

A NOTE ON THE EXISTENCE OF SOLUTIONS OF A FRACTIONAL DIFFERENTIAL INCLUSION OF HILFER-HADAMARD TYPE*

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Dedicated to the memory of Professor Mihail Megan

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Abstract

A fractional differential inclusion with multi-point boundary conditions involving Hilfer-Hadamard derivatives is considered. An existence result is established when the set-valued map is lower semicontinuous.

Keywords: differential inclusion, fractional derivative, boundary value problem.

MSC: 34A60, 34A12, 34A08.

1 Introduction

In this note we consider the following differential inclusion

$$D_{HH}^{\alpha,\beta}x(t) \in F(t, x(t)) \quad a.e. \quad ([1, e]) \quad (1)$$

with boundary conditions of the form

$$x(1 + \varepsilon) = 0, \quad D_{HH}^{1,1}x(e) = \sum_{i=1}^n \nu_i D_{HH}^{1,1}x(\eta_i), \quad (2)$$

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where $D_{HH}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha \in (1, 2]$ and type $\beta \in [0, 1]$, $\eta_i \in (1, e)$, $\nu_i \in \mathbf{R}$, $i = \overline{1, n}$, $\varepsilon \in (0, 1)$ and $F : [1, e] \times \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$ is a set-valued map.

We recall that Hilfer fractional derivative introduced in [14] is an interpolation between Riemann-Liouville and Caputo derivatives. The modification of Hilfer fractional derivative resulted in the concept of Hilfer-Hadamard fractional derivative. Similarly, the Hilfer-Hadamard fractional derivative covers the cases of Riemann-Liouville-Hadamard and Caputo-Hadamard fractional derivatives.

Several recent papers are devoted to the study to boundary value problems associated to fractional differential equations and inclusions defined by Hilfer-Hadamard fractional derivative [1, 9, 10, 18] etc. For background facts concerning fractional calculus and differential equations of fractional order we refer to [2, 12, 15, 17] etc.

The present study is motivated by a recent paper [3], where two existence results for problem (1)-(2) are provided. A result is obtained in the case when the values of the set-valued map are convex and the set-valued map is upper semicontinuous. The other result is proved in the situation when the set-valued map is Lipschitz in the state variable with non convex values.

Our aim is to complete the study in [3] by taking into account the case when the set-valued map has non convex values, but instead the Lipschitz hypothesis in [3] we assume that $F(., .)$ is lower semicontinuous. The proof of our existence result relies on a nonlinear alternative of Leray-Schauder type and on Bressan-Colombo-Fryszkowski selection theorem for lower semicontinuous set-valued maps with decomposable values.

We emphasize that such kind of approach is usual in the study of differential inclusions (e.g., [5-8, 11]), however its presentation in the framework of problem (1)-(2) is new.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, in Section 3 we prove our result.

2 Preliminaries

Let (X, d) be a metric space with the corresponding norm $|\cdot|$ and let $I \subset \mathbf{R}$ be a compact interval. Denote by $\mathcal{L}(I)$ the σ -algebra of all Lebesgue measurable subsets of I , by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X . If $A \subset I$ then $\chi_A(\cdot) : I \rightarrow \{0, 1\}$ denotes the characteristic function of A . For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A . $B_r(0)$ and $\overline{B_r(0)}$ will be the open and closed balls centered at

the origin and of radius r

We recall that the Pompeiu-Hausdorff generalized metric of the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B); a \in A\}$, where $d(x, B) = \inf_{y \in B} d(x, y)$.

As usual, if X is a separable Banach space, we denote by $C(I, X)$ the Banach space of all continuous functions $x(\cdot) : I \rightarrow X$ endowed with the norm $|x|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(\cdot) : I \rightarrow X$ endowed with the norm $|x|_1 = \int_I |x(t)| dt$.

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \setminus A$.

Consider $H : X \rightarrow \mathcal{P}(X)$ a set-valued map. $H(\cdot)$ is said to be *totally compact* if $\overline{H(X)}$ is a compact subset of X . $H(\cdot)$ is said to be *upper semi-continuous* if for any $x_0 \in X$, $H(x_0)$ is a nonempty and closed subset of X and if for each open set D of X containing $H(x_0)$ there exists an open neighborhood V_0 of x_0 such that $H(V_0) \subset D$. $H(\cdot)$ is said to be *lower semi-continuous* if for any open subset $D \subset X$, the set $\{x \in X; H(x) \cap D \neq \emptyset\}$ is open. $H(\cdot)$ is called *completely continuous* if it is upper semicontinuous and transfers bounded sets into relatively compact sets.

The result in the next section is based on the following nonlinear alternative of Leray-Schauder type (e.g, [18]).

Theorem 1. *Let D be an open subset in a normed linear space X such that $0 \in D$ and let $H : \overline{D} \rightarrow \mathcal{P}(X)$ be a completely continuous set-valued map with compact and convex values. Then either*

- i) the inclusion $x \in H(x)$ has a solution, or*
- ii) there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in H(x)$ for some $\lambda > 1$.*

In fact, we use the following consequence.

Corollary 1. *Let X be a normed linear space and let $h : \overline{B_r(0)} \rightarrow X$ be a completely continuous single valued map. Then either*

- i) the equation $x = h(x)$ has a solution, or*
- ii) there exists $x \in X$ with $|x| = r$ and $x = \lambda h(x)$ for some $\lambda < 1$.*

If $F(\cdot, \cdot) : I \times X \rightarrow \mathcal{P}(X)$ is a set-valued map with compact values we define $S_F : C(I, X) \rightarrow \mathcal{P}(L^1(I, X))$ by

$$S_F(x) := \{f \in L^1(I, X); \quad f(t) \in F(t, x(t)) \quad a.e. (I)\}.$$

We say that $F(\cdot, \cdot)$ is of *lower semicontinuous type* if $S_F(\cdot)$ is lower semicontinuous with nonempty closed and decomposable values. We need, also, the next result ([4]).

Theorem 2. *Let S be a separable metric space and $H(\cdot) : S \rightarrow \mathcal{P}(L^1(I, X))$ be a lower semicontinuous set-valued map with closed and decomposable values.*

Then $H(\cdot)$ has a continuous selection (i.e., there exists a continuous mapping $h(\cdot) : S \rightarrow L^1(I, X)$ such that $h(s) \in H(s) \quad \forall s \in S$).

Also, we recall some definitions from fractional calculus. In what follows $I = [1, e]$.

Definition 1. *a) The Hadamard fractional integral of order $q > 0$ of a Lebesgue integrable function $f : [1, \infty) \rightarrow \mathbf{R}$ is defined by*

$$I_H^q f(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\ln \frac{t}{s} \right)^{q-1} \frac{f(s)}{s} ds$$

provided the integral exists and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(q) = \int_0^\infty t^{q-1} e^{-t} dt$.

b) The Hadamard fractional derivative of order $q > 0$ of a function $f : [1, \infty) \rightarrow \mathbf{R}$ is defined by

$$D_H^q f(t) = t \frac{d^n}{dt^n} (I_H^{n-q} f)(t),$$

provided the integral exists and $n = [q] + 1$, $[q]$ is the integer part of q .

c) The Hilfer-Hadamard fractional derivative of order $\alpha \in (n-1, n)$, $n \geq 2$ and type $\beta \in [0, 1]$ of a function $f \in L^1(I, \mathbf{R})$ is defined by

$$D_{HH}^{\alpha, \beta} f(t) = (I_H^{\beta(n-\alpha)} D_H^\gamma f)(t), \quad \gamma = \alpha + n\beta - \alpha\beta.$$

When $\beta = 0$ the Hilfer-Hadamard fractional derivative gives Hadamard fractional derivative and when $\beta = 1$ the Hilfer-Hadamard fractional derivative gives Caputo-Hadamard fractional derivative.

In our case $n = [\alpha] + 1 = 2$ and $\gamma = \alpha + (2 - \alpha)\beta$. We make, also, the notations

$$\begin{aligned} \Delta &= (\gamma - 2)\mu_2[\ln(1 + \varepsilon)]^{\gamma-1} - (\gamma - 1)\mu_1[\ln(1 + \varepsilon)]^{\gamma-2}, \\ \mu_1 &= 1 - \sum_{i=1}^n \nu_i (\ln \eta_i)^{\gamma-2}, \quad \mu_2 = 1 - \sum_{i=1}^n \nu_i (\ln \eta_i)^{\gamma-3}, \\ \Omega &= (\gamma - 2)\mu_2 \ln(1 + \varepsilon) - (\gamma - 1)\mu_1. \end{aligned}$$

The next result is proved in [3].

Lemma 1. *Assume $\Omega \neq 0$ and let $h(\cdot) \in C(I, \mathbf{R})$.*

Then, the solution of problem $D_{HH}^{\alpha,\beta}x(t) = h(t)$ with boundary conditions (2) is given by

$$x(t) = I_H^\alpha h(t) + \frac{(\ln t)^{\gamma-2}((\gamma-1)\mu_1 - (\gamma-2)\mu_2 \ln t)}{\Delta} I_H^\alpha h(1+\varepsilon) +$$

$$[\ln(1+\varepsilon)]^{\gamma-2} \frac{\ln(1+\varepsilon)(\ln t)^{\gamma-2} - (\ln t)^{\gamma-1}}{\Delta} \left[\sum_{i=1}^n \nu_i I_H^\alpha h(\eta_i) - I_H^{\alpha-1} h(e) \right].$$

Definition 2. $x(\cdot) \in C(I, \mathbf{R})$ is called a solution of a problem (1)-(2) if there exists $h(\cdot) \in L^1(I, \mathbf{R})$ such that $h(t) \in F(t, x(t))$ a.e. (I) and $x(\cdot)$ is given as in Lemma 1.

Remark 1. If we denote

$$\begin{aligned} \mathcal{G}_1(t, \tau) &= \frac{(\ln t - \ln \tau)^{\alpha-1}}{\Gamma(\alpha)\tau} \chi_{[1,t]}(\tau), \\ \mathcal{G}_2(t, \tau) &= \frac{(\ln t)^{\gamma-2}[(\gamma-1)\mu_1 - (\gamma-2)\mu_2 \ln t]}{\Delta} \frac{(\ln t - \ln \tau)^{\alpha-1}}{\Gamma(\alpha)\tau} \chi_{[1,1+\varepsilon]}(\tau), \\ \mathcal{G}_3(t, \tau) &= \sum_{i=1}^n \nu_i \chi_{[1,\eta_i]}(\tau) \frac{(\ln t - \ln \tau)^{\alpha-2}}{\Gamma(\alpha-1)\tau} [\ln(1+\varepsilon)]^{\gamma-2} \frac{\ln(1+\varepsilon)(\ln t)^{\gamma-2} - (\ln t)^{\gamma-1}}{\Delta}, \\ \mathcal{G}_4(t, \tau) &= -\frac{(\ln t - \ln \tau)^{\alpha-2}}{\Gamma(\alpha-1)\tau} [\ln(1+\varepsilon)]^{\gamma-2} \frac{\ln(1+\varepsilon)(\ln t)^{\gamma-2} - (\ln t)^{\gamma-1}}{\Delta}, \\ \mathcal{G}(t, \tau) &= \mathcal{G}_1(t, \tau) + \mathcal{G}_2(t, \tau) + \mathcal{G}_3(t, \tau) + \mathcal{G}_4(t, \tau) \end{aligned}$$

then the solution $x(\cdot)$ in Lemma 1 may be put as $x(t) = \int_1^e \mathcal{G}(t, s) h(s) ds$.

Moreover,

$$|\mathcal{G}(t, s)| \leq \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)} + \frac{|(\gamma-1)\mu_1 - (\gamma-2)\mu_2 \ln t|}{|\Delta|\Gamma(\alpha)} (\ln t)^{\alpha+\gamma-3} + (1 + \sum_{i=1}^n |\nu_i|) \cdot$$

$$\frac{[\ln(1+\varepsilon)]^{\gamma-2}}{|\Delta|\Gamma(\alpha-1)} |\ln(1+\varepsilon)(\ln t)^{\alpha+\gamma-2} - (\ln t)^{\alpha+\gamma-3}| =: M(t) \quad \forall t, \tau \in I.$$

Denote $M := \max_{t \in I} M(t)$.

3 The result

We are working under the following hypothesis.

Hypothesis H i) $F(\cdot, \cdot) : I \times \mathbf{R} \rightarrow \mathcal{P}(\mathbf{R})$ has compact values, $F(\cdot, \cdot)$ is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable and $x \rightarrow F(t, x)$ is lower semicontinuous for almost all $t \in I$.

ii) There exists $a(\cdot) \in L^1(I, \mathbf{R})$ with $a(t) > 0$ a.e. (I) and there exists a nondecreasing function $b : [0, \infty) \rightarrow (0, \infty)$ such that

$$\sup\{|v|; v \in F(t, x)\} \leq a(t)b(|x|) \quad \text{a.e. (I),} \quad \forall x \in \mathbf{R}.$$

Theorem 3. Assume that $\Omega \neq 0$, Hypothesis H is satisfied and there exists $c > 0$ such that

$$c > M|a|_1 b(c). \quad (3)$$

Then problem (1)-(2) has at least one solution on I .

Proof. From Hypothesis H it follows that $F(.,.)$ is of lower semicontinuous type (e.g., [13]). Therefore, by Theorem 2 applied with $S = C(I, \mathbf{R})$ and $G(.) = S_F(.)$, we infer that there exists a continuous mapping $g(.) : C(I, \mathbf{R}) \rightarrow L^1(I, \mathbf{R})$ such that $g(x) \in S_F(x)$ for all $x \in C(I, \mathbf{R})$.

We consider the integral equation

$$x(t) = \int_1^e \mathcal{G}(t, s)g(x(s))ds, \quad t \in I \quad (4)$$

in the space $X = C(I, \mathbf{R})$. If $x(.) \in C(I, \mathbf{R})$ is a solution of the problem (4) then $x(.)$ is a solution to problem (1)-(2).

Let $c > 0$ that satisfies condition (3) and define the map $h : \overline{B_r(0)} \rightarrow C(I, \mathbf{R})$ by $(h(x))(t) := \int_1^e \mathcal{G}(t, s)g(x(s))ds$.

The integral equation (4) becomes the operatorial equation

$$x(t) = (h(x))(t), \quad t \in I.$$

It remains to prove that $h(.)$ satisfies the hypotheses of Corollary 1.

First, we show that $h(.)$ is continuous on $\overline{B_r(0)}$. Taking into account Hypotheses H ii) one has

$$|g(x(t))| \leq a(t)b(|x(t)|) \quad a.e. (I)$$

for all $x(.) \in C(I, \mathbf{R})$. Consider $y_n, y \in \overline{B_r(0)}$ with $y_n \rightarrow y$ as $n \rightarrow \infty$. Then

$$|g(y_n(t))| \leq a(t)b(c) \quad a.e. (I).$$

Lebesgue's dominated convergence theorem and the continuity of $g(.)$ allow to write, for all $t \in I$

$$\lim_{n \rightarrow \infty} (h(y_n))(t) = \int_1^e \mathcal{G}(t, s)g(y_n(s))ds = \int_1^e \mathcal{G}(t, s)g(y(s))ds = (h(y))(t)$$

which means that $h(.)$ is continuous on $\overline{B_r(0)}$.

Secondly, we prove that $h(\overline{B_r(0)})$ is compact. Take $g \in S_F(x)$ and set $w(t) = \int_1^e \mathcal{G}(t, s)g(s)ds$. For any $t \in I$, we have

$$|w(t)| \leq \int_1^e |\mathcal{G}(t, s)| \cdot |g(s)|ds \leq \int_1^e |\mathcal{G}(t, s)|a(s)b(|x(t)|)ds$$

which means that $|w|_C \leq M|a|_1b(r)$, i.e., $h(\overline{B_r(0)})$ is bounded.

Next, we show that $h(\cdot)$ maps bounded sets into equi-continuous sets.

Therefore, let $B \subset \overline{B_r(0)}$ be a bounded set. As before, take $g \in S_F(x)$ such that $w(t) = \int_1^e \mathcal{G}(t, s)g(s)ds$. For every $\tau_1, \tau_2 \in I$ we have

$$|w(\tau_1) - w(\tau_2)| \leq \left| \int_1^e \mathcal{G}(\tau_1, s)g(s)ds - \int_1^e \mathcal{G}(\tau_2, s)g(s)ds \right| \leq \int_1^e |\mathcal{G}(\tau_1, s) - \mathcal{G}(\tau_2, s)| \cdot |g(s)|ds \leq \int_1^e |\mathcal{G}(\tau_1, s) - \mathcal{G}(\tau_2, s)|a(s)b(r)ds,$$

which means that $|w(\tau_1) - w(\tau_2)| \rightarrow 0$ as $\tau_1 \rightarrow \tau_2$. Therefore, $h(\overline{B_r(0)})$ is an equi-continuous set in $C(I, \mathbf{R})$. Arzela-Ascoli's theorem allows us to deduce that $h(\cdot)$ is completely continuous on $C(I, \mathbf{R})$.

So, one may apply Corollary 1 in order to deduce that either i) the equation $x = h(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = c$ and $x = \lambda h(x)$ for some $\lambda < 1$.

Assume that ii) is true. Following the same arguments as in the second step of our proof, we get $c = |x|_C \leq M|a|_1b(c)$. This is a contradiction with (3). Thus, only i) is valid, which means that problem (1) has a solution with $|x|_C < c$. $\square$

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