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ON SOME CONCEPTS OF (h, k)-SPLITTING FOR SKEW-EVOLUTION SEMIFLOWS IN BANACH SPACES*

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Abstract

The paper treats some concepts of (h, k)-splitting for the general case of skew-evolution semiflows in Banach spaces. We obtain characterizations for these notions, as well as connections between them. As particular case, we emphasize the results for the corresponding properties of (h, k)-trichotomy.

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1 Introduction

The qualitative theory of the asymptotic behaviors of dynamical systems is a prolific research area, with an important development in the last years.

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Different types of uniform (nonuniform) asymptotic properties are approached as: stability, dichotomy and trichotomy (see [6], [8], [11] and the references therein).

Also, in the last period, we remark a special attention for more general concepts of dichotomy (trichotomy), called (h, k)-dichotomy (trichotomy), with h and k growth rates ([10], [12], [16], [21], [23]). This study is motivated for instance in [4].

A classical and well-studied subject in the field of differential equations is the theory of skew-evolution, which arise as a solution of the equation

$$\dot{v}(t) = A(\varphi(t, s, x))v(t), \quad t \geq s \geq 0,$$

where φ is an evolution semiflow on a locally compact metric space X and $A(\varphi(t, s, x))$ a bounded linear operator on a Banach space V, for each $t \geq s \geq 0$ and $x \in X$.

The pair $C = (\varphi, \Phi)$, with Φ evolution cocycle and φ evolution semiflow is called skew-evolution semiflow (see Section 2 for definitions) and it is a natural generalization of the notion of skew-product semiflow treated in [5], [7], [9], [17]-[20]. Important results concerning the qualitative theory of skew-evolution semiflows are obtained in [14], [23], [24].

The property of exponential splitting was approached for the first time in [1]-[3], [22] for differential systems and recently in [13], [15] for linear discrete-time systems, respectively evolution operators.

In this paper we study three general concepts of splitting: strong (h, k)-splitting, (h, k)-splitting and weak (h, k)-splitting, for the case of skew-evolution semiflows. Characterizations for these properties are established and in particular, we illustrate the results in the case of (h, k)-trichotomic behaviors.

Also, we emphasize the connections between the notions through some representative examples.

2 Skew-evolution semiflows

Let X be a metric space and V a Banach space. Let $\mathcal{B}(V)$ be the Banach algebra of all bounded linear operators on V. The norms on V and on $\mathcal{B}(V)$ will be denoted by $|| \cdot ||$.

We consider the set

$$\Delta = \{(t, s) \in \mathbb{R}^2 \text{ with } t \geq s \geq 0\},$$

I represents the identity operator on V and $Y = X \times V$.

Definition 1. A mapping $\varphi : \Delta \times X \to X$ is called *evolution semiflow* on X if

(es_1) $\varphi(t,t,x) = x$, for all $(t,x) \in \mathbb{R}_+ \times X$;

(es_2) $\varphi(t,s,\varphi(s,t_0,x)) = \varphi(t,t_0,x)$, for all $(t,s),(s,t_0) \in \Delta$ and $x \in X$.

Example 1. For every metric space X, the mapping

$$\varphi : \Delta \times X \to X,\ \varphi(t,s,x) = x$$

for all $(t,s,x) \in \Delta \times X$ is an evolution semiflow on X.

Example 2. We consider $\mathcal{C}(\mathbb{R},\mathbb{R})$ the set of all continuous functions $x : \mathbb{R} \to \mathbb{R}$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$. Let X be the closure in $\mathcal{C}$ of the set $\{x_t,\ t \geq 0\}$, with $x_t(u) = x(t+u)$, $u \geq 0$. Then the mapping $\varphi : \Delta \times X \to X$, given by $\varphi(t,s,x) = x_{t-s}$ is an evolution semiflow on X.

Definition 2. We say that $\Phi : \Delta \times X \to \mathcal{B}(V)$ is an *evolution cocycle* over an evolution semiflow φ if the following properties are satisfied:

(ec_1) $\Phi(t,t,x) = I$, for all $(t,x) \in \mathbb{R}_+ \times X$;

(ec_2) $\Phi(t,s,\varphi(s,t_0,x))\Phi(s,t_0,x) = \Phi(t,t_0,x)$, for all $(t,s),(s,t_0) \in \Delta$ and $x \in X$.

Definition 3. The mapping $C : \Delta \times Y \to Y$, defined by

$$C(t,s,x,v) = (\varphi(t,s,x), \Phi(t,s,x)v),$$

where Φ is an evolution cocycle over an evolution semiflow φ, is called *skew-evolution semiflow on* Y.

Example 3. Let $U : \Delta \to \mathcal{B}(V)$ be an evolution operator on the Banach space V (i.e. $U(t,t) = I$, for every $t \geq 0$ and $U(t,s)U(s,t_0) = U(t,t_0)$, for all $(t,s),\ (s,t_0) \in \Delta$).

Let $X = \mathbb{R}_+$. The mapping $\varphi : \Delta \times X \to X,\ \varphi(t,s,x) = t - s + x$ is an evolution semiflow on X and we consider the evolution cocycle on V

$$\Phi_U : \Delta \times X \to \mathcal{B}(V),$$

defined by

$$\Phi_U(t,s,x) = U(t-s+x,x).$$

Then $C_U = (\varphi, \Phi_U)$ is a skew-evolution semiflow.

3 Preliminary results

In what follows, we will introduce the notions of invariance and strong invariance for a family of projectors relative to a skew-evolution semiflow and connections between them are given.

Definition 4. A mapping $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is said to be a *family of projectors* on V if

$$P(t,x)P(t,x) = P(t,x), \quad \text{for every} \quad (t,x) \in \mathbb{R}_+ \times X.$$

Definition 5. A family of projectors $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is said to be *invariant* for a skew-evolution semiflow $C = (\varphi, \Phi)$ if

$$P(t,\varphi(t,s,x))\Phi(t,s,x) = \Phi(t,s,x)P(s,x), \quad \text{for all } (t,s,x) \in \Delta \times X.$$

Remark 1. If the evolution cocycle Φ is reversible (i.e. $\Phi(t,s,\cdot)$ is bijective for all $(t,s) \in \Delta$) then

$$P(s,x)\Phi(t,s,x)^{-1} = \Phi(t,s,x)^{-1}P(t,\varphi(t,s,x)),$$

for all $(t,s,x) \in \Delta \times X$.

Example 4. Let $X = \mathbb{R}_+$, $U : \Delta \to \mathcal{B}(V)$ be an evolution operator on V and $\tilde{P} : \mathbb{R}_+ \to \mathcal{B}(V)$ a family of projectors invariant for U (i.e. $\tilde{P}(t)U(t,s) = U(t,s)\tilde{P}(s)$ for all $(t,s) \in \Delta$). Then the mapping $P : \mathbb{R}_+^2 \to \mathcal{B}(V)$, given by $P(t,x) = \tilde{P}(t)$ is a family of projectors invariant for the skew-evolution semiflow C_U, defined in Example 3.

Proposition 1. A family of projectors P is invariant for $C = (\varphi, \Phi)$ if and only if the following relations hold:

(i) $\Phi(t,s,x)(Ker\ P(s,x)) \subset Ker\ P(t,\varphi(t,s,x))$;

(ii) $\Phi(t,s,x)(Range\ P(s,x)) \subset Range\ P(t,\varphi(t,s,x))$,

for all $(t,s,x) \in \Delta \times X$.

Proof. It is immediate. □

Definition 6. A family of projectors $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is called *strongly invariant* for a skew-evolution semiflow $C = (\varphi, \Phi)$ if it is invariant for C and for all $(t,s,x) \in \Delta \times X$, the restriction $\Phi(t,s,x)$ is an isomorphism from *Range* $P(s,x)$ to *Range* $P(t,\varphi(t,s,x))$.

Remark 2. If $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is invariant for $C = (\varphi, \Phi)$ and $\Phi(t, s, \cdot)$ is reversible for all $(t, s) \in \Delta$, then P is also strongly invariant for C.

Indeed, if Φ is reversible, then for all $y \in$ *Range* $P(t, \varphi(t, s, x))$ exists $v_0 \in V$ with $y = \Phi(t, s, x)v_0$. Then

$$y = P(t, \varphi(t, s, x))y = P(t, \varphi(t, s, x))\Phi(t, s, x)v_0 =$$

$$= \Phi(t, s, x)P(s, x)v_0 = \Phi(t, s, x)v$$

for all $(t, s, x) \in \Delta \times X$, where $v = P(s, x)v_0 \in$ *Range* $P(s, x)$.

Thus Φ is surjective from *Range* $P(s, x)$ to *Range* $P(t, \varphi(t, s, x))$ and from the reversibility of Φ we obtain that P is strongly invariant for C.

The following example emphasizes that, in general, an invariant family of projectors for a skew-evolution semiflow is not strongly invariant.

Example 5. Let $V = l^2(\mathbb{N}, \mathbb{R}) = \{v : \mathbb{N} \to \mathbb{R} : \sum\limits_{j=0}^{+\infty} |v(j)|^2 < +\infty\}$, endowed with the norm

$$||v|| = \left(\sum_{j=0}^{+\infty} |v(j)|^2 \right)^{1/2}.$$

Also, we consider $X \subset \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and $\varphi : \Delta \times X \to X$, given by $\varphi(t, s, x) = x_{t-s}$ as in Example 2.

Let $\Phi : \Delta \times X \to \mathcal{B}(V)$ be the mapping defined by

$$\Phi(t, s, x)(v) = \begin{cases} \left(\frac{h(t)}{h(s)} v_0, 0, \left(\frac{h(t)}{h(s)}\right)^2 v_2, \frac{h(t)}{h(s)} v_3, ... \right) e^{\int_0^t x(\tau)d\tau}, & \text{if } t > s = 0 \\ \left(\frac{h(t)}{h(s)} v_0, \left(\frac{h(t)}{h(s)}\right)^2 v_1, \left(\frac{h(t)}{h(s)}\right)^2 v_2, \frac{h(t)}{h(s)} v_3, ... \right) e^{\int_s^t x(\tau - s)d\tau}, & \text{if } t \geq s > 0 \text{ or } t = s = 0. \end{cases}$$

where $h : \mathbb{R}_+ \to [1, +\infty)$ is an increasing function with $\lim\limits_{t\to\infty} h(t) = +\infty$. Then $C = (\varphi, \Phi)$ is a skew-evolution semiflow and $P : \mathbb{R}_+^2 \to \mathcal{B}(V)$, given by $P(t, x) = P_0(t)$, where

$$P_0(t)(v_0, v_1, v_2, ...) = \begin{cases} (v_0, 0, v_2, v_3, 0, ...), & \text{if } t = 0 \\ (0, 0, v_0 h(t)^{-2} + v_2, 0, 0, ...), & \text{if } t > 0 \end{cases}$$

is a family of invariant projectors for C.

Let us suppose that P is also strongly invariant for C, which implies that Φ is surjective from *Range* $P(s, x)$ to *Range* $P(t, \varphi(t, s, x))$.

For $y = (0, 0, -\frac{1}{2}, 0, 0, -\frac{1}{3}, ...) \in$ *Range* $P_0(1)$ it does not exists $v = (v_0, 0, v_2, v_3, 0, ...) \in$ *Range* $P_0(0)$ with $y = \Phi(1, 0, x)v$, because we obtain

$$(0, 0, -\frac{1}{2}, 0, 0, -\frac{1}{3}, ...) = \left(\frac{h(1)}{h(0)} v_0, 0, \left(\frac{h(1)}{h(0)}\right)^2 v_2, \left(\frac{h(1)}{h(0)}\right)^2 v_3, ...\right) e^{\int_0^1 x(\tau)d\tau},$$

which is a contradiction.

So P is not strongly invariant for C.

Let $b : \mathbb{R}_+ \to [1, +\infty)$ be a nondecreasing function with $\lim_{t\to\infty} b(t) = +\infty$.

Definition 7. A family of projectors $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ is called *b-bounded* if there exist $B \geq 1$ and $\varepsilon \geq 0$ such that

$$||P(t, x)|| \leq Bb(t)^{\varepsilon}, \quad \text{for all} \quad (t, x) \in \mathbb{R}_+ \times X.$$

Remark 3. If in Definition 7 we consider $b(t) = e^t$ for all $t \geq 0$, then P is called *exponentially bounded* and if $b(t) = t + 1$ for all $t \geq 0$, then we say that P is *polynomially bounded.*

Definition 8. Let $P_1, P_2, P_3 : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be three families of projectors on V. We say that $\mathcal{P} = \{P_1, P_2, P_3\}$ is a family of *supplementary* projectors if

(s_1) $P_1(t, x) + P_2(t, x) + P_3(t, x) = I$;

(s_2) $P_i(t, x)P_j(t, x) = 0$,

for all $(t, x) \in \mathbb{R}_+ \times X,\ i, j \in \{1, 2, 3\},\ i \neq j$.

Definition 9. A family of supplementary projectors $\mathcal{P} = \{P_1, P_2, P_3\}$ is said to be *compatible* with $C = (\varphi, \Phi)$ if

(c_1) P_1 is invariant for C;

(c_2) P_2 and P_3 are strongly invariant for C.

Proposition 2. *If* $\mathcal{P} = \{P_1, P_2, P_3\}$ *is compatible with* $C = (\varphi, \Phi)$*, then there exist* $\Psi_2, \Psi_3 : \Delta \times X \to \mathcal{B}(V)$ *such that for all* $(t, s, x) \in \Delta \times X$*,* $i = 2, 3$*,* Ψ_i *is an isomorphism from Range* $P_i(t, \varphi(t, s, x))$ *to Range* $P_i(s, x)$*, with the properties*

(Ψ_i^1) $\Phi(t,s,x)\Psi_i(t,s,x)P_i(t,\varphi(t,s,x)) = P_i(t,\varphi(t,s,x))$;

(Ψ_i^2) $\Psi_i(t,s,x)\Phi(t,s,x)P_i(s,x) = P_i(s,x)$;

(Ψ_i^3) $\Psi_i(t,s,x)P_i(t,\varphi(t,s,x)) = P_i(s,x)\Psi_i(t,s,x)P_i(t,\varphi(t,s,x))$;

(Ψ_i^4) $\Psi_i(t,t_0,x)P_i(t,\varphi(t,t_0,x)) = \Psi_i(s,t_0,x)\Psi_i(t,s,\varphi(s,t_0,x))P_i(t,\varphi(t,t_0,x))$,

for all $(t,s),(s,t_0) \in \Delta,\ x \in X,\ i = 2,3$.

Proof. The relations (Ψ_i^1), (Ψ_i^2) are immediate.
(Ψ_i^3) From $P_i(t,\varphi(t,s,x))v \in Range\ P_i(t,\varphi(t,s,x))$ we obtain

$$\Psi_i(t,s,x)P_i(t,\varphi(t,s,x))v \in Range\ P_i(s,x),$$

which implies that

$$\Psi_i(t,s,x)P_i(t,\varphi(t,s,x))v = P_i(s,x)\Psi_i(t,s,x)P_i(t,\varphi(t,s,x))v,$$

for all $(t,s,x,v) \in \Delta \times Y,\ i = 2,3$.
(Ψ_i^4) For all $(t,s),(s,t_0) \in \Delta,\ x \in X,\ i = 2,3$ it results that

$$\Psi_i(t,t_0,x)P_i(t,\varphi(t,t_0,x)) = P_i(t_0,x)\Psi_i(t,t_0,x)P_i(t,\varphi(t,t_0,x)) =$$

$$= \Psi_i(s,t_0,x)\Phi(s,t_0,x)P_i(t_0,x)\Psi_i(t,t_0,x)P_i(t,\varphi(t,t_0,x)) =$$

$$= \Psi_i(s,t_0,x)P_i(s,\varphi(s,t_0,x))\Phi(s,t_0,x)P_i(t_0,x)\Psi_i(t,t_0,x)P_i(t,\varphi(t,t_0,x)) =$$

$$= \Psi_i(s,t_0,x)\Psi_i(t,s,\varphi(s,t_0,x))\Phi(t,s,\varphi(s,t_0,x))$$

$$P_i(s,\varphi(s,t_0,x))\Phi(s,t_0,x)P_i(t_0,x)\Psi_i(t,t_0,x)P_i(t,\varphi(t,t_0,x)) =$$

$$= \Psi_i(s,t_0,x)\Psi_i(t,s,\varphi(s,t_0,x))\Phi(t,t_0,x)P_i(t_0,x)\Psi_i(t,t_0,x)P_i(t,\varphi(t,t_0,x)) =$$

$$= \Psi_i(s,t_0,x)\Psi_i(t,s,\varphi(s,t_0,x))\Phi(t,t_0,x)\Psi_i(t,t_0,x)P_i(t,\varphi(t,t_0,x)) =$$

$$= \Psi_i(s,t_0,x)\Psi_i(t,s,\varphi(s,t_0,x))P_i(t,\varphi(t,t_0,x)).$$

□

4 (h, k)-splitting

An increasing function $\varphi : \mathbb{R}_+ \to [1, +\infty)$ is said to be a *growth rate*, if

$$\lim_{t\to+\infty} \varphi(t) = +\infty.$$

Let $h, k : \mathbb{R}_+ \to [1, +\infty)$ be two growth rates and $\mathcal{P} = \{P_1, P_2, P_3\}$ a family of projectors supplementary and invariant for a skew-evolution semiflow $C = (\varphi, \Phi)$.

Definition 10. We say that the pair $(C, \mathcal{P})$ admits a (h, k)*-splitting* if there exist the real constants $N \geq 1,\ \alpha < \beta,\ \gamma < \delta$ and $\varepsilon \geq 0$ such that:

(hks_1) $h(s)^\alpha||\Phi(t, s, x)P_1(s, x)v|| \leq Nh(t)^\alpha k(s)^\varepsilon||P_1(s, x)v||$;

(hks_2) $h(t)^\beta||P_2(s, x)v|| \leq Nh(s)^\beta k(t)^\varepsilon||\Phi(t, s, x)P_2(s, x)v||$;

(hks_3) $h(t)^\gamma||\Phi(t, s, x)P_3(s, x)v|| \leq Nh(s)^\gamma k(s)^\varepsilon||P_3(s, x)v||$;

(hks_4) $h(s)^\delta||P_3(s, x)v|| \leq Nh(t)^\delta k(t)^\varepsilon||\Phi(t, s, x)P_3(s, x)v||$,

for all $(t, s) \in \Delta,\ (x, v) \in Y$.

In the particular case when $\varepsilon = 0$ or k is a constant function, we say that C has a *uniform h-splitting.*

The constants $N,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ are called *splitting constants.*

Remark 4. As particular cases of (h, k)-splitting we have that

(i) if $h(t) = k(t) = e^t$ for all $t \geq 0$, then we recover the notion of *nonuniform exponential splitting* and in particular when the function k is constant or $\varepsilon = 0$, we obtain the concept of *uniform exponential splitting*;

(ii) if $h(t) = k(t) = t + 1$ for all $t \geq 0$, then we obtain the property of *nonuniform polynomial splitting* and in particular when $\varepsilon = 0$ or the function k is constant, we recover the notion of *uniform polynomial splitting*;

(iii) if $(C, \mathcal{P})$ admits a (h, k)-splitting with $\alpha < 0 < \beta,\ \gamma < 0 < \delta$, then $(C, \mathcal{P})$ is called (h, k)*-trichotomic* (or $(C, \mathcal{P})$ has a (h, k)-trichotomy).

Remark 5. It is obvious that if $(C, \mathcal{P})$ admits a uniform (h, k)-splitting, then it also admits a (h, k)-splitting. The converse is not valid, as we show in the following example.

Example 6. Let V be a Banach space, X a metric space and $h,k:\mathbb{R}_+\to[1,+\infty)$ two growth rates.

We consider the positive constants $\alpha<\beta,\ \gamma<\delta,\ \varepsilon$ and $\mathcal{P}=\{P_1,P_2,P_3\}$ a family of projectors with

$$P_i(t,x)P_i(s,x)=P_i(s,x),\quad \text{for all}\quad (t,s,x)\in\Delta\times X,\ i=\overline{1,3}$$

and it is supplementary and invariant for a skew-evolution semiflow $C=(\varphi,\Phi)$, where

$$\Phi(t,s,x)=\left(\frac{h(t)}{h(s)}\right)^{\alpha}\frac{k(s)^{\varepsilon\cos^2 s}}{k(t)^{\varepsilon\cos^2 t}}P_1(s,x)+$$

$$\left(\frac{h(t)}{h(s)}\right)^{\beta}\left(\frac{k(s)}{k(t)}\right)^{\varepsilon}P_2(s,x)+\left(\frac{h(s)}{h(t)}\right)^{\gamma}\left(\frac{k(s)}{k(t)}\right)^{\varepsilon}P_3(s,x)$$

and φ is an arbitrary evolution semiflow.

It is easy to check that the pair $(C,\mathcal{P})$ admits a (h,k)-splitting, with the splitting constants $\alpha,\ \beta,\ \gamma,\ \delta$ and ε.

Assuming that $(C,\mathcal{P})$ admits a uniform (h,k)-splitting it results that there exists $N\geq 1$ such that

$$h(s)^{\alpha}||\Phi(t,s,x)P_1(s,x)v||\leq Nh(t)^{\alpha}||P_1(s,x)v||,$$

which implies that

$$\frac{k(s)^{\varepsilon\cos^2 s}}{k(t)^{\varepsilon\cos^2 t}}\leq N,\quad \text{for all } (t,s)\in\Delta.$$

For $s=2n\pi,\ t=2n\pi+\frac{\pi}{2}$ we obtain

$$k(s)^{\varepsilon}\leq N,\quad \text{for all } s\geq 0,$$

which is absurd.

Remark 6. The pair $(C,\mathcal{P})$ is (h,k)-trichotomic if and only if there are the constants $N\geq 1,\ a,b>0$ and $\varepsilon\geq 0$ such that

(hkt_1) $h(t)^a||\Phi(t,s,x)P_1(s,x)v||\leq Nh(s)^a k(s)^{\varepsilon}||P_1(s,x)v||;$

(hkt_2) $h(t)^a||P_2(s,x)v||\leq Nh(s)^a k(t)^{\varepsilon}||\Phi(t,s,x)P_2(s,x)v||;$

(hkt_3) $h(s)^b||\Phi(t,s,x)P_3(s,x)v||\leq Nh(t)^b k(s)^{\varepsilon}||P_3(s,x)v||;$

(hkt_4) $h(s)^b||P_3(s,x)v|| \leq Nh(t)^b k(t)^\varepsilon ||\Phi(t,s,x)P_3(s,x)v||$,

for all $(t,s) \in \Delta,\ (x,v) \in Y$.

Indeed, for the necessity it is sufficient to put $a = \min\{-\alpha, \beta\},\ b = \min\{-\gamma, \delta\}$. The converse is obvious.

Remark 7. If a pair $(C, \mathcal{P})$ has a (h,k)-trichotomy, then it has a (h,k)-splitting. The converse implication is not true, as the following example shows.

Example 7. We consider V a Banach space, X a metric space and the growth rates $h, k : \mathbb{R}_+ \to [1, +\infty)$.

Let $c_1 < c_2,\ c_3 < c_4$ be positive constants and $\mathcal{P} = \{P_1, P_2, P_3\}$ a family of projectors with the property

$$P_i(t,x)P_i(s,x) = P_i(s,x), \quad \text{for all} \quad (t,s,x) \in \Delta \times X,\ i = \overline{1,3}$$

and it is supplementary and invariant for a skew-evolution semiflow $C = (\varphi, \Phi)$, where

$$\Phi(t,s,x) = \left(\frac{h(t)}{h(s)}\right)^{c_1} P_1(s,x) + \left(\frac{h(t)}{h(s)}\right)^{c_2} P_2(s,x) + \left(\frac{h(s)}{h(t)}\right)^{c_3} P_3(s,x)$$

and φ is an arbitrary evolution semiflow.

It is simple to verify that $(C, \mathcal{P})$ has a (h,k)-splitting with the splitting constants c_1, c_2, c_3, c_4.

If we suppose that $(C, \mathcal{P})$ has a (h,k)-trichotomy it results from Remark 6 that there exist $N \geq 1, a > 0, \varepsilon \geq 0$ with

$$h(t)^a||\Phi(t,s,x)P_1(s,x)v|| \leq Nh(s)^a k(s)^\varepsilon ||P_1(s,x)v||, \text{ for all } (t,s,x) \in \Delta \times X,$$

which implies

$$h(t)^{c_1+a} \leq Nh(s)^{c_1+a} k(s)^\varepsilon, \quad \text{for all} \quad (t,s) \in \Delta.$$

Considering $s = 0$ we obtain

$$h(t)^{c_1+a} \leq Nh(0)^{c_1+a} k(0)^\varepsilon, \quad \text{for all} \quad t \geq 0,$$

which is a contradiction.

Hence, $(C, \mathcal{P})$ is not (h,k)-trichotomic.

A characterization for the property of (h,k)-splitting is given by

Theorem 1. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of compatible projectors with a skew-evolution semiflow $C = (\varphi, \Phi)$. Then $(C, \mathcal{P})$ admits a (h,k)-splitting if and only if there exist the real constants $N \geq 1,\ \alpha < \beta,\ \gamma < \delta$ and $\varepsilon \geq 0$ such that*

(hks_1) $h(s)^\alpha ||\Phi(t,s,x)P_1(s,x)v|| \leq Nh(t)^\alpha k(s)^\varepsilon ||P_1(s,x)v||$;

(hks_2') $h(t)^\beta ||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v|| \leq Nh(s)^\beta k(t)^\varepsilon ||P_2(t,\varphi(t,s,x))v||$;

(hks_3) $h(t)^\gamma ||\Phi(t,s,x)P_3(s,x)v|| \leq Nh(s)^\gamma k(s)^\varepsilon ||P_3(s,x)v||$;

(hks_4') $h(s)^\delta ||\Psi_3(t,s,x)P_3(t,\varphi(t,s,x))v|| \leq Nh(t)^\delta k(t)^\varepsilon ||P_3(t,\varphi(t,s,x))v||$,

for all $(t,s) \in \Delta,\ (x,v) \in Y$.

Proof. It is sufficient to prove $(hks_2) \Leftrightarrow (hks_2')$ and $(hks_4) \Leftrightarrow (hks_4')$.

If (hks_2) from Definition 10 holds, then

$$h(t)^\beta ||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v|| =$$
$$= h(t)^\beta ||P_2(s,x)\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v|| \leq$$
$$\leq Nh(s)^\beta k(t)^\varepsilon ||\Phi(t,s,x)P_2(s,x)\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v|| =$$
$$= Nh(s)^\beta k(t)^\varepsilon ||P_2(t,\varphi(t,s,x))\Phi(t,s,x)\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v|| =$$
$$= Nh(s)^\beta k(t)^\varepsilon ||P_2(t,\varphi(t,s,x))v||,$$

for all $(t,s) \in \Delta,\ (x,v) \in Y$, which imply (hks_2').

Now we prove $(hks_2') \Rightarrow (hks_2)$. We have

$$h(t)^\beta ||P_2(s,x)v|| = h(t)^\beta ||\Psi_2(t,s,x)\Phi(t,s,x)P_2(s,x)v|| =$$
$$= h(t)^\beta ||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))\Phi(t,s,x)P_2(s,x))v|| \leq$$
$$\leq Nh(s)^\beta k(t)^\varepsilon ||P_2(t,\varphi(t,s,x))\Phi(t,s,x)P_2(s,x)v|| =$$
$$= Nh(s)^\beta k(t)^\varepsilon ||\Phi(t,s,x)P_2(s,x)v||$$

for all $(t,s) \in \Delta$ and $(x,v) \in Y$.

Similarly, it results that $(hks_4) \Leftrightarrow (hks_4')$. □

Corollary 1. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a family of compatible projectors with a skew-evolution semiflow $C = (\varphi, \Phi)$. Then $(C, \mathcal{P})$ is (h,k)-trichotomic if and only if there are some constants $N \geq 1,\ a, b > 0$ and $\varepsilon \geq 0$ such that*

(hkt_1) $h(t)^a ||\Phi(t,s,x)P_1(s,x)v|| \leq Nh(s)^a k(s)^\varepsilon ||P_1(s,x)v||$;

(hkt_2') $h(t)^a||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v|| \le Nh(s)^a k(t)^\varepsilon ||P_2(t,\varphi(t,s,x))v||$;

(hkt_3) $h(s)^b||\Phi(t,s,x)P_3(s,x)v|| \le Nh(t)^b k(s)^\varepsilon ||P_3(s,x)v||$;

(hkt_4') $h(s)^b||\Psi_3(t,s,x)P_3(t,\varphi(t,s,x))v|| \le Nh(t)^b k(t)^\varepsilon ||P_3(t,\varphi(t,s,x))v||$

for all $(t,s)\in\Delta,\ (x,v)\in Y$.

Proof. It is obvious from Theorem 1 and Remark 6. □

Proposition 3. *Let $\mathcal{P}=\{P_1,P_2,P_3\}$ be a compatible family of projectors with a skew-evolution semiflow $C=(\varphi,\Phi)$. The pair $(C,\mathcal{P})$ has a (h,k)-splitting if and only if there exist the real constants $N\ge 1,\ \alpha<\beta,\ \gamma<\delta$ and $\varepsilon\ge 0$ such that*

(hks_1') $h(s)^\alpha||\Phi(t,t_0,x)P_1(t_0,x)v|| \le Nh(t)^\alpha k(s)^\varepsilon ||\Phi(s,t_0,x)P_1(t_0,x)v||$;

(hks_2'') $h(s)^\beta||\Psi_2(t,t_0,x)P_2(t,\varphi(t,t_0,x))v|| \le$

$$\le Nh(t_0)^\beta k(s)^\varepsilon ||\Psi_2(t,s,\varphi(s,t_0,x))P_2(t,\varphi(t,t_0,x))v||;$$

(hks_3') $h(t)^\gamma||\Phi(t,t_0,x)P_3(t_0,x)v|| \le Nh(s)^\gamma k(s)^\varepsilon ||\Phi(s,t_0,x)P_3(t_0,x)v||$;

(hks_4'') $h(t_0)^\delta||\Psi_3(t,t_0,x)P_3(t,\varphi(t,t_0,x))v|| \le$

$$\le Nh(s)^\delta k(s)^\varepsilon ||\Psi_3(t,s,\varphi(s,t_0,x))P_3(t,\varphi(t,t_0,x))v||,$$

for all $(t,s),(s,t_0)\in\Delta,\ (x,v)\in Y$.

Proof. Necessity. We will use the relations from Proposition 2 and Theorem 1.

(hks_1') $\quad h(s)^\alpha||\Phi(t,t_0,x)P_1(t_0,x)v|| =$

$$= h(s)^\alpha||\Phi(t,s,\varphi(s,t_0,x))P_1(s,\varphi(s,t_0,x))\Phi(s,t_0,x)P_1(t_0,x)v|| \le$$

$$\le Nh(t)^\alpha k(s)^\varepsilon ||\Phi(s,t_0,x)P_1(t_0,x)v||;$$

(hks_2'') $\quad h(s)^\beta||\Psi_2(t,t_0,x)P_2(t,\varphi(t,t_0,x))v|| =$

$$= h(s)^\beta||\Psi_2(s,t_0,x)P_2(s,\varphi(s,t_0,x))\Psi_2(t,s,\varphi(s,t_0,x))P_2(t,\varphi(t,t_0,x))v|| \le$$

$$\le Nh(t_0)^\beta k(s)^\varepsilon ||P_2(s,\varphi(s,t_0,x))\Psi_2(t,s,\varphi(s,t_0,x))P_2(t,\varphi(t,t_0,x))v|| =$$

$$= Nh(t_0)^\beta k(s)^\varepsilon ||\Psi_2(t,s,\varphi(s,t_0,x))P_2(t,\varphi(t,t_0,x))v||,$$

for all $(t,s),(s,t_0)\in\Delta,\ (x,v)\in Y$.
Using a similar technique, we obtain that (hks_3') and (hks_4'') are satisfied.

Sufficiency. For $s=t$ in (hks_2'') and (hks_4''), respectively for $s=t_0$ in (hks_1') and (hks_3') it results the inequalities from Theorem 1.
We conclude that $(C,\mathcal{P})$ admits a (h,k)-splitting. □

Corollary 2. *Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be a compatible family of projectors with $C = (\varphi, \Phi)$. Then $(C, \mathcal{P})$ admits a (h,k)-trichotomy if and only if there exist the constants $N \geq 1,\ a, b > 0$ and $\varepsilon \geq 0$ with*

(hkt_1') $h(t)^a||\Phi(t,t_0,x)P_1(t_0,x)v|| \leq Nh(s)^a k(s)^\varepsilon||\Phi(s,t_0,x)P_1(t_0,x)v||;$

(hkt_2'') $h(s)^a||\Psi_2(t,t_0,x)P_2(t,\varphi(t,t_0,x))v|| \leq$

$\leq Nh(t_0)^a k(s)^\varepsilon||\Psi_2(t,s,\varphi(s,t_0,x))P_2(t,\varphi(t,t_0,x))v||;$

(hkt_3') $h(s)^b||\Phi(t,t_0,x)P_3(t_0,x)v|| \leq Nh(t)^b k(s)^\varepsilon||\Phi(s,t_0,x)P_3(t_0,x)v||;$

(hkt_4'') $h(t_0)^b||\Psi_3(t,t_0,x)P_3(t,\varphi(t,t_0,x))v|| \leq$

$\leq Nh(s)^b k(s)^\varepsilon||\Psi_3(t,s,\varphi(s,t_0,x))P_3(t,\varphi(t,t_0,x))v||,$

for all $(t,s),(s,t_0) \in \Delta,\ (x,v) \in Y$.

Proof. It is immediate from Proposition 3 and Remark 6. □

5 Strong (h,k)-splitting

In what follows, we consider $h, k : \mathbb{R}_+ \to [1, +\infty)$ two growth rates and $\mathcal{P} = \{P_1, P_2, P_3\}$ a compatible family of projectors with a skew-evolution semiflow $C = (\varphi, \Phi)$.

Definition 11. We say that the pair $(C, \mathcal{P})$ has a *strong (h,k)-splitting* if there exist the real constants $N \geq 1,\ \alpha < \beta,\ \gamma < \delta$ and $\varepsilon \geq 0$ such that

$(shks_1)$ $h(s)^\alpha||\Phi(t,s,x)P_1(s,x)v|| \leq Nh(t)^\alpha k(s)^\varepsilon||v||;$

$(shks_2)$ $h(t)^\beta||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v|| \leq Nh(s)^\beta k(t)^\varepsilon||v||;$

$(shks_3)$ $h(t)^\gamma||\Phi(t,s,x)P_3(s,x)v|| \leq Nh(s)^\gamma k(s)^\varepsilon||v||;$

$(shks_4)$ $h(s)^\delta||\Psi_3(t,s,x)P_3(t,\varphi(t,s,x))v|| \leq Nh(t)^\delta k(t)^\varepsilon||v||,$

for all $(t,s) \in \Delta$ and $(x,v) \in Y$.

In particular, if $\alpha < 0 < \beta,\ \gamma < 0 < \delta$, then $(C, \mathcal{P})$ is called *strongly (h,k)-trichotomic.*

Remark 8. The pair $(C, \mathcal{P})$ has a strong (h,k)-splitting if and only if there are the real constants $N \geq 1,\ \alpha < \beta,\ \gamma < \delta$ and $\varepsilon \geq 0$ such that

$(shks_1')$ $h(s)^\alpha||\Phi(t,s,x)P_1(s,x)|| \leq Nh(t)^\alpha k(s)^\varepsilon;$

$(shks'_2)$ $h(t)^{\beta}||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))|| \leq Nh(s)^{\beta}k(t)^{\varepsilon}$;

$(shks'_3)$ $h(t)^{\gamma}||\Phi(t,s,x)P_3(s,x)|| \leq Nh(s)^{\gamma}k(s)^{\varepsilon}$;

$(shks'_4)$ $h(s)^{\delta}||\Psi_3(t,s,x)P_3(t,\varphi(t,s,x))|| \leq Nh(t)^{\delta}k(t)^{\varepsilon}$,

for all $(t,s,x) \in \Delta \times X$.

Remark 9. If $(C,\mathcal{P})$ admits a strong (h,k)-splitting, then $\mathcal{P}$ is k-bounded.

Remark 10. The pair $(C,\mathcal{P})$ is strongly (h,k)-trichotomic if and only if there exist $N \geq 1$, $a,b > 0$ and $\varepsilon \geq 0$ with:

$(shkt_1)$ $h(t)^{a}||\Phi(t,s,x)P_1(s,x)v|| \leq Nh(s)^{a}k(s)^{\varepsilon}||v||$;

$(shkt_2)$ $h(t)^{a}||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v|| \leq Nh(s)^{a}k(t)^{\varepsilon}||v||$;

$(shkt_3)$ $h(s)^{b}||\Phi(t,s,x)P_3(s,x)v|| \leq Nh(t)^{b}k(s)^{\varepsilon}||v||$;

$(shkt_4)$ $h(s)^{b}||\Psi_3(t,s,x)P_3(t,\varphi(t,s,x))v|| \leq Nh(t)^{b}k(t)^{\varepsilon}||v||$,

for all $(t,s) \in \Delta$, $(x,v) \in Y$.

Remark 11. If $(C,\mathcal{P})$ has a strong (h,k)-splitting, then it also admits a (h,k)-splitting. In general the converse implication is not accomplished, as it results from the following example.

Example 8. Let $V = l^{\infty}(\mathbb{N},\mathbb{R})$ be the Banach space of all bounded real-valued sequences, endowed with the norm

$$||v|| = \sup_{n\in\mathbb{N}} |v_n|, \qquad v = (v_0, v_1, ..., v_n, ...) \in V$$

and X a metric space.

We consider $h,k : \mathbb{R}_+ \to [1,+\infty)$ growth rates and the family of projectors $\mathcal{P} = \{P_1,P_2,P_3\}$, $P_i(t,x) = \tilde{P}_i(t)$ for all $(t,x) \in \mathbb{R}_+ \times X$, $i = \overline{1,3}$, where

$$\tilde{P}_1(t)(v_0,v_1,...) = (v_0 + (e^{k(t)} - 1)v_1, 0, v_2 + (e^{k(t)} - 1)v_3, 0, ...),$$

$$\tilde{P}_2(t)(v_0,v_1,...) = ((1 - e^{k(t)})v_1, 0, (1 - e^{k(t)})v_3, 0, ...),$$

$$\tilde{P}_3(t)(v_0,v_1,...) = (0, v_1, 0, v_3, ...).$$

Let $\alpha < \beta$, $\gamma < \delta$ be real constants and the evolution cocycle is defined by

$$\Phi(t,s,x) = \left(\frac{h(t)}{h(s)}\right)^{\alpha} \tilde{P}_1(s) + \left(\frac{h(t)}{h(s)}\right)^{\beta} \tilde{P}_2(t) + \left(\frac{h(s)}{h(t)}\right)^{\gamma} \tilde{P}_3(s),$$

for all $(t,s,x)\in\Delta\times X$.

It is immediate that Φ is an evolution cocycle over all evolution semiflows φ and after some computations we obtain that $(C,\mathcal{P})$ has a (h,k)-splitting.

If we suppose that $(C,\mathcal{P})$ admits a strong (h,k)-splitting, it results from Remark 9 that $\mathcal{P}$ is k-bounded, which is a contradiction.

Theorem 2. *The pair $(C,\mathcal{P})$ has a strong (h,k)-splitting if and only if it admits a (h,k)-splitting and $\mathcal{P}$ is k-bounded.*

Proof. Necessity. As $(C,\mathcal{P})$ has a strong (h,k)-splitting, we deduce that it admits a (h,k)-splitting.
By Remark 9, it results that $\mathcal{P}$ is k-bounded.

Sufficiency. As $\mathcal{P}$ is k-bounded, there exist $B\geq 1,\ \varepsilon\geq 0$ with

$$||P_i(t,x)||\leq Bk(t)^{\varepsilon},$$

for all $(t,x)\in\mathbb{R}_+\times X,\ i\in\{1,2,3\}$.

According to Theorem 1 we deduce

$$h(s)^{\alpha}||\Phi(t,s,x)P_1(s,x)v||\leq Nh(t)^{\alpha}k(s)^{\varepsilon}||P_1(s,x)v||\leq$$

$$\leq BNh(t)^{\alpha}k(s)^{2\varepsilon}||v||=\tilde{N}h(t)^{\alpha}k(s)^{\tilde{\varepsilon}}||v||,$$

where $\tilde{N}=BN,\ \tilde{\varepsilon}=2\varepsilon$;

$$h(t)^{\beta}||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v||\leq Nh(s)^{\beta}k(t)^{\varepsilon}||P_2(t,\varphi(t,s,x))v||\leq$$

$$\leq\tilde{N}h(s)^{\beta}k(t)^{\tilde{\varepsilon}}||v||;$$

$$h(t)^{\gamma}||\Phi(t,s,x)P_3(s,x)v||\leq Nh(s)^{\gamma}k(s)^{\varepsilon}||P_3(s,x)v||\leq$$

$$\leq\tilde{N}h(s)^{\gamma}k(s)^{\tilde{\varepsilon}}||v||;$$

$$h(s)^{\delta}||\Psi_3(t,s,x)P_3(t,\varphi(t,s,x))v||\leq Nh(t)^{\delta}k(t)^{\varepsilon}||P_3(t,\varphi(t,s,x))v||\leq$$

$$\leq\tilde{N}h(t)^{\delta}k(t)^{\tilde{\varepsilon}}||v||,$$

for all $(t,s)\in\Delta,\ (x,v)\in Y$.

It results that $(C,\mathcal{P})$ has a strong (h,k)-splitting. □

Corollary 3. *The pair $(C,\mathcal{P})$ has a strong (h,k)-trichotomy if and only if it has a (h,k)-trichotomy and $\mathcal{P}$ is k-bounded.*

Proof. It is a particular case of Theorem 2. □

6 Weak (h,k)-splitting

Let $h,k:\mathbb{R}_+\to[1,+\infty)$ be two growth rates and $\mathcal{P}=\{P_1,P_2,P_3\}$ a compatible family of projectors with a skew-evolution semiflow $C=(\varphi,\Phi)$.

Definition 12. The pair $(C,\mathcal{P})$ admits a *weak (h,k)-splitting* if there exist the real constants $N\geq 1,\ \alpha<\beta,\gamma<\delta$ and $\varepsilon\geq 0$ such that

$(whks_1)$ $h(s)^\alpha||\Phi(t,s,x)P_1(s,x)||\leq Nh(t)^\alpha k(s)^\varepsilon||P_1(s,x)||$;

$(whks_2)$ $h(t)^\beta||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))||\leq Nh(s)^\beta k(t)^\varepsilon||P_2(t,\varphi(t,s,x))||$;

$(whks_3)$ $h(t)^\gamma||\Phi(t,s,x)P_3(s,x)||\leq Nh(s)^\gamma k(s)^\varepsilon||P_3(s,x)||$;

$(whks_4)$ $h(s)^\delta||\Psi_3(t,s,x)P_3(t,\varphi(t,s,x))||\leq Nh(t)^\delta k(t)^\varepsilon||P_3(t,\varphi(t,s,x))||$,

for all $(t,s,x)\in\Delta\times X$.

In particular, if $\alpha<0<\beta,\ \gamma<0<\delta$, then we say that $(C,\mathcal{P})$ admits a *weak (h,k)-trichotomy.*

Remark 12. If the pair $(C,\mathcal{P})$ admits a (h,k)-splitting, then it admits also a weak (h,k)-splitting.

Remark 13. The pair $(C,\mathcal{P})$ admits a weak (h,k)-trichotomy if and only if there exist $N\geq 1,\ a,b>0$ and $\varepsilon\geq 0$ such that

$(whkt_1)$ $h(t)^a||\Phi(t,s,x)P_1(s,x)||\leq Nh(s)^a k(s)^\varepsilon||P_1(s,x)||$;

$(whkt_2)$ $h(t)^a||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))||\leq Nh(s)^a k(t)^\varepsilon||P_2(t,\varphi(t,s,x))||$;

$(whkt_3)$ $h(s)^b||\Phi(t,s,x)P_3(s,x)v||\leq Nh(t)^b k(s)^\varepsilon||P_3(s,x)||$;

$(whkt_4)$ $h(s)^b||\Psi_3(t,s,x)P_3(t,\varphi(t,s,x))v||\leq Nh(t)^b k(t)^\varepsilon||P_3(t,\varphi(t,s,x))||$,

for all $(t,s,x)\in\Delta\times X$.

The main result of this section is given by

Theorem 3. *Let $\mathcal{P}=\{P_1,P_2,P_3\}$ be a family of projectors k-bounded, compatible with a skew-evolution semiflow $C=(\varphi,\Phi)$. The following statements are equivalent:*

(i) *$(C,\mathcal{P})$ admits a strong (h,k)-splitting;*

(ii) *$(C,\mathcal{P})$ admits a (h,k)-splitting;*

(iii) $(C, \mathcal{P})$ *admits a weak* (h, k)*-splitting.*

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.

We show $(iii) \Rightarrow (i)$. As $\mathcal{P}$ is k-bounded, there exist $B \geq 1$ and $\varepsilon \geq 0$ such that

$$||P_i(t,x)|| \leq Bk(t)^{\varepsilon} \quad \text{for all} \quad (t,x) \in \mathbb{R}_+ \times X,\ i = 1,2,3.$$

Thus,

$$h(s)^{\alpha}||\Phi(t,s,x)P_1(s,x)v|| \leq Nh(t)^{\alpha}k(s)^{\varepsilon}||P_1(s,x)v|| \leq$$

$$\leq BNh(t)^{\alpha}k(s)^{2\varepsilon}||v||;$$

$$h(t)^{\beta}||\Psi_2(t,s,x)P_2(t,\varphi(t,s,x))v|| \leq Nh(s)^{\beta}k(t)^{\varepsilon}||P_2(t,\varphi(t,s,x))v|| \leq$$

$$\leq BNh(s)^{\beta}k(t)^{2\varepsilon}||v||;$$

$$h(t)^{\gamma}||\Phi(t,s,x)P_3(s,x)v|| \leq Nh(s)^{\gamma}k(s)^{\varepsilon}||P_3(s,x)v|| \leq$$

$$\leq BNh(s)^{\gamma}k(s)^{2\varepsilon}||v||;$$

$$h(s)^{\delta}||\Psi_3(t,s,x)P_3(t,\varphi(t,s,x))v|| \leq Nh(t)^{\delta}k(t)^{\varepsilon}||P_3(t,\varphi(t,s,x))v|| \leq$$

$$\leq BNh(t)^{\delta}k(t)^{2\varepsilon}||v||$$

for all $(t,s) \in \Delta,\ (x,v) \in Y$.

We conclude that $(C, \mathcal{P})$ has a strong (h,k)-splitting.

□

Corollary 4. *Let* $\mathcal{P} = \{P_1, P_2, P_3\}$ *be a family of projectors* k*-bounded, compatible with a skew-evolution semiflow* $C = (\varphi, \Phi)$*. The following statements are equivalent:*

(i) $(C, \mathcal{P})$ *admits a strong* (h, k)*-trichotomy;*

(ii) $(C, \mathcal{P})$ *admits a* (h, k)*-trichotomy;*

(iii) $(C, \mathcal{P})$ *admits a weak* (h, k)*-trichotomy.*

Proof. It is a particular case of Theorem 3. □

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