

A GENERAL TRANSVERSALITY CRITERION FOR HOPF BIFURCATION IN CHARACTERISTIC EQUATIONS WITH DISTRIBUTED DELAY*

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Dedicated to the memory of Professor Mihail Megan

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Abstract

A general formula is given for the Hopf transversality condition associated with characteristic equations of the form $R(z) = H(z, \tau)$, where R is analytic and $H(\cdot, \tau)$ is the Laplace transform of a distributed-delay kernel. Our main result expresses the crossing direction of a simple purely imaginary characteristic root and provides a geometric interpretation of the transversality condition. We obtain as special cases the classical discrete-delay formula and explicit criteria for self-similar kernels, including the exponential and Erlang families. Several consequences for stability switching are also discussed and examples are given.

Keywords: distributed delay, Hopf bifurcation, characteristic equation, transversality condition, stability switching.

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1 Introduction

Time delays are often incorporated in mathematical models whenever the present rate of change of a system depends on its past states. However, in many applications, the delay is not concentrated at a single instant but spread over an interval of past times. This leads to equations with distributed delay, where the memory term is weighted by a kernel that describes the relative influence of the past. Such mathematical models may arise from population dynamics, epidemiology, physiology, economics, neural systems, and chemical kinetics, among many other areas [5, 7, 10, 11, 15].

From the viewpoint of qualitative theory, retarded functional differential equations (RFDEs) form an infinite-dimensional dynamical system on a phase space of history segments. For a fixed maximal delay $r > 0$, the natural state at time t is the segment $x_t \in C([-r, 0], \mathbb{R}^n)$ defined by $x_t(\theta) = x(t + \theta)$. The general RFDE

$$\dot{x}(t) = \mathcal{F}(x_t, \mu), \quad (1)$$

with parameter $\mu \in \mathbb{R}$, is well posed under standard smoothness assumptions, generates a local semiflow on the phase space, and admits a linearization principle analogous to that for ordinary differential equations [5, 7]. If x^* is an equilibrium and $L_\mu = D\mathcal{F}(x^*, \mu)$ denotes the Fréchet derivative, then the local dynamics near x^* is governed by the linear RFDE $\dot{y}(t) = L_\mu y_t$. For retarded equations, characteristic roots are isolated, have finite algebraic multiplicity, and the spectral bound determines local asymptotic stability of the equilibrium [5, 7].

The nonlinear dynamics near a critical spectral configuration can be reduced by center-manifold and normal-form techniques. In particular, if the linearization has a simple pair of purely imaginary roots $\pm i\omega_0$ and no other roots on the imaginary axis, then a local center manifold exists and the dynamics near the equilibrium is locally topologically equivalent to a finite-dimensional system. This reduction makes it possible to apply the classical Hopf bifurcation theory to RFDEs: if the critical pair crosses the imaginary axis with nonzero velocity as the parameter varies, then a branch of small-amplitude periodic solutions bifurcates from the equilibrium [6–8]. Therefore, in practical applications, a Hopf bifurcation analysis usually requires two steps: first, the determination of the critical parameter values for which purely imaginary characteristic roots exist, and the verification of the transversality condition, namely that the corresponding characteristic roots cross the imaginary axis with nonzero velocity.

For equations with distributed delay, the characteristic equation involves Laplace transforms of delay kernels. For example, for the linearized equation:

$$\dot{y}(t) = A_0 y(t) + \int_0^\infty K(s, \tau) y(t-s) ds, \quad (2)$$

where $K(\cdot, \tau)$ is a matrix kernel, seeking solutions of the form $y(t) = e^{zt}v$ leads to the characteristic matrix:

$$\Delta(z, \tau) = zI - A_0 - \int_0^\infty e^{-zs} K(s, \tau) ds.$$

In a large number of models, after scalar reduction, factorization, or elimination, a scalar transcendental equation of the form

$$R(z) = H(z, \tau), \quad (3)$$

is obtained, where R is rational or, more generally, analytic, and $H(\cdot, \tau)$ is the Laplace transform of a delay distribution kernel. Characteristic equations of this type are often encountered in the stability and Hopf analysis of distributed-delay systems [2, 3, 9, 12, 14].

In the discrete-delay case, several criteria are available. Cooke and Grossman [3] and Cooke and van den Driessche [4] derived methods for locating purely imaginary roots and identifying stability switches by means of auxiliary scalar equations. For delay-dependent parameters, Beretta and Kuang [1] developed a geometric approach which expresses stability switching in terms of the intersection of explicitly computable curves. In the distributed-delay setting, important results were obtained by Kuang [12] on the nonoccurrence of stability switching in certain classes of models, by Bernard, Bélair and Mackey [2] on stability criteria based on properties of the kernel, and by Ruan and Wolkowicz [14] in the bifurcation analysis of chemostat models with distributed delay. Recent work continues to study stability and Hopf bifurcation for general distributed kernels and confirms the continuing relevance of the topic [9].

The purpose of the present paper is to isolate a general transversality criterion for (3). Our starting point is the following: write the characteristic root as a local smooth branch $z(\tau)$ and compute $\frac{dz}{d\tau}$ by implicit differentiation. The main observation is that, once $R(i\omega)$ and $H(i\omega, \tau)$ are written in polar form, the sign of $\frac{d}{d\tau} \operatorname{Re} z(\tau)$ can be expressed in terms of two scalar equations in the variables (ω, τ) , more precisely, the modulus equation and the phase equation. This leads to a precise Jacobian formula for the crossing direction and provides a simple geometric interpretation: the spectral

transversality condition is equivalent to the transversality of the two curves defined by the modulus and phase relations.

If we consider $R(i\omega) = \tilde{\rho}(\omega)e^{i\tilde{\theta}(\omega)}$ and $H(i\omega, \tau) = \rho(\omega, \tau)e^{-i\theta(\omega, \tau)}$, then purely imaginary characteristic roots are determined by

$$\tilde{\rho}(\omega) = \rho(\omega, \tau), \quad \tilde{\theta}(\omega) + \theta(\omega, \tau) = 2k\pi, \quad k \in \mathbb{Z}.$$

Our main result shows that the crossing direction is given by the sign of the Jacobian determinant of this system with respect to (ω, τ) . When the phase equation can be solved locally as $\tau = T_k(\omega)$, the criterion reduces to the sign of the derivative of the auxiliary function

$$F_k(\omega) = \tilde{\rho}(\omega) - \rho(\omega, T_k(\omega)).$$

For self-similar kernels of the form $H(z, \tau) = \hat{H}(\tau z)$, this reduction becomes particularly explicit and includes the classical discrete-delay formula as a limiting case.

The paper is organized as follows. In Section 2 we briefly recall the classical Hopf framework for RFDEs and formulate the class of characteristic equations under consideration. Section 3 contains the main transversality theorem, its proof, and several equivalent formulations. Section 4 discusses consequences for stability switching. In Section 5 we specialize the theory to the Dirac, exponential, and Erlang kernels and we work out a family of second-order characteristic equations. The last section summarizes the main conclusions.

2 Background and problem formulation

2.1 Classical Hopf background

Let $r > 0$ and let $C([-r, 0], \mathbb{R}^n)$ be endowed with the supremum norm. For a continuous function $x : [t_0 - r, t_1] \rightarrow \mathbb{R}^n$ and $t \in [t_0, t_1]$, we define $x_t \in C([-r, 0], \mathbb{R}^n)$ by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$. Consider the parameter-dependent delay differential equation

$$\dot{x}(t) = \mathcal{F}(x_t, \mu), \tag{4}$$

where $\mathcal{F} : C([-r, 0], \mathbb{R}^n) \times I \rightarrow \mathbb{R}^n$ is sufficiently smooth and $I \subset \mathbb{R}$ is an interval. Assume that $x \equiv 0$ is an equilibrium for all $\mu \in I$, that is, $\mathcal{F}(0, \mu) = 0$. Denoting $L_\mu = D_x \mathcal{F}(0, \mu)$, The linearized equation in a neighborhood of $x = 0$ is

$$\dot{y}(t) = L_\mu y_t. \tag{5}$$

The operator L_μ admits a Riesz representation in terms of a matrix-valued measure, and the associated characteristic roots are the zeros of a characteristic equation of the form $\Delta(\lambda, \mu) = 0$ [5, 7].

The following theorem summarizes the classical Hopf scenario for delay differential equations.

Theorem 1 (see [6–8]). *Consider (4) and assume that $\mathcal{F}$ is of class C^k , $k \geq 3$, in a neighborhood of $(0, \mu_0)$. Suppose that the characteristic equation of (5) at $\mu = \mu_0$ has a simple pair of roots $\lambda_{1,2}(\mu_0) = \pm i\omega_0$ with $\omega_0 > 0$, that no other characteristic root lies on the imaginary axis, and that*

$$\left. \frac{d}{d\mu} \operatorname{Re} \lambda_1(\mu) \right|_{\mu=\mu_0} \neq 0.$$

Then a Hopf bifurcation occurs at $(0, \mu_0)$: there exists a local branch of nontrivial periodic solutions bifurcating from the equilibrium as μ passes through μ_0 .

In applications, the main technical difficulty is usually the evaluation of the crossing quantity $\frac{d}{d\mu} \operatorname{Re} \lambda(\mu_0)$, which provides the transversality condition. The aim of the present paper is to derive an explicit formula for the crossing quantity when the characteristic equation can be written in the general form given in the following subsection.

2.2 Characteristic equations generated by distributed delays

We consider the scalar characteristic equation:

$$R(z) = H(z, \tau), \quad (6)$$

where $z \in \mathbb{C}$ and $\tau \in I \subset (0, \infty)$ is the bifurcation parameter. In applications R is rational and $H(\cdot, \tau)$ is the Laplace transform of a delay kernel. More precisely, if $g(\cdot, \tau)$ is a probability density function on $[0, \infty)$ with finite first moment (mean) τ , then:

$$H(z, \tau) = \int_0^\infty e^{-zs} g(s, \tau) ds, \quad \int_0^\infty g(s, \tau) ds = 1, \quad \int_0^\infty sg(s, \tau) ds = \tau. \quad (7)$$

We consider the following assumptions.

Assumption 1. *There exist an open interval $J \subset (0, \infty)$, an open interval $I \subset (0, \infty)$, and a point $(\omega_*, \tau_*) \in J \times I$ such that:*

- (A1) R is holomorphic in a neighborhood of $iJ := \{i\omega : \omega \in J\}$;
- (A2) H is defined in a neighborhood of $iJ \times I$, is holomorphic in z for each fixed $\tau \in I$, and is of class C^1 with respect to τ ;
- (A3) $\Delta(z, \tau) := R(z) - H(z, \tau)$ satisfies

$$\Delta(i\omega_*, \tau_*) = 0, \quad R(i\omega_*) = H(i\omega_*, \tau_*) \neq 0;$$

- (A4) the purely imaginary root $z_* = i\omega_*$ is simple, that is,

$$\Delta_z(i\omega_*, \tau_*) = R'(i\omega_*) - H_z(i\omega_*, \tau_*) \neq 0.$$

Assumption 1 (A4) implies, by the implicit function theorem, that there exists a unique C^1 branch $z(\tau)$ of characteristic roots near τ_* such that

$$\Delta(z(\tau), \tau) = 0, \quad z(\tau_*) = i\omega_*.$$

The transversality condition relevant for the Hopf bifurcation is

$$\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} \neq 0. \quad (8)$$

3 A general transversality theorem

By Assumption 1 (A3), both $R(i\omega_*)$ and $H(i\omega_*, \tau_*)$ are nonzero. We choose local C^1 polar representations

$$R(i\omega) = \tilde{\rho}(\omega)e^{i\tilde{\theta}(\omega)}, \quad H(i\omega, \tau) = \rho(\omega, \tau)e^{-i\theta(\omega, \tau)}, \quad (9)$$

where $\tilde{\rho}(\omega) > 0$ for $\omega \in J$ and $\rho(\omega, \tau) > 0$ for $(\omega, \tau) \in J \times I$. Define the scalar functions

$$G_1(\omega, \tau) := \tilde{\rho}(\omega) - \rho(\omega, \tau), \quad G_2(\omega, \tau) := \tilde{\theta}(\omega) + \theta(\omega, \tau) - 2k\pi, \quad (10)$$

where the integer k is chosen so that

$$\tilde{\theta}(\omega_*) + \theta(\omega_*, \tau_*) = 2k\pi.$$

Then the condition $\Delta(i\omega, \tau) = 0$ is equivalent to the system

$$G_1(\omega, \tau) = 0, \quad G_2(\omega, \tau) = 0. \quad (11)$$

Theorem 2. Assume that Assumption 1 holds. Let $z(\tau)$ be the unique local root branch satisfying $z(\tau_*) = i\omega_*$. Then

$$\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} = \frac{\rho(\omega_*, \tau_*)}{|\Delta_z(i\omega_*, \tau_*)|^2} \det \begin{pmatrix} \partial_\omega G_1 & \partial_\tau G_1 \\ \partial_\omega G_2 & \partial_\tau G_2 \end{pmatrix}_{(\omega_*, \tau_*)}. \quad (12)$$

In particular,

$$\operatorname{sgn} \left(\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} \right) = \operatorname{sgn} \det DG(\omega_*, \tau_*), \quad G = (G_1, G_2). \quad (13)$$

Therefore the spectral transversality condition is equivalent to the transversal intersection of the modulus and phase curves in the (ω, τ) -plane.

Proof. Since $\Delta(z(\tau), \tau) = 0$, differentiation with respect to τ yields

$$\Delta_z(z(\tau), \tau) z'(\tau) + \Delta_\tau(z(\tau), \tau) = 0.$$

Because $\Delta_\tau = -H_\tau$, we obtain at $\tau = \tau_*$

$$z'(\tau_*) = \frac{H_\tau(i\omega_*, \tau_*)}{R'(i\omega_*) - H_z(i\omega_*, \tau_*)} = \frac{H_\tau(i\omega_*, \tau_*)}{\Delta_z(i\omega_*, \tau_*)}. \quad (14)$$

Taking real parts gives

$$\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} = \frac{\operatorname{Re} \left(\overline{H_\tau(i\omega_*, \tau_*)} \Delta_z(i\omega_*, \tau_*) \right)}{|\Delta_z(i\omega_*, \tau_*)|^2}. \quad (15)$$

We now compute the derivatives in terms of the polar representation (9).

Since

$$\frac{d}{d\omega} R(i\omega) = iR'(i\omega), \quad \frac{\partial}{\partial \omega} H(i\omega, \tau) = iH_z(i\omega, \tau),$$

a direct calculation yields

$$R'(i\omega) = (\tilde{\rho}(\omega)\tilde{\theta}'(\omega) - i\tilde{\rho}'(\omega))e^{i\tilde{\theta}(\omega)}, \quad (16)$$

$$H_z(i\omega, \tau) = -(\rho(\omega, \tau)\theta_\omega(\omega, \tau) + i\rho_\omega(\omega, \tau))e^{-i\theta(\omega, \tau)}, \quad (17)$$

$$H_\tau(i\omega, \tau) = (\rho_\tau(\omega, \tau) - i\rho(\omega, \tau)\theta_\tau(\omega, \tau))e^{-i\theta(\omega, \tau)}. \quad (18)$$

At the critical pair (ω_*, τ_*) , the identity $R(i\omega_*) = H(i\omega_*, \tau_*)$ implies

$$\tilde{\rho}(\omega_*) = \rho(\omega_*, \tau_*), \quad \tilde{\theta}(\omega_*) + \theta(\omega_*, \tau_*) = 2k\pi. \quad (19)$$

Hence $e^{i\tilde{\theta}(\omega_*)} = e^{-i\theta(\omega_*, \tau_*)}$, and therefore

$$\begin{aligned}\overline{H_\tau} \Delta_z &= (\rho_\tau + i\rho\theta_\tau) e^{i\theta} \left[(\tilde{\rho}\tilde{\theta}' - i\tilde{\rho}') e^{i\tilde{\theta}} + (\rho\theta_\omega + i\rho_\omega) e^{-i\theta} \right]_{(\omega_*, \tau_*)} \\ &= (\rho_\tau + i\rho\theta_\tau) \left[\rho(\tilde{\theta}' + \theta_\omega) + i(\rho_\omega - \tilde{\rho}') \right]_{(\omega_*, \tau_*)}.\end{aligned}$$

Taking real parts, we obtain

$$\operatorname{Re}(\overline{H_\tau} \Delta_z) = \rho \left[\rho_\tau(\tilde{\theta}' + \theta_\omega) + \theta_\tau(\tilde{\rho}' - \rho_\omega) \right]_{(\omega_*, \tau_*)}. \quad (20)$$

On the other hand, from (10) we have

$$\partial_\omega G_1 = \tilde{\rho}' - \rho_\omega, \quad \partial_\tau G_1 = -\rho_\tau, \quad \partial_\omega G_2 = \tilde{\theta}' + \theta_\omega, \quad \partial_\tau G_2 = \theta_\tau.$$

Therefore

$$\det DG = (\tilde{\rho}' - \rho_\omega)\theta_\tau + \rho_\tau(\tilde{\theta}' + \theta_\omega). \quad (21)$$

Combining (15), (20), and (21) proves (12). Since $\rho(\omega_*, \tau_*) > 0$, the sign identity (13) follows immediately. $\square$

Remark 1. *The Jacobian determinant in (12) is independent of the particular local branches chosen for the arguments in (9). Indeed, replacing $\tilde{\theta}$ or θ by another local branch only adds an integer multiple of 2π , which does not affect derivatives.*

Theorem 2 is especially convenient when one of the two equations in (11) can be solved locally with respect to τ .

Corollary 1. *Assume that Assumption 1 holds and that*

$$\theta_\tau(\omega_*, \tau_*) \neq 0. \quad (22)$$

Then there exist a neighborhood U of ω_ and a unique C^1 function $T_k : U \rightarrow I$ such that $T_k(\omega_*) = \tau_*$ and*

$$\tilde{\theta}(\omega) + \theta(\omega, T_k(\omega)) = 2k\pi, \quad \omega \in U. \quad (23)$$

Define

$$F_k(\omega) := \tilde{\rho}(\omega) - \rho(\omega, T_k(\omega)). \quad (24)$$

Then $F_k(\omega_) = 0$ and*

$$\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} = \frac{\rho(\omega_*, \tau_*) \theta_\tau(\omega_*, \tau_*) F'_k(\omega_*)}{|\Delta_z(i\omega_*, \tau_*)|^2}. \quad (25)$$

In particular,

$$\operatorname{sgn} \left(\frac{d}{d\tau} \operatorname{Re} z(\tau) \Big|_{\tau=\tau_*} \right) = \operatorname{sgn}(\theta_\tau(\omega_*, \tau_*)) \operatorname{sgn}(F'_k(\omega_*)). \quad (26)$$

Proof. Condition (22) allows us to solve (23) locally for $\tau = T_k(\omega)$ by the implicit function theorem. Differentiating (23) with respect to ω gives

$$T'_k(\omega_*) = -\frac{\tilde{\theta}'(\omega_*) + \theta_\omega(\omega_*, \tau_*)}{\theta_\tau(\omega_*, \tau_*)}. \quad (27)$$

Differentiating (24) and evaluating at ω_* , we obtain

$$F'_k(\omega_*) = \tilde{\rho}'(\omega_*) - \rho_\omega(\omega_*, \tau_*) - \rho_\tau(\omega_*, \tau_*) T'_k(\omega_*).$$

Using (27), we find

$$\theta_\tau(\omega_*, \tau_*) F'_k(\omega_*) = \theta_\tau(\omega_*, \tau_*) (\tilde{\rho}' - \rho_\omega) + \rho_\tau (\tilde{\theta}' + \theta_\omega),$$

all quantities being evaluated at (ω_*, τ_*) . The conclusion follows from (12). $\square$

Corollary 2. Assume that Assumption 1 holds and that

$$\rho_\tau(\omega_*, \tau_*) \neq 0. \quad (28)$$

Then there exist a neighborhood U of ω_* and a unique C^1 function $S_k : U \rightarrow I$ such that $S_k(\omega_*) = \tau_*$ and

$$\tilde{\rho}(\omega) - \rho(\omega, S_k(\omega)) = 0, \quad \omega \in U. \quad (29)$$

Define

$$P_k(\omega) := \tilde{\theta}(\omega) + \theta(\omega, S_k(\omega)) - 2k\pi. \quad (30)$$

Then $P_k(\omega_*) = 0$ and

$$\frac{d}{d\tau} \operatorname{Re} z(\tau) \Big|_{\tau=\tau_*} = \frac{\rho(\omega_*, \tau_*) \rho_\tau(\omega_*, \tau_*) P'_k(\omega_*)}{|\Delta_z(i\omega_*, \tau_*)|^2}. \quad (31)$$

Hence

$$\operatorname{sgn} \left(\frac{d}{d\tau} \operatorname{Re} z(\tau) \Big|_{\tau=\tau_*} \right) = \operatorname{sgn}(\rho_\tau(\omega_*, \tau_*)) \operatorname{sgn}(P'_k(\omega_*)). \quad (32)$$

Proof. The proof is analogous to that of Corollary 1. Differentiating (29) yields

$$S'_k(\omega_*) = \frac{\tilde{\rho}'(\omega_*) - \rho_\omega(\omega_*, \tau_*)}{\rho_\tau(\omega_*, \tau_*)},$$

and therefore

$$\rho_\tau(\omega_*, \tau_*) P'_k(\omega_*) = \rho_\tau(\tilde{\theta}' + \theta_\omega) + \theta_\tau(\tilde{\rho}' - \rho_\omega).$$

The claim now follows from (12). $\square$

A particularly important class of kernels is given by self-similar families.

Remark 2 (Self-similar kernels). *Assume that*

$$H(z, \tau) = \widehat{H}(\tau z) \quad (33)$$

for some analytic function $\widehat{H}$, and let

$$\widehat{H}(i\xi) = \widehat{\rho}(\xi) e^{-i\widehat{\theta}(\xi)}. \quad (34)$$

Then

$$\rho(\omega, \tau) = \widehat{\rho}(\omega\tau), \quad \theta(\omega, \tau) = \widehat{\theta}(\omega\tau), \quad (35)$$

and

$$\begin{aligned} \rho_\omega(\omega, \tau) &= \tau \widehat{\rho}'(\omega\tau), & \rho_\tau(\omega, \tau) &= \omega \widehat{\rho}'(\omega\tau), \\ \theta_\omega(\omega, \tau) &= \tau \widehat{\theta}'(\omega\tau), & \theta_\tau(\omega, \tau) &= \omega \widehat{\theta}'(\omega\tau). \end{aligned} \quad (36)$$

Moreover, if $\widehat{\theta}'(\omega_*\tau_*) \neq 0$, from Corollary 1 we have

$$\theta_\tau(\omega_*, \tau_*) = \omega_* \widehat{\theta}'(\omega_*\tau_*) \neq 0,$$

and hence:

$$\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} = \frac{\rho(\omega_*, \tau_*) \omega_* \widehat{\theta}'(\omega_*\tau_*) F'_k(\omega_*)}{|\Delta_z(i\omega_*, \tau_*)|^2}. \quad (37)$$

Hence

$$\operatorname{sgn} \left(\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} \right) = \operatorname{sgn}(\widehat{\theta}'(\omega_*\tau_*)) \operatorname{sgn}(F'_k(\omega_*)). \quad (38)$$

In particular, if $\widehat{\theta}'$ is positive on the relevant interval, then the crossing direction is determined solely by the sign of $F'_k(\omega_*)$.

In the particular case of a discrete time delay, we consider the Dirac kernel, and we obtain:

Corollary 3 (Discrete delay). *Suppose that $H(z, \tau) = e^{-\tau z}$. Then*

$$\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} = \frac{\omega_* \tilde{\rho}'(\omega_*)}{|\Delta_z(i\omega_*, \tau_*)|^2}, \quad \operatorname{sgn} \left(\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} \right) = \operatorname{sgn} \tilde{\rho}'(\omega_*). \quad (39)$$

If, moreover, $R = P/Q$ with polynomials P and Q , then at a critical frequency ω_* the crossing direction satisfies

$$\operatorname{sgn} \left(\left. \frac{d}{d\tau} \operatorname{Re} z(\tau) \right|_{\tau=\tau_*} \right) = \operatorname{sgn} \left. \frac{d}{d\omega} (|P(i\omega)|^2 - |Q(i\omega)|^2) \right|_{\omega=\omega_*}. \quad (40)$$

Proof. In this case $\hat{H}(\xi) = e^{-\xi}$, so $\hat{\rho} \equiv 1$ and $\hat{\theta}(\xi) = \xi$. Hence $F_k(\omega) = \tilde{\rho}(\omega) - 1$, and (39) follows from Remark 2. If $R = P/Q$, then

$$\tilde{\rho}(\omega) = \frac{|P(i\omega)|}{|Q(i\omega)|}.$$

Therefore

$$\tilde{\rho}'(\omega) = \frac{|Q(i\omega)| \frac{d}{d\omega} |P(i\omega)| - |P(i\omega)| \frac{d}{d\omega} |Q(i\omega)|}{|Q(i\omega)|^2}.$$

At a critical frequency we have $\tilde{\rho}(\omega_*) = 1$, hence $|P(i\omega_*)| = |Q(i\omega_*)|$, and consequently

$$\operatorname{sgn} \tilde{\rho}'(\omega_*) = \operatorname{sgn} \left(\left. |P(i\omega_*)| \frac{d}{d\omega} |P(i\omega)| \right|_{\omega=\omega_*} - \left. |Q(i\omega_*)| \frac{d}{d\omega} |Q(i\omega)| \right|_{\omega=\omega_*} \right).$$

This is equivalent to (40). $\square$

4 Consequences for stability switching

The previous section provides a local formula for the crossing direction of a simple pair of imaginary roots. We now provide a standard consequence for the variation of the unstable dimension.

Proposition 1. *Let $\nu(\tau)$ denote the number of characteristic roots of (6) in the open right half-plane, counted with algebraic multiplicity. Assume that at $\tau = \tau_*$ the only characteristic roots on the imaginary axis are the simple*

conjugate pair $\pm i\omega_*$ and that Assumption 1 holds. Then there exists $\varepsilon > 0$ such that

$$\nu(\tau) = \nu(\tau_*^-) + 2 \operatorname{sgn} \left(\frac{d}{d\tau} \operatorname{Re} z(\tau) \Big|_{\tau=\tau_*} \right) \quad \text{for } \tau \in (\tau_*, \tau_* + \varepsilon),$$

while $\nu(\tau) = \nu(\tau_*^-)$ for $\tau \in (\tau_* - \varepsilon, \tau_*)$. In other words, the unstable dimension changes by $+2$ or -2 according to the sign of the crossing direction.

Proof. The simplicity of the roots $\pm i\omega_*$ implies that they admit local continuations as a conjugate pair of characteristic roots. Since all other roots remain a positive distance away from the imaginary axis, the number of roots in the right half-plane can change only through this pair. The sign of

$$\frac{d}{d\tau} \operatorname{Re} z(\tau) \Big|_{\tau=\tau_*}$$

determines whether the pair moves from left to right or from right to left as τ increases. This provides the stated jump in $\nu(\tau)$. $\square$

An immediate consequence of Proposition 1 is:

Corollary 4 (Monotone crossing criterion). *Assume that all critical values of τ are simple Hopf points and that no two distinct pairs of characteristic roots cross the imaginary axis simultaneously. If at every critical pair (ω_j, τ_j) one has*

$$\operatorname{sgn} \left(\frac{d}{d\tau} \operatorname{Re} z(\tau) \Big|_{\tau=\tau_j} \right) = +1,$$

then $\nu(\tau)$ is increasing. In particular, an equilibrium which has become unstable cannot regain stability as τ increases.

The self-similar case gives a convenient sufficient condition for the avoidance of stability switching.

Corollary 5. *Assume that the hypotheses of Remark 2 hold, and suppose that $\widehat{\theta}'(\xi) > 0$ for every admissible critical value $\xi = \omega\tau$. If, in addition, each critical frequency is a simple zero of the corresponding auxiliary function F_k and satisfies*

$$F'_k(\omega_*) > 0,$$

then every simple Hopf crossing occurs from left to right as τ increases. Consequently, the unstable dimension is increasing and restabilization cannot occur.

Proof. By (38), the sign of the crossing direction is the sign of $F'_k(\omega_*)$. The conclusion follows from Corollary 4. $\square$

5 Examples

5.1 Erlang kernels

For $n \in \mathbb{N}$, consider the Erlang (gamma) family

$$g_n(s, \tau) = \frac{n^n}{(n-1)! \tau^n} s^{n-1} e^{-ns/\tau}, \quad s \geq 0, \quad (41)$$

which is a probability density on $[0, \infty)$ with mean τ . Its Laplace transform is

$$H_n(z, \tau) = \int_0^\infty e^{-zs} g_n(s, \tau) ds = \left(1 + \frac{\tau z}{n}\right)^{-n}. \quad (42)$$

The cases $n = 1$ and $n = 2$ correspond to the classical weak and strong gamma kernels, respectively.

Then, we can write the polar form:

$$H_n(i\omega, \tau) = \hat{\rho}_n(\omega\tau) e^{-i\hat{\theta}_n(\omega\tau)}, \quad (43)$$

where

$$\hat{\rho}_n(\xi) = \left(1 + \frac{\xi^2}{n^2}\right)^{-n/2}, \quad \hat{\theta}_n(\xi) = n \arctan\left(\frac{\xi}{n}\right). \quad (44)$$

Moreover,

$$\hat{\theta}'_n(\xi) = \frac{1}{1 + \xi^2/n^2} > 0. \quad (45)$$

Consequently, Remark 2 applies. Solving

$$\hat{\theta}_n(\omega T_{k,n}(\omega)) = 2k\pi - \tilde{\theta}(\omega)$$

gives

$$T_{k,n}(\omega) = \frac{n}{\omega} \tan\left(\frac{2k\pi - \tilde{\theta}(\omega)}{n}\right) \quad (46)$$

and

$$F_{k,n}(\omega) = \tilde{\rho}(\omega) - \cos^n\left(\frac{2k\pi - \tilde{\theta}(\omega)}{n}\right). \quad (47)$$

At any simple critical frequency ω_* satisfying $F_{k,n}(\omega_*) = 0$ one has

$$\operatorname{sgn}\left(\left.\frac{d}{d\tau} \operatorname{Re} z(\tau)\right|_{\tau=\tau_*}\right) = \operatorname{sgn}(F'_{k,n}(\omega_*)). \quad (48)$$

Remark 3. For $n = 1$ one obtains the exponential or weak kernel,

$$H_1(z, \tau) = \frac{1}{1 + \tau z}, \quad \widehat{\rho}_1(\xi) = \frac{1}{\sqrt{1 + \xi^2}}, \quad \widehat{\theta}_1(\xi) = \arctan \xi.$$

For $n = 2$ one obtains the strong kernel,

$$H_2(z, \tau) = \left(1 + \frac{\tau z}{2}\right)^{-2}, \quad \widehat{\rho}_2(\xi) = \left(1 + \frac{\xi^2}{4}\right)^{-1}, \quad \widehat{\theta}_2(\xi) = 2 \arctan\left(\frac{\xi}{2}\right).$$

Both kernels satisfy $\widehat{\theta}'_n > 0$, so the crossing direction is determined by $F'_{k,n}$.

5.2 A second-order characteristic equation with weak kernel

Consider the family

$$\lambda^2 + a\lambda + b + \frac{c}{1 + \tau\lambda} = 0, \quad a, b, c > 0. \quad (49)$$

This equation appears, for instance, after linearization of second-order systems with exponentially distributed delay. The second-order terms capture inertial or oscillatory behavior, which can be found in mechanical systems, biological and economic processes. The rational term $\frac{1}{1 + \tau\lambda}$ to the Laplace transform of the weak, or exponential, delay kernel (see, for example, the references [5], [10], [11], [15]).

Equation (49) can be rewritten as (6) with

$$R(\lambda) = -\frac{\lambda^2 + a\lambda + b}{c}, \quad H(\lambda, \tau) = \frac{1}{1 + \tau\lambda}. \quad (50)$$

Let

$$d(\omega) := \sqrt{(\omega^2 - b)^2 + a^2\omega^2}. \quad (51)$$

Then

$$R(i\omega) = \frac{\omega^2 - b - ia\omega}{c}, \quad \widetilde{\rho}(\omega) = \frac{d(\omega)}{c}. \quad (52)$$

Choosing the local argument branch in the lower half-plane, we also have

$$\cos \widetilde{\theta}(\omega) = \frac{\omega^2 - b}{d(\omega)}, \quad \sin \widetilde{\theta}(\omega) = -\frac{a\omega}{d(\omega)}. \quad (53)$$

Proposition 2. *For equation (49), the purely imaginary roots $\pm i\omega$ are determined by the relations*

$$\tau = \frac{a}{\omega^2 - b}, \quad \omega^2 > b, \quad (54)$$

and

$$p(\omega^2) = 0, \quad p(x) := x^2 + (a^2 - 2b - c)x + b^2 + bc. \quad (55)$$

Moreover, if $\omega_* > 0$ is a simple positive root of (55), then the corresponding crossing direction is given by

$$\operatorname{sgn} \left(\frac{d}{d\tau} \operatorname{Re} \lambda(\tau) \Big|_{\tau=\tau_*} \right) = \operatorname{sgn}(2\omega_*^2 + a^2 - 2b - c). \quad (56)$$

Proof. Since the weak kernel is self-similar with

$$\widehat{\rho}_1(\xi) = \frac{1}{\sqrt{1 + \xi^2}}, \quad \widehat{\theta}_1(\xi) = \arctan \xi,$$

Remark 2 applies. The phase condition is

$$\widetilde{\theta}(\omega) + \arctan(\omega\tau) = 0$$

for the relevant local branch. Taking the tangent and using (53), we obtain

$$\omega\tau = -\tan \widetilde{\theta}(\omega) = \frac{a\omega}{\omega^2 - b},$$

which yields (54). Next, the amplitude equation is

$$\widetilde{\rho}(\omega) = \widehat{\rho}_1(\omega\tau) = \cos(\arctan(\omega\tau)) = \cos \widetilde{\theta}(\omega) = \frac{\omega^2 - b}{d(\omega)}.$$

Hence

$$\frac{d(\omega)}{c} = \frac{\omega^2 - b}{d(\omega)}, \quad \text{i.e.} \quad d(\omega)^2 = c(\omega^2 - b).$$

Using (51), this becomes exactly (55). Finally,

$$F(\omega) = \widetilde{\rho}(\omega) - \widehat{\rho}_1(\omega T(\omega)) = \frac{d(\omega)}{c} - \frac{\omega^2 - b}{d(\omega)} = \frac{p(\omega^2)}{c d(\omega)}.$$

If $p(\omega_*^2) = 0$ and $\omega_* > 0$, then

$$\operatorname{sgn} F'(\omega_*) = \operatorname{sgn} \left(\frac{d}{d\omega} p(\omega^2) \Big|_{\omega=\omega_*} \right) = \operatorname{sgn}(2\omega_*^2 + a^2 - 2b - c).$$

Since $\widehat{\theta}'_1 > 0$, formula (56) follows from Remark 2. $\square$

Remark 4. For the discrete-delay we have:

$$\lambda^2 + a\lambda + b + ce^{-\tau\lambda} = 0. \quad (57)$$

This equation relies on the Dirac kernel, meaning the system's memory is localized at a single past moment. The exponential term $e^{-\tau\lambda}$ can model a strict transport lag, a fixed processing time in neural signal propagation, or a specific maturation period in population models, where the historical influence is strictly delayed rather than continuously distributed across an interval.

Corollary 3 gives the critical-frequency equation

$$q(\omega^2) = 0, \quad q(x) := x^2 + (a^2 - 2b)x + b^2 - c^2, \quad (58)$$

and the crossing direction

$$\operatorname{sgn} \left(\frac{d}{d\tau} \operatorname{Re} \lambda(\tau) \Big|_{\tau=\tau_*} \right) = \operatorname{sgn} (2\omega_*^2 + a^2 - 2b). \quad (59)$$

For instance, if $a = 1$, $b = 3$, and $c = 2$, then $q(x) = x^2 - 5x + 5$, so the critical values are $\omega_{1,2}^2 = \frac{5 \pm \sqrt{5}}{2}$. The corresponding crossings have opposite directions because $2\omega_1^2 - 5 < 0$ and $2\omega_2^2 - 5 > 0$. Thus, the characteristic equation admits two simple crossings with different orientation, a typical mechanism behind stability switching.

Using the formula (54) we calculate the critical value of the time delay. In Proposition 2 we have the condition $\omega^2 > b$ so that τ is positive. In our example $b = 3$, the frequency ω_2 ($\omega_2^2 \approx 3.618$) leads to a valid critical value $\tau_* = 1.618$.

6 Concluding remarks

A general formula for the Hopf transversality condition has been derived, associated with characteristic equations of the form $R(z) = H(z, \tau)$, where H is the Laplace transform of a distributed-delay kernel. The main contribution is the Jacobian identity (12), which shows that the spectral crossing direction is exactly the Jacobian determinant of the modulus and phase equations in the (ω, τ) -plane. This gives a direct geometric interpretation of the transversality condition and places a number of existing criteria into a single framework.

In addition, several particular cases have been discussed. When the phase equation can be solved locally for the delay parameter, the crossing direction is reduced to the derivative of an auxiliary scalar function F_k multiplied by the sign of θ_τ . An analogous reduction is available when the modulus equation is solvable. For self-similar kernels, the formula becomes particularly simple and immediately provides the classical discrete-delay criterion as a special case.

From an application perspective, the results are useful for two reasons. First, they provide a unified formula for $\frac{d}{d\tau} \operatorname{Re} z(\tau)$. Second, they make it possible to formulate monotonicity criteria for stability switching directly in terms of the functions governing the amplitude and phase relations on the imaginary axis. Several examples, including the Dirac, exponential, and Erlang kernels are given to illustrate the theoretical results.

Directions for future research include considering determinant-type characteristic equations associated with higher-dimensional systems, kernels depending on several shape parameters, and situations in which the characteristic equation cannot be reduced globally to a single branch of the phase relation. Combining the present criterion with center-manifold and normal-form computations can also be investigated, in order to obtain a unified treatment of existence, direction, and stability of Hopf bifurcations in general classes of distributed-delay models.

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