Annals of the Academy of Romanian Scientists
Series on Engineering Sciences
ISSN 2066 - 8570 Volume **4**, Number **1**/**2012**

MATHEMATICAL METHODS FOR THE OPTIMIZATION OF THE AEOLIAN AND HYDRAULICS ENERGIES WITH APPLICATIONS IN HYDRO-AERODYNAMICS

Mircea LUPU[1]

Rezumat. *Viața și activitatea oamenilor in natura si societate este dependenta primordial de aer, apă, lumina, căldură, sol și utilizarea energiilor eoliene, hidraulice, mecanice, electrice, generate din dinamica acestor medii. Dinamica acestor fenomene din natura este liniară ți majoritar neliniară, probabilistica – inducând o modelare matematică – pentru control optimal, prin ecuații de o mare complexitate. În lucrare autorul prezintă noi metode și modele matematice în optimizarea acestor fenomene cu aplicații tehnice efective: optimizarea palelor (cupelor) turbinelor hidraulice, eoliene sau pentru eliminarea noxelor aerului sau purificarea reziduală a apelor; acționări hidropneumatice (robotică) pentru echilibrarea stabilității navelor în ruliu; optimizarea provelor sau a velelor fluviale (acționate de vânt) pentru rezistenta sau forța de propulsie extremă; optimizarea profilelor avioanelor pentru extremarea forțelor de rezistență sau portanță, direcționarea navigației, parașute de frânare, impermeabile etc. Rezultatele științifice sunt însoțite de calcule numerice, integrându-se în literatura de specialitate din țară și străinătate.*

Abstract. *The people's life and activity in nature and society depends primary by air, water, light, climate, ground and by using the aeolian, hydraulic, mechanic and electrical energies, generated by the dynamics of these environments. The dynamics of these phenomena from the nature is linear and majority nonlinear, probabilistic – inducing a mathematical modeling – for the optimal control, with the equations with a big complexity. In the paper the author presents new mathematical models and methods in the optimization of these phenomena with technical applications: the optimization of the hydraulic, aeolian turbine's blades or for the eliminating air pollutants and residual water purification; the actions hydropneumatics (robotics) to balance the ship in roll stability, optimizing the sails (wind powered) for extreme durability or propelling force, optimizing aircraft profiles for the drag or the lift forces, directing navigation, parachute brake, the wall, etc. The scientific results are accompanied by numerical calculations, integrating in the specialized literature from our country and foreign.*

1. Theoretical and practical methods regarding the absorbitors of oscillations

It was modeled a hydraulic - system, regarding the dissipation of some discontinuous oscillations for obtaining the asymptotic stability. The amortization system was make by using the sensors with blocks of calculus and electronic control for the mathematical system at the input and output.

In the first part of this paper it's made the study regarding the dissipation of some discontinuous oscillations for obtaining the asymptotic stability. It is modeled a

[1]Prof. Ph.D., corresponding member of the Academy of Romanian Scientists (m.lupu@unitbv.ro).

hydraulic - pneumatic system for the oscillations of rolling (level) absorption, like a response for the fluid and dry amortization.

The amortization system is a pumping servo-mechanism of the alternative fluid in two tanks: using the sensors with blocks of calculus and electronic control for the mathematical system at the input and output.

The Tantal's basin – an application from automatics and hydraulics

We'll start to present the utility of the hydraulic automatic regulation with a first legendary example: "The Tantal's basin" from the Antique Greece. This was the first hydraulic system for the automatic regulation of the flow process based on auto-oscillations. The filling and discharge of the basin has a practical importance by these periodical effects. The legend: The monk Tantal has violated the rules and he was punished, tied by his legs near the basin, so the maximum level overtakes under his chin. In the maximum momentum, when he wants drink some water, the basin starts to discharge; so he make many bows.

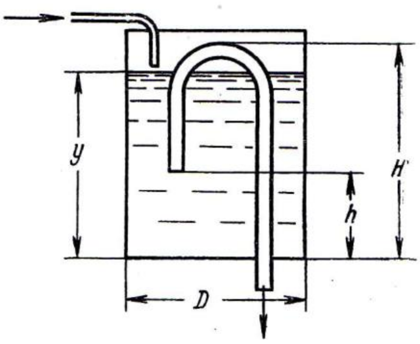


Fig. 1. The Tantal's basin.

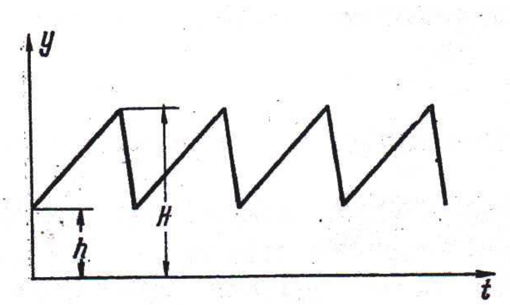


Fig. 2. The auto-oscillations graphic of the fluid level in the Tantal's basin.

The entire process has two phases:

1. In the basin the flow rate q is constant

2. When the fluid reaches the level H, in the basin it starts the discharge through the second pipe with the flow rate $Q > q$. When it reaches the level h both phases are repeating

The amortization of the ships

It is known that in their movement, the maritime or aviation ships and the dynamical systems from mechanics are perturbed (by the waves, wind or mechanical parameters). In this way are forced to make the rolling oscillations represented by the angle of rotation $\theta(t)$ side by a fixed mark. The attenuation (the dissipation) of these oscillations to obtain again the stabile regime of movement it is made by using some pumps leaded servo-mechanics and electronic - to transfer the fluid by contra balance in two tanks mounted symmetrically of the longitudinal and vertical axis. In this mode of discharge and filling of the tanks in contra balance with the amplitude it is obtain a spell effect by the friction and the variation of the fluid mass (left - right), involving the asymptotic stability with intermittences. In the phase plane the spiral trajectories are wavy with the change of the movement sense tend to the stable asymptotic focus.

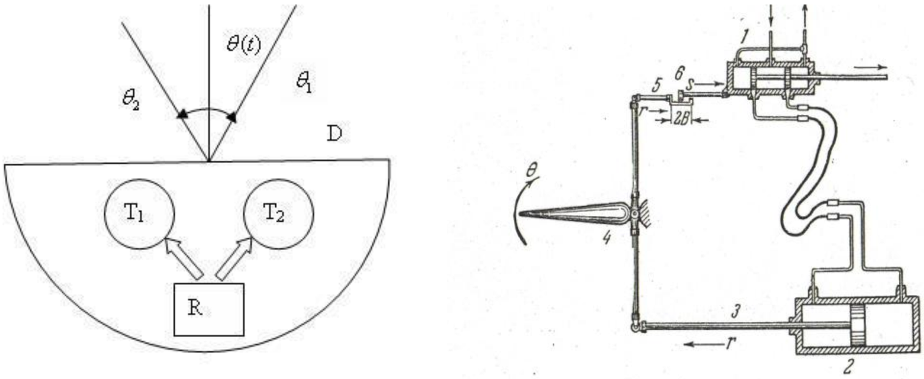


Fig. 3. a)A pump connected to two tanks; **Fig. 3.b)** The scheme of the hydraulic regulator.

It is thought that R is the hydraulic regulator (fig. 3b)) is the schematic figure and in the figure (fig. 3a) R is lied by the two tanks T_1, T_2 connected by the pipes acting by the pumps P.

Simplifying the model, the equation of the system with the free degree $\theta(t)$ will be:

$$J\ddot{\theta} = -M\dot{\theta} - Kr \tag{1}$$

Where J is the inertia momentum of the ship, relating the vertical axis which goes through the load centre, $\theta(t)$ is the rolling angle side by the $\theta = 0$; $M\dot{\theta}$ is the linear momentum of the amortization by the friction of the viscous fluid; r is the displacement of the mechanism arm owing to the rolling. This is proportional of the θ angle, so that $Kr = N\theta$. This is considering to be the reaction momentum necessary that the ship come back to the normal course. The rolling is signalized by the θ angle in the gyroscopically quadrant mounted on the ship. So, through the variation of this angle θ, the server (1) is involved by the pump (ex: to the right) and is opening the input of the fluid in the half right of the servo-motors (2). This produce the movement r of the bar (3) proportional with the angle θ, which will be signalize by the rolling indicator (4).

Because of the momentum obtained $N\theta$ like a response, the ship comes back to the normal course, making an amortized oscillation. For accelerating this amortization the hydraulic absorber must delimitate the rolling angle $\theta(t)$. In this way, this system is adding up an "inverse link" (5). The cylinder of the server acquires beside the rolling rotation to the right a double pressure. The server's valves are closing and the acting of the server is stopped. The reasoning is repeat acting to the left with the discharge of the right tank and the filling of the left one. So, we'll have a bigger amortization because of the momentum $N\theta$ and after a finite number of oscillations witch are amortizing by decreasing the amplitude, the ship is stabilized to the null solution.

The link is with "free bearing clearance" and for coulisse (6) an the $2B$ distance and the body' s server oscillating on the space $\pm B$, this mean $r \pm B \approx \theta$ or $r = \theta \pm B$ with the $+$ sign for $\dot{\theta} > 0$ and $-$ for $\dot{\theta} < 0$.

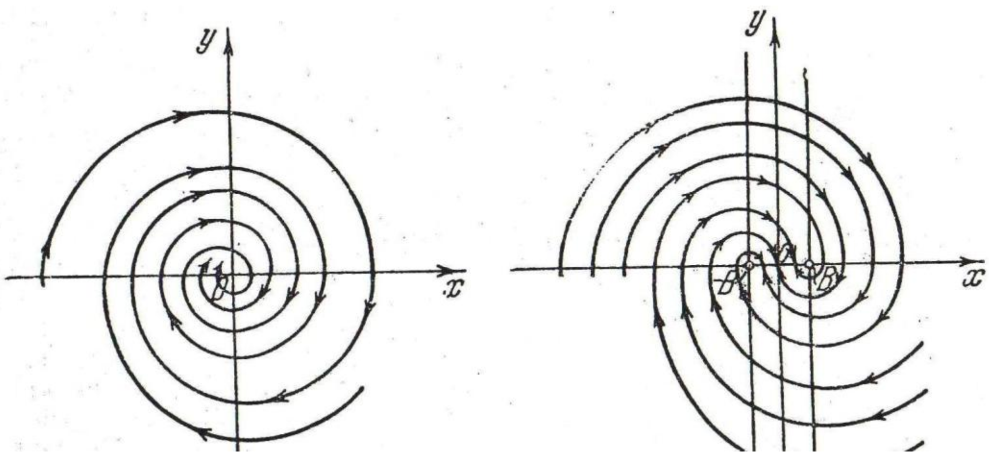


Fig. 4. The trajectory in the phases plane:
a)the homogeneous case; b)the non-homogeneous case.

$$\ddot{\theta} + 2r\dot{\theta} + k^2\theta = \pm k^2 B \tag{2}$$

In the phases plane the study of the movement is make thus: we are considering the movement from the right to the left with $\theta \in (-B, B)$. Noting $\theta = x_1 + B$:

$$\ddot{x}_1 + 2r\dot{x}_1 + k^2 x_1 = 0 \tag{3}$$

For $n^2 < k^2$ we have a low resistance and the solution is amortized linearly side by $x \equiv 0$ Having a displacing to the right face to the origin with B. The decrement of the oscillation is $\delta = e^{-n\pi/k_1}, k_1 = \sqrt{k^2 - n^2}$.

If the displacement is from the left to the right $\theta = x_2 - B$, we obtain for x_2:

$$\ddot{x}_2 + 2n\dot{x}_2 + k^2 x_2 = 0 \tag{4}$$

The amortization law is the same like (3), but the oscillation will be displaced to the left with B.

So, the trajectory in the phases space will be obtain traced at the first time the logarithmic spiral in the (x, y) plane, $\dot{x} = y$ with the solution for the case (fig. 4a)

$$x = e^{-nt}\left(C_1 \cos k_1 t + C_2 \sin k_2 t\right) \tag{5}$$

And then, $\theta = x \pm B$ (fig. 4b) for the non-homogeneous equation (2); graphically it was made a cut on the $x'Ox$ axis and we are displacing the upper figure to the right with B and the figure from the under half plane to the left with B. the spirals are merged by continuity and will be undulate to the origin thus: considering the initial position at $t = 0$, $\dot{x}_0^i > 0$ then with recurrence to the left:

$$x_0^{i+1} = (x_0^i - B)\delta - B = x_0^i\delta - B(1+\delta) \tag{6}$$

And the point will go to the left side of the origin if: $x_0^i\delta - B(1+\delta) > 0$ or $x_0^i > B\left(1+\frac{1}{j}\right) > 2B$.

The (6) formulae is conditioned by $0 < x_0^i \le B$. If $B < x_0^i \le B\left(1+\frac{1}{j}\right)$ the trajectory falls to the origin through the inferior side without cross in the interval $(-B, B)$, see (fig. 4b).

The presented previous solution has need of the exact knowledge of the process parameters (the tanks, pumps, pipes, etc.) a difficult fact of realize. The practical control of the level in the ballast tanks, can have as solution an adaptive multi-model system. The functionality of such system implies solving the next problems: the choice of the better model, the identification in the close loop of the adaptive model, the recalculation of the regulation algorithm. This article details some of the essential aspects of these problems.

2. New theoretical and practical methods in the profiles optimization in hydro aerodynamics

The mathematical problems of the advanced resistance theory of the profiles by hydro aerodynamics leads us to equations and nonlinear operators; in general the effective solving of the problems and the optimization of the solutions use numerical methods programmed with the help of the computer. In this paper we will present a series of methods which give the possibility to obtain the exact analytical solutions trough inverse methods [2] and for singular integral equations, [4], [5]. These methods are applied for the determination of the stem (the board attack – the wing) and of the flow regime for the hydro aerodynamics profiles. Applying the boundary problems theory in the complex field, with the help of analytical functions and of the conformal mapping, singular integral equations are obtained, which give the speed and pressure distributions. It is determined the advance resistance and profile shapes for the minimal or maximal resistance. The applications for the maximal resistance are in the case of the deflectors, the aeolian turbine cups, and of the sail propulsion forms. These new results have been promoted in research, appreciated on the international plan and in the education system for the mentioned domains. The optimization of the integral operators is used for obtaining the exact solutions using the integral Jensen inequality. The Newton problem for the stem optimization with the minimal advanced resistance is also solved in this paper is trough the mentioned methods. The problems are approached for the incompressible plane case and are the basis for the transition to compressible or spatial problems with axial symmetry.

In this part of paper we will use in order to solve and optimization mathematical models for naval sail systems. We consider rigid sails, plates rights or curves and search optimal shape for maximal propulsion in two limit cases, wind rectangular on the plate and wind paralel with profile chord. For optimization of performances, we study flaps sail model. These researches have start point theoretical and practical experiments of Naval Academy "Mircea cel Bătrân" team and the Reserch Institut for Wind Energy from Braşov. We present now theoretical model for plate sail used the hidrodynamic potential theory with free surfaces. The stationary potential plane flow of an inviscid fluid is considered in the absence of mass forces. Relating the velocity field $\vec{v} = u\vec{i} + v\vec{j}, u = u(x, y), v = v(x, y)$, to the frame in the physical flow domain $D_z, z = x + iy$, then within the hypothesis as well as from the continuity equation $div\vec{v} = 0$ and the condition for an irrotational flow $(rot\vec{v} = 0)$, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{7}$$

In this case, $\vec{v} = grad\varphi(x, y)$, and the velocity potential $\varphi = \varphi(x, y)$ is a harmonic function, $\Delta\varphi = 0$ in D_z. By introducing the stream function $\psi = \psi(x, y)$, the harmonic conjugate of φ in D_z, $\Delta\psi = 0$ in D_z, the relations (8) are obtained

$$u = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, v = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{8}$$

Thus, the complex potential of the flow is considered to be $f = f(z)$ and the complex velocity $w = w(z)$:

$$f(z) = \varphi(x, y) + i\psi(x, y), \overline{w} = \frac{df}{dz} = u - iv \tag{9}$$

In the hodograph plane (V,θ), where $V=\sqrt{(u^2+v^2)}$ is the magnitude of the velocity and $\theta = argw = arctg\dfrac{v}{u}$ the complex velocity angle with $x'Ox$ axis, the following relations can be written:

$$W = V + i\theta, u = V\cos\theta, v = V\sin\theta, w = Ve^{i\theta} \tag{10}$$

With (10), the transition relation $f = f(z)$ is obtained:

$$d\varphi + id\psi = (u - iv)(dx + idy) \tag{11}$$

In the case of free surface flow, the domain D_z is generally bounded by polygonal rigid walls, curvilinear obstacles and stream lines detaching from walls or obstacles. Along these free lines the velocity, the pressure and the density are $V^0, p^0, \rho = \rho^0 = const$. Applying Bernoulli's law for incompressible fluids $(\rho = const)$ along a streamline, $\psi = const$, we have, [1], [3]

$$\frac{1}{2}V^2 + \frac{p}{\rho} = \frac{1}{2}V^{0^2} + \frac{p^0}{\rho} \tag{12}$$

In the hypothesis (Hyp), we generally consider the plane parallel to infinity flow of an inviscid fluid which encounters a curvilinear obstacle $\lim_{|z|\to\infty} \vec{v} = V^0\vec{i}$. The Ox axis is the symmetry axis. This is the "Helmholtz model" of the symmetrical obstacle in unlimited fluid, [3], [7]. The repose zone is downstream and is delimited by obstacle and free lines. In the case of a curvilinear domain D_z it is generally difficult to obtain directly $f = f(z)$ and $w = w(z)$ by solving the boundary problem, therefore it should be introduced a canonic auxiliary domain $D_\zeta = \xi + i\eta$. In this paper, we choose the half-plane $D_\zeta^+ = \xi + i\eta, \eta > 0$, as canonic domain, we give some theoretic and applied results and we emphasize the computation techniques for analytic functions or nonlinear operators. We try to determine the analytic function $f = f(\zeta)$ which is the conformal mapping $D_f^+ \leftrightarrow D_\zeta^+$, with

$$f(\bar{\zeta}) = 0, \varphi_\xi = \psi_\eta, \varphi_\eta = -\psi_\xi \tag{13}$$

In order to obtain the analyticity conditions for the velocity $W(V,\theta)$ in D_ζ^+, we introduce the Jukovski's function ω, by considering $V = V^0$ along the free lines:

$$\omega = t + i\theta, \overline{w} = V^0 e^{-\omega}, t = \ln\frac{V^0}{V}, 0 \le V \le V^0 \tag{14}$$

$$\theta_\psi = t_\varphi, \theta_\varphi = -t_\psi, \varphi_\theta = -\psi_t, \varphi_t = \psi_\theta, \omega_{\bar{f}} = 0, f_{\bar{\omega}} = 0 \tag{15}$$

Now, we consider the following theorems, [4], [5].

Th.1 *In the hypothesis* (Hyp)*, if there is a conformal mapping*

$f = f(\zeta)$, $f_{\bar{\zeta}} = 0$, with $D_f^+ \leftrightarrow D_\zeta^+$, then $z = z(\zeta)$ is analytic with $D_z^+ \leftrightarrow D_\zeta^+$.

Th.2 *In the hypothesis* (Hyp)*, if the function* f *is analytic in* ζ *and realizes a conformal mapping* $D_f^+ \leftrightarrow D_\zeta^+$*, then* $\omega = \omega(\zeta)$ *is analytic and it is the conformal mapping* $D_\omega^+ \leftrightarrow D_\zeta^+$.

Writing the relation (11) along a stream line $\psi = const$ and using Theorem 1 one obtains the equations of this stream line (obstacle, free lines) and with $\eta = 0$ $\frac{\partial \varphi}{\partial \eta}\Big|_{\eta=0} = 0$, we have

$$x(\xi) = \int_{\xi_0}^{\xi} \varphi_\xi \frac{\cos\theta}{V} d\xi + x_0, y(\xi) = \int_{\xi_0}^{\xi} \varphi_\xi \frac{\cos\theta}{V} d\xi + y_0 \tag{16}$$

In order to obtain the functions $V = V(\xi)$ and $\theta = \theta(\xi)$, we carry out Theorem 2 and solve a mixed Riemann-Hilbert or Volterra problem for $\omega = \omega(\zeta)$ By (14), we then obtain $w = w(\zeta)$. Thus, the movement $f = f(z), w = w(z)$ (or parametric $f = f(\zeta), w = w(\zeta)$) is obtained by the composition $D_f^+, D_z^+, D_\omega^+ \leftrightarrow D_\zeta^+$.

Next we realize Theorems 1 and 2 for the "Helmholtz model" obtaining the integral singular equations for direct and inverse problems too. So, in the conditions stated above, we consider the plane flow of an unlimited fluid, moving infinitely upstream in an uniform translation movement of velocity $\vec{V}^0 = V^0\vec{i}$.

The fluid hits a symmetrical curvilinear obstacle (BOB') and in the points B and B', the free streamlines (BC) and $(B'C')$ of V^0 velocity are detached. The $x'Ox$ axis is the symmetry axis A_0O. The point A_0 is at infinity upstream, $\vec{V}(A_0) = V^0\vec{i}$. The free lines are asymptotically parallels to $x'Ox$ axis, $\vec{V}(C) = \vec{V}(C') = V^0\vec{i}$. $(CBOB'C')$ will be the repose zone behind the obstacle and $\vec{V}(O) = 0$.

We consider the correspondence between the domains D_z^+, D_f^+, D_w^+ with the half-plane $D_\zeta^+, \eta > 0$, so that the boundary (A_0OBC) will be replaced by the $\eta = 0$ axis, $\xi \in (-\infty, \infty) : A_0(-\infty), O(-1).B(1), C(+\infty)$ (Fig.6). So, the obstacle (OB) is the segment $(-1,1)$ and, in the physical plane D_z, the length of (OB) is L.

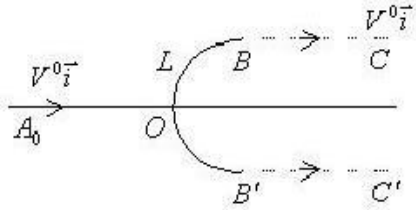


Fig. 5.

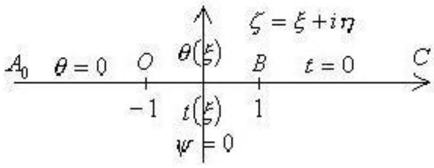


Fig. 6.

Obtaining of the integral equations for general shape of profiles

In order to determine the complex potential $f = \varphi + i\psi$ in the half-plane $D_\zeta^+, \eta > 0$, we have to solve the following Dirichlet problem: find an analytic function $f(\zeta) = \varphi + i\psi$ in $\eta > 0$ such that $\psi = 0$ on $\eta = 0, \xi \in (-\infty, \infty)$. The solution of this problem is:

$$f(\zeta) = A\zeta, A > 0, \frac{\partial\varphi}{\partial\xi}\Big|_{\eta=0} = A, \frac{\partial\varphi}{\partial\eta}\Big|_{\eta=0} = 0 \tag{17}$$

We determine $\omega = \omega(\zeta)$ in two manners. We have to find the analytic function $\omega = \omega(\zeta = t + i\theta$ in $\eta > 0$ knowing the following values on the boundary $\eta = 0 : \theta = 0, \xi \in (-\infty, -1), \theta = \theta(\xi)$ or $t = t(\xi), \xi \in (-1,1); t = 0, \xi \in (1, \infty)$. These are mixed problems and we transform them into Dirichlet problems for the analytic functions S_1, S_2 in $\eta > 0$:

$$S_1(\zeta) = R_1 + iI_1 = \omega(\varsigma)\big/\sqrt{(\zeta+1)},$$
$$R_1 = 0, \xi \in (-\infty,-1) \cup (1,+\infty); R_1 = t(\xi)\big/\sqrt{\xi+1}, \xi \in (-1,1)$$
$$S_2(\zeta) = R_2 + iI_2 = \omega(\zeta)\big/\sqrt{(\zeta-1)}:$$
$$R_2 = 0, \xi \in (-\infty,-1) \cup (1,+\infty); R_2 = t(\xi)\big/\sqrt{1-\xi}, \xi \in (-1,1)$$

From the Cisotti formula we have, [4], [5]

$$\omega(\zeta) = \frac{\sqrt{\zeta+1}}{\pi i}\int_{-1}^{1}\frac{t(s)}{\sqrt{s+1}}\frac{ds}{s-\zeta} + C_1 i, \zeta \in D_\zeta^+ \tag{18}$$

$$\omega(\zeta) = \frac{\sqrt{\zeta-1}}{\pi i}\int_{-1}^{1}\frac{\theta(s)}{\sqrt{1-s}}\frac{ds}{s-\zeta} + C_2 i, \zeta \in D_\zeta^+ \tag{19}$$

with the constants $C_1 = 0, C_2 = 0$ if $V(\xi = -1) = 0$ and $t(\xi = -1) = +\infty$.

Applying the Sohotski-Plemelj formula to the Cauchy integrals (18), (19) when $\zeta = \xi, \eta = 0^+$ for $\xi \in (-1,1)$ we obtain, [4], [5]

$$\theta(\xi) = \frac{\sqrt{\zeta+1}}{\pi}\int_{-1}^{1}\frac{t(s)}{\sqrt{s+1}}\frac{ds}{s-\zeta}\,\zeta \in (-1,1) \tag{20}$$

$$t(\xi) = \frac{\sqrt{\zeta-1}}{\pi}\int_{-1}^{1}\frac{\theta(s)}{\sqrt{1-s}}\frac{ds}{s-\zeta}, \zeta \in (-1,1) \tag{21}$$

The singular integrals are taken in the sense of Cauchy's principal value. By (11), the arc element on (OB) is

$$dS = \varphi_\xi \frac{d\xi}{V(\xi)}, S = \int_{-1}^{\xi} A\frac{d\xi}{V(\xi)}, \xi \in (-1,1) \tag{22}$$

and from (16) and (14) the length (OB) is

$$L = A\int_{-1}^{1}\frac{d\xi}{V(\xi)} = \frac{A}{V^0}\int_{-1}^{1}e^{t(s)}ds. \tag{23}$$

If the length L and the distribution of the velocity along (OB) are given, then the parameter A can be found. With these results we can emphasize the inverse problems for the "Helmholtz model" with curvilinear obstacle. If the distribution of the velocity $V = V(\theta)$, i.e. $t = t(\theta(\xi))$, or the pressure $p = p(\theta)$ on the profile (OB) are given, then (21) is a singular integral equation with the unknown function $\theta = \theta(\xi)$. Next, the functions $\omega = \omega(\zeta), w = w(\zeta)$ may be deduced (see (18), (19)). The relations (16) give us the equation of the profile (OB): $z = z(\xi) = \int_{\xi_0}^{\xi}\varphi_\xi \frac{e^{i\theta}}{V}d\xi + z_0$. We remark that the relations (20), (21) are inversion formulae for $\theta(\xi) \leftrightarrow t(\xi), \xi \in (-1,1)$.

The general problem of the prow

We consider the inverse problem, when the velocity distribution or the angle velocity distribution on the profile is known .Let the distribution of the velocity on the obstacle (OB), fig.1, be

$$V = V(\xi) = V^0\sqrt{\frac{1+\xi}{2}}, \quad \xi \in (-1,1), t = t(\xi) = \ln\frac{V^0}{V} = \ln\sqrt{\frac{2}{1+\xi}}, \tag{24}$$

where $V(O) = V(\xi=-1) = 0,\ \ V(B) = V(\xi=1) = V^0.$ The choice of this distribution is motivated by the fact that the function $V = V(\xi)$ from relations (22),

(23) must verify $V(-1)=0$ and assure the convergence of the integrals. So, it is necessary to have $V(\xi)=(1+\xi)^{\alpha}g(\xi), 0\prec\alpha\prec 1, g(-1)\neq 0.$

Thus, (18) is a choice with $\alpha=1/2$ and $V'(\xi)\succ 0.$ From (24) and (20) we obtain the velocity angle along (OB)

$$\theta(\xi)=\frac{\pi}{2}-T\left(\sqrt{\frac{1+\xi}{2}}\right), \xi\in(-1,1). \tag{25}$$

Here,

$$T(a)=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{a^{2n+1}}{(2n+1)^2}=\frac{1}{\pi}\left[Li_2(a)-Li_2(-a)\right], Li_2(a)=\sum_{n=1}^{\infty}\frac{a^n}{n^2}, \qquad |a\prec 1| \tag{26}$$

It`s easy to see that, [6]

$$\frac{dT}{da}=\frac{1}{\pi a}\ln\frac{1+a}{1-a}, T(a)=\frac{1}{\pi}\int_0^a \ln\frac{1+a}{1-a}\frac{da}{a}, \quad T(\pm 1)=\pm\frac{\pi}{4}, \quad T(0)=0$$

From (19) we have $\theta(O)=\theta(\xi=-1)=\frac{\pi}{2}, \theta(B)=\theta(\xi=1)=\frac{\pi}{4},$ and we observe that the curve (BOB') has a continuous tangent with $\theta'(\xi)\prec 0$ which assure the downstream convexity.

We remark that (24) and (25) are inversion formulae for the integral singular equations (20) and (21), and we have in the hodograph plane (V,θ) on the profile

$$\theta(V)=\frac{\pi}{2}-T\left(\frac{V}{V^0}\right), \quad 0\leq V\leq V^0, \theta=\theta(V), \quad V=V(\theta) \tag{27}$$

We consider the general case,

$$V=V(\xi)=V^0\left(\frac{1+\xi}{2}\right)^{\alpha}; \xi\in(-1,1), t=t(\xi)=\ln\frac{V^0}{V}=\left(\frac{2}{1+\xi}\right)^{\alpha}; \tag{28}$$

From (14) and (22) we obtain the velocity angle along (OB)

$$\theta(\xi)=\alpha\pi-2\alpha T\left[\sqrt{\frac{1+\xi}{2}}\right]; \xi\in[-1,1] \tag{29}$$

From (29) we have $\theta(0)=\theta(\xi=-1)=\alpha\pi; \theta(B)=\theta(\xi=1)=\frac{\alpha\pi}{2}.$

Knowing L and V^0 from (23) and (29) it is possible to determine the value of parameter A:

$$L = \frac{A}{V^0}\int_{-1}^{1} e^t d\xi = \frac{A}{V^0}\frac{2}{1-\alpha}, \alpha \in \left[0, \frac{1}{2}\right] \tag{30}$$

In the physical plane D_z the equations of the obstacle (OB) are obtained and the graph can be numerically drawn:

$$X(\xi) = \frac{x(\xi)}{L} = \frac{\sqrt{2}}{4}\int_{-1}^{\xi}\frac{\cos\theta(s)}{\sqrt{1+s}}ds, Y(\xi) = \frac{y(\xi)}{L} = \frac{\sqrt{2}}{4}\int_{-1}^{\xi}\frac{\sin\theta(s)}{\sqrt{1+s}}ds, \quad \xi \in (-1,1) \tag{31}$$

One similarly obtains the equations of the free line (BC).

Next, we compute the resultant of pressures for the whole profile (BOB') and the drag coefficient C_x.

$$P = \frac{\rho(V^0)^2 L}{2}\frac{16\alpha^2(1-\alpha)}{\pi} \tag{32}$$

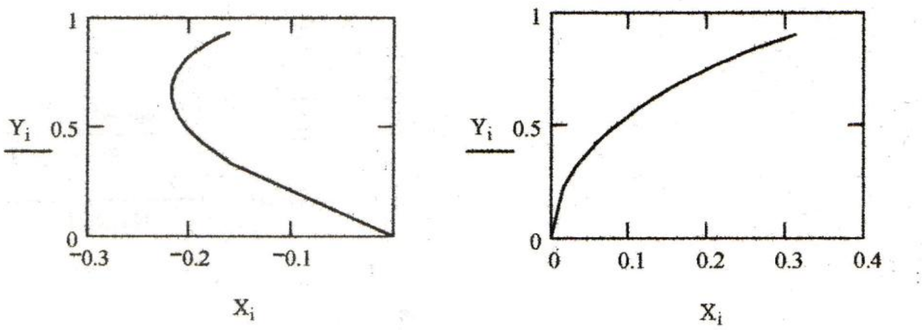
The case in which we have just one parameter with $C_x(\alpha) \Rightarrow C_x(\alpha)$ is maximal.

$$C_x = \frac{P}{\frac{\rho(V^0)^2 L}{2}} = \frac{16\alpha^2(1-\alpha)}{\pi}; C_x^1(\alpha) \equiv 0 \tag{33}$$

If we take $\alpha \in [0, 1/2]$ the $C_{\max} = C_x(\alpha = \frac{1}{2}) = \frac{2}{\pi} = 0{,}638$ and for $\alpha \in (0,1)$ we obtaining

$$C_{\max} = C_x(\alpha = \frac{2}{3}) = \frac{64}{27\pi} = 0{,}75\,. \tag{34}$$

The graphics of the profiles with maximal drag are:



2.3 The problem of the normal plate in unlimited flow

Let us consider in the same hypothesis the plane flow of an unlimited fluid, moving infinitely upstream in an uniform translation movement of velocity $\vec{V}^0 = V^0\vec{i}$ which encounters the symmetric plate (BOB'),see annexe 1. Similarly, the free streamlines (BC) and $(B'C')$ are detached in the points B and B'. This is the Helmholtz's problem. We will solve this direct problem by means of the results prescribing along the plate $\theta(\xi) = \frac{\pi}{2}, \xi \in (-1,1)$. Due to the symmetry, we consider the physical half-plane $D_z^+, y > 0$, delimited by (A_0OBC) with the same correspondence in the half-plane $D_\zeta^+, \eta > 0$. Thus, replacing $\theta(\xi) = \frac{\pi}{2}$ in (21) we get the distribution of the velocity on the plate (OB)

$$t(\xi) = \frac{1}{2}\ln\frac{\sqrt{2}+\sqrt{1-\xi}}{\sqrt{2}-\sqrt{1-\xi}}, V(\xi) = V^0\left(\frac{\sqrt{2}-\sqrt{1-\xi}}{\sqrt{2}+\sqrt{1-\xi}}\right)^{\frac{1}{2}}, \xi \in (-1,1) \tag{35}$$

It's clear that $V(O) = V(\xi = -1) = 0$ and $V(B) = V(\xi = 1) = V^0$. Computing length of the plate (BOB') and using the relations (35), (23) we have $L = \frac{A}{V^0}\frac{\pi+4}{2}, A = \frac{2LV^0}{\pi+4}$. Knowing F and V^0 one determines the parameter A. To compute the distribution of the velocity along (A_0O) with $\theta = 0$, we remake the computations of Section 3 with distribution (24) and we get

$$t(\xi) = \frac{\sqrt{-1-\xi}}{\pi}\int_{-1}^{1}\frac{t(s)}{\sqrt{1+s}}\frac{ds}{s-\xi} = \frac{1}{2}\ln\frac{\sqrt{1-\xi}+\sqrt{2}}{\sqrt{1-\xi}-\sqrt{2}}, \xi \in (-\infty,-1) \tag{36}$$

$$V = V^0\left(\frac{\sqrt{1-\xi}-\sqrt{2}}{\sqrt{1-\xi}+\sqrt{2}}\right)^{\frac{1}{2}}, \xi \in (-\infty,-1) \tag{37}$$

$$P = \frac{\rho V^{0^2}}{2}\int_{-1}^{1}\left[1-\left(\frac{V}{V^0}\right)^2\right]\frac{A}{V(\xi)}d\xi = \frac{\rho V^{0^2}}{2}A\int_{-1}^{1}\sqrt{\frac{1-\xi}{1+\xi}}d\xi = \pi A\frac{\rho V^{0^2}}{2} \tag{38}$$

Then using (25), the drag coefficient is given by, [3]

$$C_x^P = \frac{2\pi}{\pi+4} \cong 0{,}87980 \tag{39}$$

Propulsion force will be equal with $C_x^P \cdot S$, where S is plate aria.

2.4 The problem of the curve plate in unlimited flow

Let us consider that the unlimited fluid encounters the symmetrical, curvilinear, upstream convex obstacle (BOB'). The free lines $(BC),(B'C')$ are detached in the points B, B' and will be infinitely downstream parallel to the Ox axis. Our purpose is to find the shape of the obstacle with maximal drag. The length of the curve $(B'OB)$ is given and is equal to $2L$. These profiles of maximal drag are called "deflectors" or "impermeable parachutes" , and they still correspond to the "Helmholtz model". They are very important in relation with applications to the thrust reversal devices or the direction control of the reactive vehicles.

We notice also other applications to the slowing by fluid jets or to the jet flaps systems from the airplanes wings. Within the hypothesis of Section 3 and V^0, L being given we ask the condition of maximum P and we want to determine the distribution of the velocity on the profile (OB), i.e. $V = V(\xi)$ or $t = t(\xi), \xi \in (-1,1)$.

The resultant P is,

$$P = \frac{i\rho V^{0^2} L}{2} \frac{\oint_{K_\zeta} e^{\omega(\zeta)} d\zeta}{\int_{-1}^{1} e^{t(s)} ds} = \frac{\rho V^{0^2} L}{2\pi} \frac{\left(\int_{-1}^{1} \frac{t(s)}{\sqrt{1+s}} ds \right)^2}{\int_{-1}^{1} e^{t(s)} ds} \tag{40}$$

We write: $P = \frac{\rho V^{0^2} L}{2} J[t]$ where the nonlinear functional

$$J[t] = \frac{\left(\int_{-1}^{1} \frac{t(s)}{\sqrt{1+s}} ds \right)^2}{\int_{-1}^{1} e^{t(s)} ds} \tag{41}$$

must be maximized. To assure the convergence of the integrals and using (23) and $V(\xi = -1) = 0$ we put $V(\xi) = \frac{(1+\xi)^\alpha}{2} g(\xi)$ with $0 < \alpha < 1$ and $g(-1) \neq 0$.

Without losing the generality, we choose $\alpha = \frac{1}{2}$. From $t(\xi) = \ln \frac{V^0}{V(\xi)}$ we have

$$t(\xi) = G(\xi) + \ln \sqrt{\frac{2}{1+\xi}}, \xi \in (-1,1), \tag{42}$$

where the term $G(\xi)$ is generated by $g(\xi)$. Introducing (42) in (41) we get

$$J[G]=\frac{\left[\int_{-1}^{1}\frac{G(s)}{\sqrt{1+s}}ds+2\sqrt{2}\right]^2}{\sqrt{2}\int_{-1}^{1}\frac{e^{G(s)}}{\sqrt{1+s}}ds} \qquad (43)$$

In order to find $G(\xi)$, the functional $J[G]$ is maximized to a functional $H[G](J[G]\le H[G])$ whose maximum point may be easily computed and where the two functionals have the same value. The Jensen's inequality is: if $f(x)\ge 0$, $g(x)$ are integrable functions on $[a,b]$, then

$$\int_a^b f(x)e^{g(x)}dx \ge \left(\int_a^b f(x)dx\right)e^{\frac{\int_a^b f(x)g(x)dx}{\int_a^b f(x)dx}} \qquad (44)$$

where the equality case holds if and only if $g(x)$ is a constant function. Applying the Jensen's inequality for (43) we have, [5], [6]

$$J[G]\le\frac{2(U+1)^2}{e^U}\equiv H[U(G)], U(G)\equiv\frac{\sqrt{2}}{4}\int_{-1}^{1}\frac{G(s)}{\sqrt{1+s}}ds. \qquad (45)$$

In (39), the equality holds if and only if $G(\xi)\equiv G_0=const$. The functional $H[U]$ has a maximum point in $U_0\equiv 1$, obtained by differentiation and $H_{max}=H[U_0=1]=\frac{8}{e}$. For $G(\xi)\equiv G_0=1$ in (45) we have equality and then $J_{max}=J[G_0=1]=\frac{8}{e}$. Using (42) we obtaine:

$$t(\xi)=1+\ln\sqrt{\frac{2}{1+\xi}}, V(\xi)=\frac{V^0}{e}\sqrt{\frac{1+\xi}{2}}, \xi\in(-1,1) \qquad (46)$$

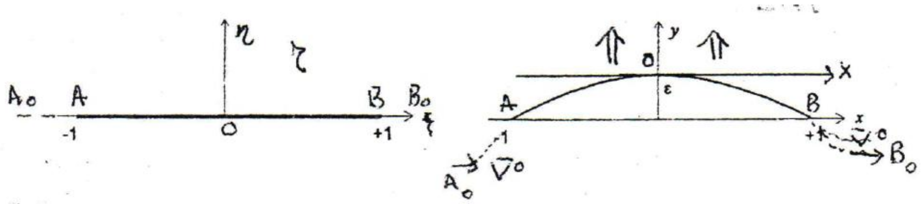
The result will be $P_{max}=\frac{\rho V^{0^2}L}{2\pi}J_{max}=\frac{4\rho V^{0^2}L}{\pi e}$. The maximal drag coefficient will be $C_x^{II}=\frac{2P}{\rho v^{0^2}L}$, i.e.

$$C_x^{II}=\frac{8}{\pi e}=0{,}936797. \qquad (47)$$

This result is in agreement with that obtained by Maklakov using the Levi-Civita method, [6].

3. The deduction of the plate sail for maxim lift in a wind parallel with profile chord

We consider a symmetrical curve plate (AB) axis x'x on profile chord and O in the middle of the chord and the wind is parallel with the chord. (Annexe 2) Let be $L(AB)$ length of (AB) and l length of chord known again A_0A, BB_0 free lines with A_0AMBB_0 stream line $\psi = 0$. We denote by $k = \frac{L-l}{l}$ and we will must to determine optimal geometrical shape for maximum lift P(rectangular on chord]. We consider T1,T2 theorems with integral equations (20),(21) and we will determine potential function $f = f(\zeta)$ and $\overline{w} = \overline{w}(\xi)$ in the upper half plane, $\eta \geq 0$; the plate (AB) being lateral acting of wind with the speed $V^0\vec{i}$.



Let be $f(\zeta)$ complex potential and $\overline{w} = \frac{df}{dz} = \frac{df}{d\zeta}\frac{d\zeta}{dz}$,

$$f(\zeta) = AV^0\zeta; dz = \varphi\xi e^{i\theta} d\xi, \psi = 0, \eta = 0 \tag{48}$$

$$z(\xi) = \int_{-1}^{\xi} \varphi_\xi \frac{e^{i\theta}}{V(\xi)} d\xi, dS = \varphi_\xi \frac{d\xi}{V(\xi)} \tag{49}$$

$$L = AV^0 \int_{-1}^{1} \frac{d\xi}{V(\xi)} = A\int_{1}^{1} e^{t(S)} dS. \tag{50}$$

The resultant of pressures is

$$X + iY = i\rho V^{0^2} A\oint e^{\omega(\zeta)} d\zeta \tag{51}$$

and because the symmetry, $X = 0, Y = \rho V^{0^2} A\int_{-1}^{1} t(S)dS$.

Here, $\omega(\zeta) = t + i\theta = -\frac{1}{\pi i}\int_{-1}^{1}\frac{t(s)}{s-\zeta}ds$ for the inferior half plane. The lift will be:

$$Y = \rho V^{0^2} LJ(t); J(t) = \frac{2\int_{-1}^{1} t(S)dS}{\int_{-1}^{1} e^{t(S)} dS} \tag{52}$$

Applying the Jensen's inequality at denominator of $J \le I$, we search the velocity distribution on (AB) so that the functional J(t) to be maximum; if we note $H = \int_{-1}^{1} t(S)dS$ we will obtain $J \le I = He^{-\frac{H}{2}}$ in the case equal the functional is $I_{\max}$.

For the maximum, the derived $I'(H) = 0$, with $H \succ 0, I' = e^{-\frac{H}{2}}\left(1 - \frac{H}{2}\right)$. For $H = 2, I_{\max} = \frac{2}{e}$ and $t(\xi) = 1$.We obtain with Sohotski and Plemelj

$$A = \frac{L}{2e}, V = \frac{V^0}{e}, \theta(\xi) = -\frac{1}{\pi}\int_{-1}^{\xi}\frac{dS}{(S-1)} = \frac{1}{\pi}\ln\frac{\sqrt{2}+\sqrt{1+\xi}}{\sqrt{2}-\sqrt{1+\xi}}, \xi \in (-1,1)$$

and with
$$\frac{l}{L} = \int_{-1}^{1}\cos\theta(\xi)d\xi = \frac{2e}{e^2-1}, k = sh(e-1).$$

From Y with I_{max} we obtain the lift coefficient (Annexe 1)

$$C_y = \frac{Y}{\rho {V^0}^2 L} \le C_{y\max}, C_{y\max} = \frac{2}{e}(1+k).$$

The optimal lift for plate will be $P_{\max} = C_{y\max} \cdot S, k = 0{,}175, C_{y\max} \approx 0{,}876$.

Wu and Whitney have study this problem with application at the flight of "para-slope". Also, Maklakov has found this solution.

R E F E R E N C E S

[1] Dragos, L., Fluid Mechanics Ed. Ed. Acad. Romane, Bucuresti, **1999** (Romanian).

[2] Elizarov, A. M., Il'inski, N. B., Potashev, A. V., Inverse boundary value problems in aerohidrodynamics, Fiz-Mat, Moskva, **1994** (Russian).

[3] Iacob, C., Introduction mathematique a la mecanique des fluides, Gauthier - Villars, Bucharest - Paris, **1959**.

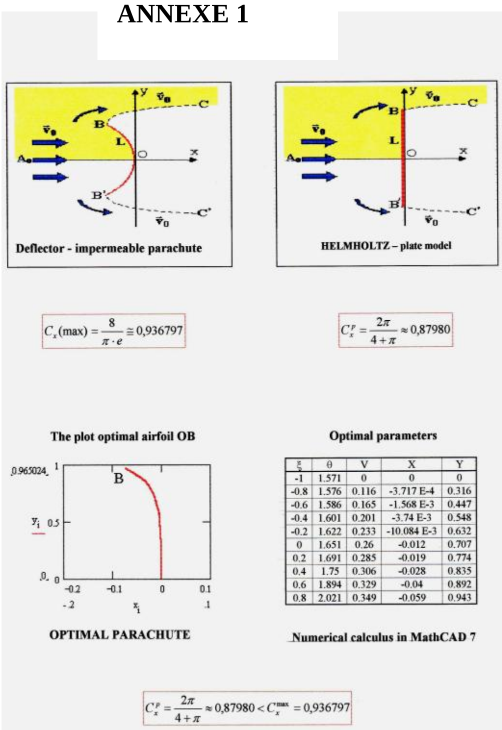
[4] Lupu, M., Scheiber, E., Analytical methods for airfoils optimization in the case of nonlinear problems in jet aerodynamicsMath. Reports. Acad. Romana, 1 (**2001**), pp. 33-43

[5] Lupu, M, Scheiber, E., Exact solution for an optimal impermeable parachute problemJ. Comput. and Applied Math., Elsevier, 147(**2002**), pp 227 – 286.

[6] Maklakov, D. V., Nonlinear problems in hydrodynamics, Nonlinear problems in hydrodynamics.

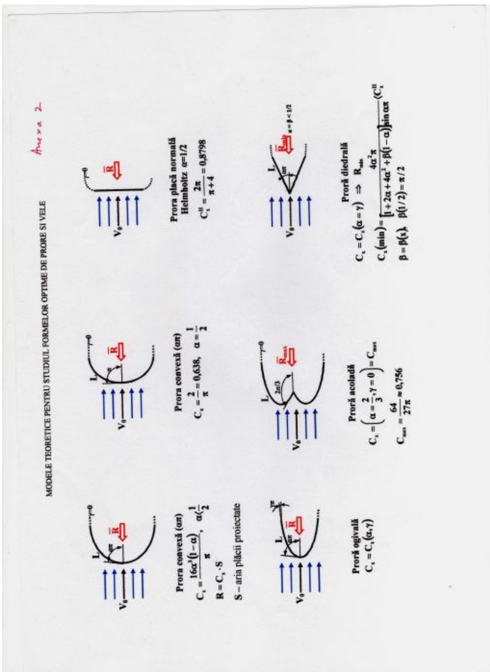
[7] Marcov, N., Carabineanu, A., The study of the fluid flows with free lines by nonlinear Integral equations, Math. Reports. Acad. Romana, 3(**1999**), pp. 359-377.

ANNEXE 1



ξ	θ	V	X	Y
-1	1.571	0	0	0
-0.8	1.576	0.116	-3.717 E-4	0.316
-0.6	1.586	0.165	-1.568 E-3	0.447
-0.4	1.601	0.201	-3.74 E-3	0.548
-0.2	1.622	0.233	-10.084 E-3	0.632
0	1.651	0.26	-0.012	0.707
0.2	1.691	0.285	-0.019	0.774
0.4	1.75	0.306	-0.028	0.835
0.6	1.894	0.329	-0.04	0.892
0.8	2.021	0.349	-0.059	0.943

ANNEXE 2



ANNEXE 3