delta_p$ from X into X^* is well defined and bounded. Taking $v = u$ shows that (8) holds true.

Since $-\Delta_p$ in (5) is the time realization on the space X of the negative p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, the operator $-\Delta_p : X \to X^*$ is demicontinuous (see [3, Theorem 2.153]). The strict monotonicity of $-\Delta_p$ in (5) is valid because the mapping $\xi \mapsto |\xi|^{p-2}\xi$ is strictly monotone on $\mathbb{R}^N$. The proof is complete. □

Recall that the Nemytskii operator N_f associated with the Carathéodory function $f : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is defined as $N_f(u) := f(\cdot, \cdot, u(\cdot, \cdot))$.

Lemma 2. *Assume condition (H) holds. Then the mapping $N_f : X \to X^*$ is well defined, bounded and continuous.*

Proof. The growth condition in (H) guarantees that

$$f(\cdot, \cdot, u(\cdot, \cdot)) \in L^{\frac{p}{p-1}}(\Omega \times (0,T)) \quad \text{for all } u \in X. \tag{9}$$

The continuity and density of the embedding $X \subset L^p(\Omega \times (0,T))$ implies that $L^{\frac{p}{p-1}}(\Omega \times (0,T)) \subset X^*$ continuously. According to (9), through Krasnoselskii's theorem (see [7]), the mapping N_f is bounded and continuous from $L^p(\Omega \times (0,T))$ to $L^{\frac{p}{p-1}}(\Omega \times (0,T))$. Then the restriction $N_f : X \to X^*$ is bounded and continuous, which completes the proof. □

Next we briefly discuss the generalized derivative u' associated with the evolution triple (2) for a function $u \in X$. More details can be found in [3, 16, 18, 19].

The function $u' \in X^*$ is called the generalized derivative of $u \in X$ if there holds

$$\int_0^T \varphi'(t) u(t) dt = -\int_0^T \varphi(t) u'(t) dt \quad \text{for all } \varphi \in C_c^\infty(0,T).$$

A necessary and sufficient condition is that

$$\int_0^T (u(t), v)_{L^2(\Omega)} \varphi'(t) dt = -\int_0^T \langle u'(t), v \rangle_{W_0^{1,p}(\Omega)} \varphi(t) dt \tag{10}$$

for all $\varphi \in C_c^\infty(0,T)$ and $v \in W_0^{1,p}(\Omega)$.

Definition 1 shows that each generalized solution to problem (P) belongs to

$$W := \{w \in X : w' \in X^*\}, \tag{11}$$

which is a Banach space endowed with the norm $\|w\|_W = \|w\|_X + \|w'\|_{X^*}$. There is the continuous embedding

$$W \subset C([0,T], L^2(\Omega)) \tag{12}$$

(see [18, Proposition 23.23]). Due to inclusion (12), the initial condition $u(0) = u_0 \in L^2(\Omega)$ in problem (P) makes sense.

Thanks to the Lions-Aubin theorem, we have the compact embedding

$$W \subset L^p(\Omega \times (0,T)) \tag{13}$$

(see [16, p. 106]).

Lemma 3. *Assume condition (H). Then the mapping $N_f : W \to X^*$ is completely continuous (i.e., $u_n \rightharpoonup u$ in W implies $N_f(u_n) \to N_f(u)$ in X^*).*

Proof. As noted in the proof of Lemma 2, the mapping N_f is bounded and continuous from $L^p(\Omega \times (0,T))$ to X^*. In view of the compact embedding (13), the restriction of N_f to W is completely continuous, so achieving the result. □

3 Finite dimensional approximation for problem (P)

Since the space $W_0^{1,p}(\Omega)$ is separable, there is a countable family $\{w_k\}_{k \geq 1}$ of linearly independent vectors in $W_0^{1,p}(\Omega)$ such that if E_n is the linear span of $\{w_1, \ldots, w_n\}$, we have

$$\overline{\bigcup_{n=1}^{\infty} E_n} = W_0^{1,p}(\Omega). \tag{14}$$

Hence E_n is a vector subspace of $W_0^{1,p}(\Omega)$ of finite dimension n with the basis $\{w_1, \ldots, w_n\}$ that we endow with the scalar product of $L^2(\Omega)$. Given the initial condition $u_0 \in L^2(\Omega)$, we fix a sequence $\{u_{0,n}\}_{n\geq 1}$ with $u_{0,n} \in E_n$ for each n satisfying $u_{0,n} \to u_0$ in $L^2(\Omega)$. This is possible according to (14) (note $p \geq 2$). We can uniquely express $u_{0,n} = \sum\limits_{1\leq k\leq n} c_{k,n} w_k$, with $c_{k,n} \in \mathbb{R}$, $1 \leq k \leq n$.

On the interval $[0, T]$ we consider the Cauchy problem

$$\begin{cases} \sum\limits_{k=1}^{n} \theta_k'(t)(w_k, w_i)_{L^2(\Omega)} \\ \qquad\qquad +\langle(-\Delta_p + \Delta_q - N_f)(\sum\limits_{k=1}^{n} \theta_k(t) w_k), w_i\rangle_{W_0^{1,p}(\Omega)} = 0, \\ \theta_i(0) = c_{i,n}, \quad 1 \leq i \leq n, \end{cases} \tag{S_P}$$

with the unknowns $(\theta_1(t), \ldots, \theta_n(t)) \in \mathbb{R}^n$.

Proposition 1. *Assume that condition (H) holds. Then the system of ordinary differential equations (S_P) admits an absolutely continuous solution $(\theta_1(t), \ldots, \theta_n(t))$ which is defined on the entire interval $[0, T]$.*

Proof. The Gram determinant $\det(w_k, w_i)_{L^2(\Omega)}$ is nonzero because the vectors $\{w_1, \ldots, w_n\}$ are linearly independent, so the matrix $((w_k, w_i)_{L^2})_{1\leq k,i\leq n}$ is invertible. Hypothesis (H) permits to apply Carathéodory's theorem (see [19, p. 1044]) to the ordinary differential system (S_P) noting that the term involving N_f can be locally bounded by an integrable function in t due to the imposed growth condition. Consequently, there exists an absolutely continuous solution $(\theta_1(t), \cdots, \theta_n(t))$ of system (S_P) on a maximal subinterval $[0, T_n)$ of $[0, T]$. We are going to show that $T_n = T$.

Set

$$u_n(t) = \sum_{k=1}^{n} \theta_k(t) w_k. \tag{15}$$

Then system (S_P) takes the form

$$\begin{cases} \sum\limits_{k=1}^{n} \theta_k'(t)(w_k, w_i)_{L^2(\Omega)} \\ \qquad\qquad +\langle(-\Delta_p + \Delta_q - N_f)(u_n(t)), w_i\rangle_{W_0^{1,p}(\Omega)} = 0 \\ \theta_i(0) = c_{i,n}, \quad 1 \leq i \leq n. \end{cases} \tag{16}$$

It follows that

$$(u_n'(t), u_n(t))_{L^2(\Omega)} + \langle(-\Delta_p + \Delta_q - N_f)(u_n(t)), u_n(t)\rangle_{W_0^{1,p}(\Omega)} = 0 \tag{17}$$

for almost all $t \in [0, T_n)$. Integration over the interval $[0, t]$ results in

$$\frac{1}{2}\|u_n(t)\|^2_{L^2(\Omega)} + \int_0^t \langle(-\Delta_p + \Delta_q - N_f)(u_n(s)), u_n(s)\rangle_{W_0^{1,p}(\Omega)} ds = \frac{1}{2}\|u_{0,n}\|^2_{L^2(\Omega)}$$

for all $t \in [0, T_n)$. By hypothesis (H) and Hölder's inequality, in conjunction with (1), it turns out

$$\begin{aligned}
& \frac{1}{2}\|u_n(t)\|^2_{L^2(\Omega)} + \int_0^t \|u_n(s)\|^p_{W_0^{1,p}(\Omega)} ds - \int_0^t \|u_n(s)\|^q_{W_0^{1,q}(\Omega)} ds \\
\leq\ & \frac{1}{2}\|u_{0,n}\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \sigma(x) u_n(x,s) dx ds \\
& + a \int_0^t \int_\Omega |u_n(x,s)|^p dx ds \\
\leq\ & \frac{1}{2}\|u_{0,n}\|^2_{L^2(\Omega)} + \|\sigma\|_{L^{\frac{p}{p-1}}(\Omega\times(0,T))} \lambda_{1,p}^{-\frac{1}{p}} \left(\int_0^t \|u_n(s)\|^p_{W_0^{1,p}(\Omega)} ds \right)^{\frac{1}{p}} \\
& + a\lambda_{1,p}^{-1} \int_0^t \|u_n(s)\|^p_{W_0^{1,p}(\Omega)} ds.
\end{aligned}$$

Taking into account (7), this amounts to saying that

$$\begin{aligned}
& \frac{1}{2}\|u_n(t)\|^2_{L^2(\Omega)} + \left(1 - a\lambda_{1,p}^{-1}\right) \int_0^t \|u_n(s)\|^p_{W_0^{1,p}(\Omega)} ds \\
\leq\ & \frac{1}{2}\|u_{0,n}\|^2_{L^2(\Omega)} + C_1 \left(\int_0^t \|u_n(s)\|^p_{W_0^{1,p}(\Omega)} ds \right)^{\frac{1}{p}} \\
& + C_2 \left(\int_0^t \|u_n(s)\|^p_{W_0^{1,p}(\Omega)} ds \right)^{\frac{q}{p}},
\end{aligned} \tag{18}$$

with positive constants C_1 and C_2 independent of t. Using that $1 - a\lambda_{1,p}^{-1} > 0$ and $p > q > 1$, it is seen from (18) that the set $\{u_n(t) : t \in [0, T_n)\}$ is uniformly bounded in $L^2(\Omega)$. Hence the solution $(\theta_1(t), \cdots, \theta_n(t))$ of system (16) is uniformly bounded on $[0, T_n)$, which renders $T_n = T$. This completes the proof. □

Corollary 1. *The function $u_n : [0, T] \to W_0^{1,p}(\Omega)$ in (15) fulfills $u_n \in W$, where W is the space described in (11).*

Proof. Proposition 1 combined with (3) entails that $u_n \in X$. From (15) and Proposition 1 we infer that the derivative $u_n' = \sum_{k=1}^{n} \theta_k' w_k$, defined almost everywhere on $[0, T]$, is an element of X^*. Altogether, we have $u_n \in W$. □

4 The sequence of approximate solutions

Now, we focus on the sequence $\{u_n\}_{n\geq 1} \subset W$ introduced in (15). The next proposition relies on key uniform estimates for the Galerkin approximations u_n and their time derivatives u_n'.

Proposition 2. *Assume that hypothesis (H) holds. Then there exists $u \in W$ such that*

(i) $u_n \rightharpoonup u$ *in* X;

(ii) $u_n' \rightharpoonup u'$ *in* X^*, *so* $u_n \rightharpoonup u$ *in* W *and in* $C([0,T], L^2(\Omega))$;

(iii) $-\Delta_p u_n + \Delta_q u_n \rightharpoonup N_f(u) - u'$ *in* X^*;

(iv) $u(0) = u_0$ *and* $u_n(0) \rightharpoonup u_0$ *and* $u_n(T) \rightharpoonup u(T)$ *in* $L^2(\Omega)$.

Proof. From (18) we infer for $t = T$ the estimate

$$\begin{aligned} & \frac{1}{2}\|u_n(T)\|^2_{L^2(\Omega)} + \left(1 - a\lambda_{1,p}^{-1}\right)\|u_n\|^p_X \\ \leq\ & \frac{1}{2}\|u_{0,n}\|^2_{L^2(\Omega)} + C_1\|u_n\|_X + C_2\|u_n\|^q_X, \end{aligned}$$

with positive constants C_1 and C_2. Since $1 - a\lambda_{1,p}^{-1} > 0$ and $p > q > 1$, it follows that the sequence $\{u_n\}_{n\geq 1}$ is bounded in X and the sequence $\{u_n(T)\}_{n\geq 1}$ is bounded in $L^2(\Omega)$. Thanks to the reflexivity of X and of $L^2(\Omega)$ we can pass to a subsequence for which there exist $u \in X$ and $h \in L^2(\Omega)$ such that

$$u_n \rightharpoonup u \text{ in } X \text{ and } u_n(T) \rightharpoonup h \text{ in } L^2(\Omega). \tag{19}$$

The first weak limit in (19) proves item (i).

By virtue of Lemmas 1 and 2, the sequences $\{-\Delta_p u_n\}_{n\geq 1}$, $\{-\Delta_q u_n\}_{n\geq 1}$ and $\{N_f(u_n)\}_{n\geq 1}$ are bounded in X^*. Due to the reflexivity of X^*, up to subsequences there hold

$$-\Delta_p u_n \rightharpoonup \xi,\ -\Delta_q u_n \rightharpoonup \eta,\ N_f(u_n) \rightharpoonup \zeta \quad \text{in } X^*, \tag{20}$$

with some $\xi \in X^*$, $\eta \in X^*$ and $\zeta \in X^*$.

From (16) we get for almost all $t \in |0,T]$ and all $\varphi \in C^1([0,T])$ and $v \in E_n$ that

$$(u_n'(t), \varphi(t)v)_{L^2(\Omega)} + \varphi(t)\langle(-\Delta_p + \Delta_q - N_f)(u_n(t)), v\rangle_{W_0^{1,p}(\Omega)} = 0. \tag{21}$$

Let $v \in \bigcup_{n=1}^{\infty} E_n$. Fix an m with $v \in E_m$ and thus (21) is valid for all $n \geq m$. Integration by parts over $|0,T]$ in (21) entails

$$-\int_0^T \varphi'(t)(u_n(t),v)_{L^2(\Omega)}dt + \varphi(T)(u_n(T),v)_{L^2(\Omega)} - \varphi(0)(u_{0,n},v)_{L^2(\Omega)} + \int_0^T \varphi(t)\langle(-\Delta_p + \Delta_q - N_f)(u_n(t)),v\rangle_{W_0^{1,p}(\Omega)}dt = 0, \quad \forall n \geq m. \tag{22}$$

Letting $n \to \infty$, by means of (19) and (20) we find

$$-\int_0^T \varphi'(t)(u(t),v)_{L^2(\Omega)}dt + \varphi(T)(h,v)_{L^2(\Omega)} - \varphi(0)(u_0,v)_{L^2(\Omega)} + \int_0^T \varphi(t)\langle\xi - \eta - \zeta, v\rangle_{W_0^{1,p}(\Omega)}dt = 0. \tag{23}$$

Thanks to the density in (14), we have that (23) holds for all $v \in W_0^{1,p}(\Omega)$. In particular, taking $\varphi \in C_c^{\infty}(0,T)$, we arrive at

$$\int_0^T \varphi'(t)(u(t),v)_{L^2(\Omega)}dt = -\int_0^T \varphi(t)\langle(-\xi + \eta + \zeta, v\rangle_{W_0^{1,p}(\Omega)}dt$$

for all $\varphi \in C_c^{\infty}(0,T)$ and $v \in W_0^{1,p}(\Omega)$. According to (10), there exists the generalized derivative $u' \in X^*$ of u with

$$u' = -\xi + \eta + \zeta. \tag{24}$$

From (11) we conclude that $u \in W$.

We pass now to uniformly estimate the time derivatives u_n'. Return to (22) that can be written in the form

$$\int_0^T (u_n'(t),v)_{L^2(\Omega)}\varphi(t)dt + \int_0^T \varphi(t)\langle(-\Delta_p + \Delta_q - N_f)(u_n(t)),v\rangle_{W_0^{1,p}(\Omega)}dt = 0$$

whenever $\varphi \in C^1([0,T])$, $v \in E_m$ and $n \geq m$. In the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty}\left[\int_0^T (u_n'(t),\varphi(t)v)_{L^2(\Omega)}dt + \int_0^T \langle(-\Delta_p + \Delta_q - N_f)(u_n(t)),\varphi(t)v\rangle_{W_0^{1,p}(\Omega)}dt\right] = 0$$

for all $\varphi \in C^1([0,T])$ and $v \in \bigcup_{n=1}^{\infty} E_n$. Using the density properties described in (4) and (14), we get

$$\lim_{n\to\infty} \left[\int_0^T (u_n'(t), w)_{L^2(\Omega)} dt + \int_0^T \langle (-\Delta_p + \Delta_q - N_f)(u_n(t)), w \rangle_{W_0^{1,p}(\Omega)} dt \right] = 0 \quad (25)$$

for all $w \in X$. On the basis of (20) and (24), this yields $u_n' \rightharpoonup u'$ in X^*. In view of (11) and assertion (i), we infer that $u_n \rightharpoonup u$ in W. The continuous embedding (12) implies $u_n \rightharpoonup u$ in $C([0,T], L^2(\Omega))$, thereby assertion (ii) follows.

The fact that $u_n \rightharpoonup u$ in W allows us to apply Lemma 3 that provides the strong convergence $N_f(u_n) \to N_f(u)$ in X^*. Combining with (20) and (24) leads to item (iii) in the statement of proposition.

Insert (24) (with $\zeta = N_f(u)$) into (23), which gives

$$-\int_0^T \varphi'(t)(u(t), v)_{L^2(\Omega)} dt + \varphi(T)(h, v)_{L^2(\Omega)} - \varphi(0)(u_0, v)_{L^2(\Omega)} - \int_0^T \varphi(t) \langle u'(t)), v \rangle_{W_0^{1,p}(\Omega)} dt = 0$$

for all $\varphi \in C^1([0,T])$ and $v \in W_0^{1,p}(\Omega)$. The integration by parts provides

$$\varphi(T)(h, v)_{L^2(\Omega)} - \varphi(0)(u_0, v)_{L^2(\Omega)} = \varphi(T)(u(T), v)_{L^2(\Omega)} - \varphi(0)(u(0), v)_{L^2(\Omega)}.$$

Since $\varphi \in C^1([0,T])$ and $v \in W_0^{1,p}(\Omega)$ are arbitrary, and $W_0^{1,p}(\Omega)$ is dense in $L^2(\Omega)$, we deduce that $u(0) = u_0$ and $u(T) = h$. Hence item (iv) is proven. The proof is complete. □

Corollary 2. *Assume that condition (H) holds. Then the time derivative u' of the function $u \in W$ constructed in Proposition 2 can be estimated as follows*

$$\|u'\|_{X^*} \le \limsup_{n\to\infty} \| - \Delta_p u_n\|_{X^*} + \limsup_{n\to\infty} \| - \Delta_q u_n\|_{X^*} + \|N_f(u)\|_{X^*}.$$

Proof. From item (iii) in Proposition 2 and the weak lower semicontinuity of the norm we infer that

$$\begin{aligned} \|u'\|_{X^*} &\le \liminf_{n\to\infty} \| - \Delta_p u_n - \Delta_q u_n\|_{X^*} + \|N_f(u)\|_{X^*} \\ &\le \limsup_{n\to\infty} \| - \Delta_p u_n - \Delta_q u_n\|_{X^*} + \|N_f(u)\|_{X^*}. \end{aligned}$$

The stated conclusion holds true. □

5 The generalized solution

Proof of Theorem 1. Proposition 2 ensures the existence of an $u \in W$ for which assertions (i)-(iv) are satisfied with the sequence $\{u_n\}_{n\geq 1} \subset W$ constructed in Proposition 1. We check that the requirements in Definition 1 for a generalized solution to problem (P) are fulfilled. We know from Proposition 2 that $u \in L^p(0,T;W_0^{1,p}(\Omega))$ with $u' \in L^{\frac{p}{p-1}}(0,T;W^{-1,\frac{p}{p-1}}(\Omega))$. Item (iv) in Proposition 2 entails $u(0) = u_0$. Part (i) in Definition 1 is the direct consequence of items (i) and (iv) of Proposition 2. Parts (ii) and (iii) in Definition 1 are verified by the statements (ii) and (iii) of Proposition 2; respectively.

It remains to justify property (iv) in Definition 1. By (17) we have that

$$\langle u_n' - \Delta_p u_n + \Delta_q u_n - N_f(u_n), u_n\rangle_X = 0 \tag{26}$$

for all n. On the other hand, items (ii) and (iii) of Proposition 2 provide

$$\langle u_n' - \Delta_p u_n + \Delta_q u_n - N_f(u_n), u\rangle_X \to 0 \text{ as } n \to \infty. \tag{27}$$

Then (26) and (27) ensure that

$$\langle u_n' - \Delta_p u_n + \Delta_q u_n - N_f(u_n), u_n - u\rangle_X \to 0 \text{ as } n \to \infty.$$

Since $u_n \rightharpoonup u$ in W, Lemma 3 implies that

$$\langle N_f(u_n), u_n - u\rangle_X \to 0 \text{ as } n \to \infty.$$

Therefore property (iv) in Definition 1 holds true. The proof of Theorem 1 is complete.

Finally, we present an example illustrating the applicability of Theorem 1.

Example 1. Consider the initial-boundary value problem

$$\begin{cases} u_t - \Delta_p u + \Delta_q u = \rho(x,t) + \frac{|u|^{\alpha+1}}{1+|u|} & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x) = u_0(x) & \text{on } \partial\Omega \end{cases} \tag{28}$$

on a bounded domain Ω in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$ and for fixed real numbers $T > 0$, $p \geq 2$, $q \in (1,p)$, $\alpha \in [0, p-1)$, and given functions $\rho \in L^{\frac{p}{p-1}}(\Omega \times (0,T))$ and $u_0 \in L^2(\Omega)$.

Problem (28) matches the statement of (P) by setting

$$f(x,t,s) = \rho(x,t) + \frac{|s|^{\alpha+1}}{1+|s|}$$

whenever $(x,t,s) \in \Omega \times (0,T) \times \mathbb{R}$. Using the assumption $\alpha \in [0, p-1)$, we obtain through Young's inequality that for each $\varepsilon > 0$ there is a constant $c(\varepsilon) > 0$ such that

$$|f(x,t,s)| \le \rho(x,t) + |s|^{\alpha} \le |\rho(x,t)| + \varepsilon |s|^{p-1} + c(\varepsilon).$$

Choose $\varepsilon \in (0, \lambda_{1,p})$. It results that hypothesis (H) is verified with $\sigma(x,t) = |\rho(x,t)| + c(\varepsilon)$ and $a = \varepsilon$, which permits to apply Theorem 1 to problem (28).

References

[1] L. Boccardo and F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, *Nonlinear Anal.* 19 (1992), 581-597.

[2] S. Carl and V.K. Le, On systems of parabolic variational inequalities with multivalued terms, *Monatsh. Math.* 194 (2021), 227-260.

[3] S. Carl, V.K. Le and D. Motreanu, *Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications*, Springer Monographs in Mathematics, Springer, New York, 2007.

[4] S. Carl and D. Motreanu, Extremal solutions for quasilinear parabolic systems in trapping regions, *Pure Appl. Funct. Anal.* 3 (2018), 57-74.

[5] S. Carl and P. Winkert, Parabolic double phase obstacle problems, *Nonlinear Anal. Real World Appl.* 80 (2024), 104169.

[6] J. Cen, S.A. Marano and S. Zeng, Differential inclusion systems with double phase competing operators, convection, and mixed boundary conditions, *Appl. Math. Lett.* 167 (2025), 109556.

[7] D.G. de Figueiredo, *Lectures on the Ekeland Variational Principle with Applications and Detours*, Springer-Verlag, Berlin, 1989.

[8] J. Diblík, M. Galewski, I. Kossowski and D. Motreanu, On competing (p,q)-Laplacian Dirichlet problem with unbounded weight, *Differ. Integr. Equations* 38 (2025), 23-42.

[9] L.C. Evans, *Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 2010.

[10] M. Galewski and D. Motreanu, *Competing Operators and Their Applications to Boundary Value Problems*, Springer Briefs in Mathematics. Springer, Cham, 2026.

[11] L. Gambera, S.A. Marano and D. Motreanu, Quasilinear Dirichlet systems with competing operators and convection, *J. Math. Anal. Appl.* 530 (2024), 127718.

[12] Z. Liu, R. Livrea, D. Motreanu and S. Zeng, Variational differential inclusions without ellipticity condition, *Electron. J. Qual. Theory Differ. Equations* 43 (2020), 1-17.

[13] D. Motreanu, Quasilinear Dirichlet problems with competing operators and convection, *Open Math.* 18 (2020), 1510-1517.

[14] D. Motreanu and Z. Peng, Doubly coupled systems of parabolic hemivariational inequalities: existence and extremal solutions, *Nonlinear Anal.* 181 (2019), 101-118.

[15] A. Razani and G.M. Figueiredo, Degenerated and competing anisotropic (p, q)-Laplacians with weights, *Appl. Anal.* 102 (2023), 4471-4488.

[16] R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 1997.

[17] F. Vetro and R. Efendiev, Systems of differential inclusions with competing operators and variable exponents, *Opuscula Math.* 45 (2025), 665-684.

[18] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/A – Linear Monotone Operators*, Springer, New York, 1990.

[19] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B – Nonlinear Monotone Operators*, Springer-Verlag, New York, 1990.

[20] Z. Zhong, F. Vetro and B. Chen, A class of logarithmic double phase competing problems with convection term and mixed boundary conditions, *J. Fixed Point Theory Appl.* 27 (2025), 83.