Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 9, No. 2/2017

A BEREZIN-TYPE MAP ON $L^2_a(\mathbb{C}_+)$*

Namita Das†

Abstract

In this paper we introduce a map E defined on the Bergman space $L^2_a(\mathbb{C}_+, d\widetilde{A})$ as $(Ef)(w) = \int_{\mathbb{C}_+} f(s)|b_{\overline{w}}(s)|^2 d\widetilde{A}(s), w \in \mathbb{C}_+$, where $\mathbb{C}_+$ is the right half plane, $d\widetilde{A}(s) = dxdy$ is the area measure and $b_{\overline{w}}(s) = \frac{1}{\sqrt{\pi}} \frac{1+w}{1+\overline{w}} \frac{2Rew}{(s+w)^2}, s \in \mathbb{C}_+$. We refer the map E as a Berezin-type map on $L^2_a(\mathbb{C}_+)$. In this work we first investigate the boundedness of the map E on various L^p space and show that the sequence $\{E^n\}$ converges to 0 in norm in the space $L^2(\mathbb{C}_+, d\mu)$ where $d\mu(w) = |B(\overline{w}, w)|d\widetilde{A}(w), w \in \mathbb{C}_+$. We then discuss certain algebraic and ergodicity properties of the map E involving subharmonic functions.

MSC: 47B35, 32M15

keywords: Bergman space, the right half plane, Berezin transform, automorphisms, subharmonic functions.

1 Introduction

Let $\mathbb{C}_+ = \{s = x + iy \in \mathbb{C} : \text{Re}s > 0\}$ be the right half plane. Let $d\widetilde{A}(s) = dxdy$ be the area measure. Let $L^2(\mathbb{C}_+, d\widetilde{A})$ be the space of complex-valued, square-integrable, measurable functions on $\mathbb{C}_+$ with respect to the area measure. Let $L^2_a(\mathbb{C}_+)$ be the closed subspace [1] of $L^2(\mathbb{C}_+, d\widetilde{A})$ consisting of those functions in $L^2(\mathbb{C}_+, d\widetilde{A})$ that are analytic. The space $L^2_a(\mathbb{C}_+)$ is called the Bergman space of the right half plane. The functions $H(s, w) = \frac{1}{(s+\overline{w})^2}, s \in \mathbb{C}_+, w \in \mathbb{C}_+$ are the reproducing kernels [2] for

*Accepted for publication in revised form on June 30-th, 2017

†namitadas440@yahoo.co.in P.G. Department of Mathematics, Utkal University Vanivihar, Bhubaneswar, 751004, Odisha, India

$L^2_a(\mathbb{C}_+)$. Let $\mathbf{h}_w(s) = \frac{H(s,w)}{\sqrt{H(w,w)}} = \frac{2\text{Re}w}{(s+\overline{w})^2}$. The functions $\mathbf{h}_w, w \in \mathbb{C}_+$ are the normalized reproducing kernels for $L^2_a(\mathbb{C}_+)$. Let $L^\infty(\mathbb{C}_+)$ be the space of complex-valued, essentially bounded, Lebesgue measurable functions on $\mathbb{C}_+$. Define for $f \in L^\infty(\mathbb{C}_+), ||f||_\infty = \text{ess} \sup_{s\in\mathbb{C}_+} |f(s)| < \infty$. The space $L^\infty(\mathbb{C}_+)$ is a Banach space with respect to the essential supremum norm .

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $L^2(\mathbb{D}, dA)$ be the space of complex- valued, square-integrable, measurable functions on $\mathbb{D}$ with respect to the normalized area measure $dA(z) = \frac{1}{\pi}dxdy$. Let $L^2_a(\mathbb{D})$ be the space consisting of those functions of $L^2(\mathbb{D}, dA)$ that are analytic.The space $L^2_a(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA)$ and is called the Bergman space of the open unit disk $\mathbb{D}$. The sequence of functions $\{e_n(z)\}_{n=0}^\infty = \{\sqrt{n+1}z^n\}_{n=0}^\infty$ form an orthonormal basis for $L^2_a(\mathbb{D})$. Since point evaluation at $z \in \mathbb{D}$ is a bounded linear functional on the Hilbert space $L^2_a(\mathbb{D})$, the Riesz representation theorem implies that there exists a unique function K_z in $L^2_a(\mathbb{D})$ such that

$$f(z) = \int_{\mathbb{D}} f(w)\overline{K_z(w)}dA(w).$$

for all f in $L^2_a(\mathbb{D})$. Let $K(z,w)$ be the function on $\mathbb{D} \times \mathbb{D}$ defined by

$$K(z,w) = \overline{K_z(w)}.$$

The function $K(z,w)$ is analytic in z and co-analytic in w. In fact $K(z,w) = \frac{1}{(1-z\overline{w})^2}$, $z, w \in \mathbb{D}$ and is the reproducing kernel [7], [4] of $L^2_a(\mathbb{D})$. For $a \in \mathbb{D}$, let $k_a(z) = \frac{K(z,a)}{\sqrt{K(a,a)}} = \frac{(1-|a|^2)}{(1-\bar{a}z)^2}$. The function k_a is called the normalized reproducing kernel for $L^2_a(\mathbb{D})$. It is clear that $||k_a||_2 = 1$. Let P denote the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $L^2_a(\mathbb{D})$. Let $Aut(\mathbb{D})$ be the Lie group of all automorphisms (biholomorphic mappings) of $\mathbb{D}$. We can define for each $a \in \mathbb{D}$ an automorphism ϕ_a in $Aut(\mathbb{D})$ such that

(i) $(\phi_a \circ \phi_a)(z) = z$;

(ii) $\phi_a(0) = a, \phi_a(a) = 0$;

(iii) ϕ_a has a unique fixed point in $\mathbb{D}$.

In fact, $\phi_a(z) = \frac{a-z}{1-\bar{a}z}$ for all a and z in $\mathbb{D}$. An easy calculation shows that the derivative of ϕ_a at z is equal to $-k_a(z)$. It follows that the real Jacobian determinant of ϕ_a at z is $J_{\phi_a}(z) = |k_a(z)|^2 = \frac{(1-|a|^2)}{|1-\bar{a}z|^4}$. For any $f \in L^1(\mathbb{D}, dA)$, we define a function Bf on $\mathbb{D}$ by

$$Bf(z) = \int_{\mathbb{D}} f(\phi_z(w))dA(w) = \int_{\mathbb{D}} f(w)|k_z(w)|^2 dA(w).$$

The map B is called the the Berezin transform [7], [4] on $\mathbb{D}$. The layout of this paper is as follows: In section 2, we construct the Berezin-type map E on $L^2_a(\mathbb{C}_+)$ by defining some elementary functions and discuss certain algebraic properties of the operator. Section 3 is devoted to establish that the operator E is not a bounded operator on $L^1(\mathbb{C}_+, d\widetilde{A})$ and show that the sequence $\{E^n\}$ converges to 0 in norm in the space $L^2(\mathbb{C}_+, d\mu)$ where $d\mu(w) = |B(\overline{w}, w)|d\widetilde{A}(w), w \in \mathbb{C}_+$. Further, we prove that the integral operator D given by $(Df)(s) = \int_{\mathbb{C}_+} f(w)|b_{\overline{w}}(s)|^2 d\widetilde{A}(w), s \in \mathbb{C}_+$ is a contraction on $L^1(\mathbb{C}_+, d\widetilde{A})$ which maps $L^\infty(\mathbb{C}_+)$ boundedly into $L^p(\mathbb{C}_+, d\widetilde{A})$ for $1 \leq p < \infty$. In section 4, we show that if $f \in L^1(\mathbb{C}_+, d\widetilde{A})$ is harmonic, then $Ef = Jf$ where $(Jf)(w) = f(\overline{w})$ and if $f \in L^2(\mathbb{C}_+, d\mu)$ and $Ef = Jf$ then $f \equiv 0$. Further, if $f \in L^1(\mathbb{C}_+, d\widetilde{A})$ is a real-valued subharmonic function on $\mathbb{C}_+$ which admits an integrable harmonic majorant then the functions $E^n f$ are subharmonic for all $n \in \mathbb{N}$ and $E^n f \to Ju$ where u is the least harmonic majorant of f.

2 The Berezin-type map

In this section, we construct the Berezin-type map E on $L^2_a(\mathbb{C}_+)$ and discuss certain algebraic properties of the operator. But first we introduce some elementary functions and their basic properties.

Define $M : \mathbb{C}_+ \to \mathbb{D}$ by $Ms = \frac{1-s}{1+s}$. Then M is one-one, onto and $M^{-1} : \mathbb{D} \to \mathbb{C}_+$ is given by $M^{-1}(z) = \frac{1-z}{1+z}$. Thus M is its self-inverse. Let $W : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{C}_+)$ be defined by $Wg(s) = \frac{2}{\sqrt{\pi}} g(Ms) \frac{1}{(1+s)^2}$. Then $W^{-1} : L^2_a(\mathbb{C}_+) \to L^2_a(\mathbb{D})$ is given by $W^{-1}G(z) = 2\sqrt{\pi} G(Mz) \frac{1}{(1+z)^2}$, where $Mz = \frac{1-z}{1+z}$. If $a \in \mathbb{D}$ and $a = c + id,\ c, d \in \mathbb{R}$, then $t_a(s) = \frac{-ids+(1-c)}{(1+c)s+id}$ is an automorphism from $\mathbb{C}_+$ onto $\mathbb{C}_+$ and

(i) $(t_a \circ t_a)(s) = s$.

(ii) $t'_a(s) = -l_a(s)$, where $l_a(s) = \frac{1-|a|^2}{((1+c)s+id)^2}$.

Let $w \in \mathbb{C}_+$ and $w = M\overline{a}, a \in \mathbb{D}$. For $f \in L^1(\mathbb{C}_+, d\widetilde{A})$, define $(Ef)(w) = \widetilde{f}(w) = \int_{\mathbb{C}_+} f(s)|b_{\overline{w}}(s)|^2 d\widetilde{A}(s), w \in \mathbb{C}_+$ where $b_{\overline{w}}(s) = \frac{1}{\sqrt{\pi}} \frac{1+w}{1+\overline{w}} \frac{2Rew}{(s+w)^2}$. Notice that $b_{\overline{w}} \in L^\infty(\mathbb{C}_+)$ for all $w \in \mathbb{C}_+$. Let $B(s,w) = B_{\overline{w}}(s) = \frac{1}{\pi} \frac{(1+a)^2}{(1-\overline{a}Ms)^2} \frac{1}{(1+s)^2}$ and $d\mu(w) = |B(\overline{w}, w)|d\widetilde{A}(w), w \in \mathbb{C}_+$.

Lemma 2.1. *Let $s, w \in \mathbb{C}_+$. The following relations hold:*

(i) $(b_{\overline{w}}(\overline{w}))^2 = B(\overline{w}, w)$.

(ii) $|b_{\overline{w}}(s)|\ ||B_{\overline{w}}|| = |B_{\overline{w}}(s)|$.

Proof. Let $w \in \mathbb{C}_+$ and $w = M\overline{a} = \frac{1-\overline{a}}{1+\overline{a}}$. Since

$$\begin{aligned}
b_{\overline{w}}(s) &= \frac{1}{\sqrt{\pi}} \frac{1+w}{1+\overline{w}} \frac{2Rew}{[s+w]^2} \\
&= \frac{2}{\sqrt{\pi}} \frac{1-|a|^2}{(1+\overline{a})^2} \frac{1}{[s+w]^2}, \text{where } \frac{1-\overline{a}}{1+\overline{a}} = w, \\
&= \frac{2}{\sqrt{\pi}} \frac{1-|a|^2}{(1+\overline{a})^2} \frac{1}{\left[s + \frac{1-\overline{a}}{1+\overline{a}}\right]^2} \\
&= \frac{2}{\sqrt{\pi}} \frac{1-|a|^2}{[1-\overline{a}(Ms)]^2} \frac{1}{(1+s)^2},
\end{aligned}$$

we obtain

$$b_{\overline{w}}(\overline{w}) = \frac{2}{\sqrt{\pi}} \frac{(1-|a|^2)}{(1-\overline{a}M\overline{w})^2} \frac{1}{(1+\overline{w})^2}$$

$$= \frac{1}{2\sqrt{\pi}} \frac{(1+a)^2}{(1-|a|^2)}.$$

Thus

$$\begin{aligned}
b_{\overline{w}}(s) b_{\overline{w}}(\overline{w}) &= \frac{2}{\sqrt{\pi}} \frac{(1-|a|^2)}{(1-\overline{a}Ms)^2} \frac{1}{(1+s)^2} \frac{1}{2\sqrt{\pi}} \frac{(1+a)^2}{(1-|a|^2)} \\
&= \frac{(-1)}{2\pi} \frac{(1+a)^2}{(1-\overline{a}Ms)^2} M' \\
&= B(s,w).
\end{aligned}$$

Hence $b_{\overline{w}}(s) = \frac{B(s,w)}{b_{\overline{w}}(\overline{w})}$ and $(b_{\overline{w}}(\overline{w}))^2 = B(\overline{w}, w)$. This proves (i). To prove (ii), notice that

$$\begin{aligned} ||B_{\overline{w}}||^2 &= \int_{\mathbb{C}_+} |B_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\ &= \int_{\mathbb{C}_+} |B(s,w)|^2 d\widetilde{A}(s) \\ &= \int_{\mathbb{C}_+} |b_{\overline{w}}(\overline{w})|^2 |b_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\ &= |b_{\overline{w}}(\overline{w})|^2 \int_{\mathbb{C}_+} |b_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\ &= |b_{\overline{w}}(\overline{w})|^2 ||b_{\overline{w}}||_2^2 = |b_{\overline{w}}(\overline{w})|^2, \end{aligned}$$

since $||b_{\overline{w}}||_2 = 1$. Thus $||B_{\overline{w}}|| = |b_{\overline{w}}(\overline{w})|$ and $|b_{\overline{w}}(s)|\ ||B_{\overline{w}}|| = |B_{\overline{w}}(s)|$.

□

Theorem 1. *Let $f \in L^1(\mathbb{C}_+, d\widetilde{A})$. The following hold:*

(i) If f is bounded, then so is $Ef = \widetilde{f}$ and $\|\widetilde{f}\|_\infty \leq \|f\|_\infty$. In other words, E is a contraction in $L^\infty(\mathbb{C}_+)$.

(ii) The norm of E on $L^\infty(\mathbb{C}_+, d\widetilde{A})$ is equal to 1.

(iii) If $f \geq 0$, then $\widetilde{f} \geq 0$; if $f \geq g$, then $\widetilde{f} \geq \widetilde{g}$.

(iv) The mapping $E : f \mapsto \widetilde{f}$ is a contractive linear operator on each of the spaces $L^p(\mathbb{C}_+, d\mu(z)), 1 \leq p \leq \infty$ where $d\mu(w) = |B(\overline{w}, w)| d\widetilde{A}(w)$.

(v) For arbitrary $f \in L^1(\mathbb{C}_+, d\widetilde{A})$, $\widetilde{f}(w) = \frac{1}{\pi} \int_{\mathbb{C}_+} (f \circ t_a \circ M)(s) d\widetilde{A}(s)$ where $a = M\overline{w}$.

(vi) $\widetilde{f}$ is an infinitely differentiable function on $\mathbb{C}_+$.

(vii) For $f \in L^1(\mathbb{C}_+, d\widetilde{A})$, define $T_a f = f \circ t_a$ for $a \in \mathbb{D}$. Then $(ET_a f)(w) = (T_{\overline{a}} Ef)(w)$.

Proof. (i) For proof of (i), assume $f \in L^\infty(\mathbb{C}_+)$. Then

$$|\widetilde{f}(w)| = \langle f b_{\overline{w}}, b_{\overline{w}} \rangle \leq \|f b_{\overline{w}}\|_2 \|b_{\overline{w}}\|_2 \leq \|f\|_\infty \|b_{\overline{w}}\|_2^2 = \|f\|_\infty.$$

(ii) Since $f = \widetilde{f}$ when f is a constant function, hence the norm of E on $L^\infty(\mathbb{C}_+, d\widetilde{A})$ is equal to 1.

(iii) The operator E is an integral operator with positive kernel. Thus if $f \geq 0$ then $\widetilde{f} \geq 0$. If $f \geq g$, let $h = f - g$. Then $h \geq 0$ and therefore $\widetilde{h} \geq 0$. Hence $\widetilde{f} \geq \widetilde{g}$.

(iv) Since $L^1(\mathbb{C}_+, d\mu) \subset L^1(\mathbb{C}_+, d\widetilde{A})$, the operator E is defined on the former space, and

$|\widetilde{f}(w)| = |\int_{\mathbb{C}_+} f(s)|b_{\overline{w}}(s)|^2 d\widetilde{A}(s)| \leq E(|f|)(s).$

Hence if $B_{\overline{w}}(s) = \frac{(-1)}{2\pi}\frac{(1+a)^2}{(1-\overline{a}Ms)^2}M'(s)$ then

$$\begin{aligned}
&\int_{\mathbb{C}_+} |\widetilde{f}(w)||B(\overline{w}, w)|d\widetilde{A}(w) \\
&\leq \int_{\mathbb{C}_+} \left(\int_{\mathbb{C}_+} |f(s)||b_{\overline{w}}(s)|^2 d\widetilde{A}(s)\right) |B(\overline{w}, w)|d\widetilde{A}(w) \\
&= \int_{\mathbb{C}_+} |f(s)| \int_{\mathbb{C}_+} |B_{\overline{w}}(s)|^2 d\widetilde{A}(w)d\widetilde{A}(s) \\
&= \int_{\mathbb{C}_+} |f(s)|\langle B_{\overline{w}}, B_{\overline{w}}\rangle d\widetilde{A}(s) \\
&= \int_{\mathbb{C}_+} |f(s)||B(\overline{s}, s)|d\widetilde{A}(s),
\end{aligned}$$

the change of order of integration being justified by the positivity of the integrand. It thus follows that E is a contraction on $L^1(\mathbb{C}_+, d\mu)$. The same is true for $L^\infty(\mathbb{C}_+)$, and so the result follows from the Marcinkiewicz interpolation theorem.

(v) Let $f \in L^1(\mathbb{C}_+, d\widetilde{A})$ and let $a = M\overline{w} \in \mathbb{D}$. Then

$$\begin{aligned}
\widetilde{f}(w) &= \int_{\mathbb{C}_+} f(s)|b_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{C}_+} (f \circ t_a)(s)|b_{\overline{w}}(t_a(s))|^2 |l_a(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{C}_+} (f \circ t_a)(s)|V_a b_{\overline{w}}(s)|^2 d\widetilde{A}(s) \\
&= \int_{\mathbb{C}_+} (f \circ t_a)(s)|\frac{(-1)}{\sqrt{\pi}}M'(s)|^2 d\widetilde{A}(s) \\
&= \frac{1}{\pi}\int_{\mathbb{C}_+} (f \circ t_a \circ M)(s)|(M' \circ M)(s)|^2 |M'(s)|^2 d\widetilde{A}(s) \\
&= \frac{1}{\pi}\int_{\mathbb{C}_+} (f \circ t_a \circ M)(s)d\widetilde{A}(s).
\end{aligned}$$

(vi) The function $\tilde{f}$ is an infinitely differentiable function on $\mathbb{C}_+$. To see this, let $f \in L^1(\mathbb{C}_+, d\tilde{A})$. Then $f \circ M \in L^1(\mathbb{D}, dA)$ and $\tilde{f}(w) = B(f \circ M)(a)$ where $w = M\bar{a} \in \mathbb{C}_+$ and $a \in \mathbb{D}$. Now

$$B(f \circ M)(a) = \langle (f \circ M)k_a, k_a \rangle = \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}(f \circ M)(z)dA(z) \quad (1)$$

where k_a is the normalized reproducing kernel at $a \in \mathbb{D}$ and $k_a(z) = \frac{(1-|a|^2)}{|1-\bar{a}z|^2}$. Let $a = x + iy \in \mathbb{D}, x, y \in \mathbb{R}$. Denote $\frac{\partial}{\partial a} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial \bar{a}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. It is not difficult to verify that $\frac{\partial}{\partial a}\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} = \frac{2(\bar{z}-\bar{a})(1-\bar{a}z)(1-|a|^2)}{|1-\bar{a}z|^6}$. Since $|z-a| \cdot |1-\bar{a}z| = |1-\bar{a}z|^2|\frac{z-a}{1-\bar{a}z}| \leq |1-\bar{a}z|^2$, it follows that

$$|f(z) \cdot \frac{\partial}{\partial a}\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}| \leq \|f\|_\infty \frac{2(1-|a|^2)}{|1-\bar{a}z|^4} \leq \frac{2\|f\|_\infty}{(1-|a|)^4}.$$

This is uniformly bounded when $a \in B(a_0, \epsilon), a_0 \in \mathbb{D}$ and $\epsilon > 0$ is sufficiently small. Consequently, it is allowed to differentiate under the integral sign in the formula (1), which gives $\frac{\partial B(f \circ M)}{\partial a} = \int_{\mathbb{D}} (f \circ M)(z)\frac{2(\bar{z}-\bar{a})(1-\bar{a}z)(1-|a|^2)}{|1-\bar{a}z|^6}dA(z)$. Thus $\tilde{f}(w) = B(f \circ M)(a)$ is infinitely differentiable.

(vii) To prove (vii), we shall first verify that $(Bf)(\phi_a(z)) = B(f \circ \phi_a)(z)$ for all $a, z \in \mathbb{D}$. Let $G_0 = \{\psi \in \text{Aut}(\mathbb{D}) : \psi(0) = 0\}$. For any a and b in $\mathbb{D}$, let $U = \phi_b \circ \phi_a \circ \phi_{\phi_a(b)}$. Then $U(0) = \phi_b \circ \phi_a(\phi_a(b)) = \phi_b(b) = 0$. Thus $U \in G_0$ is a unitary and $\phi_b \circ \phi_a = U\phi_{\phi_a(b)}$. Now by a change of variable, we obtain

$$Bf(\phi_a(z)) = \int_{\mathbb{D}} f(w)|k_{\phi_a(z)}(w)|^2 dA(w)$$

$$= \int_{\mathbb{D}} f(\phi_a(w))|k_{\phi_a(z)} \circ \phi_a(w)|^2|k_a(w)|^2 dA(w).$$

Hence there exists a unitary U with

$$\phi_{\phi_a(z)} \circ \phi_a = U\phi_{\phi_a \circ \phi_a(z)} = U\phi_z.$$

Taking the real Jacobian determinants of the above equation, we get

$$|k_{\phi_a(z)} \circ \phi_a(w)|^2|k_a(w)|^2 = |k_z(w)|^2$$

for all a, z and w in $\mathbb{D}$. Therefore,

$$Bf(\phi_a(z)) = \int_{\mathbb{D}} f(\phi_a(w))|k_z(w)|^2 dA(w)$$

$$= B(f \circ \phi_a)(z).$$

For $f \in L^1(\mathbb{C}_+, d\widetilde{A})$, $\widetilde{f}(w) = \int_{\mathbb{C}_+} f(s)|b_{\overline{w}}(s)|^2 d\widetilde{A}(s) = B(f \circ M)(a)$. Thus $(Ef)(M\overline{a}) = B(f \circ M)(a)$. Now for $z \in \mathbb{D}$,

$$B((f \circ M) \circ \phi_a)(z) = [B(f \circ M) \circ \phi_a](z) = B(f \circ M)(\phi_a(z)).$$

That is,

$$E(f \circ M \circ \phi_a \circ M)(M\bar{z}) = (Ef)(M\overline{\phi_a(z)}).$$

Hence $E(f \circ t_a)(M\bar{z}) = (Ef)(\overline{(M \circ \phi_a)(z)}) = (Ef)(\overline{(t_a \circ M)(z)})$. Thus $E(f \circ t_a)(Mz) = (Ef)(\overline{(t_a \circ M)(\bar{z})})$ and therefore

$$\begin{aligned} E(f \circ t_a)(w) &= (Ef)(\overline{(t_a \circ M)(M\bar{w})}) \\ &= (Ef)(\overline{t_a(\overline{w})}) \\ &= (Ef)(t_{\overline{a}}(w)). \end{aligned} \tag{2}$$

Thus from (2) it follows that $(ET_a f)(w) = T_{\overline{a}}(Ef)(w) = (T_{\overline{a}}Ef)(w)$. □

3 Boundedness of the Berezin-type map

In this section we establish that the operator E is not a bounded operator on $L^1(\mathbb{C}_+, d\widetilde{A})$. Further, we prove that the integral operator D given by $(Df)(s) = \int_{\mathbb{C}_+} f(w)|b_{\overline{w}}(s)|^2 d\widetilde{A}(w), s \in \mathbb{C}_+$ is a contraction on $L^1(\mathbb{C}_+, d\widetilde{A})$ which maps $L^\infty(\mathbb{C}_+)$ boundedly into $L^p(\mathbb{C}_+, d\widetilde{A})$ for $1 \le p < \infty$.

Proposition 3.1. *The operator E is not a bounded operator on $L^1(\mathbb{C}_+, d\widetilde{A})$.*

Proof. : If it were, its adjoint $E^d \equiv D$, where

$$(Df)(s) = \int_{\mathbb{C}_+} f(w)|b_{\overline{w}}(s)|^2 d\widetilde{A}(w), s \in \mathbb{C}_+ \tag{3}$$

would be a bounded operator on $L^\infty(\mathbb{C}_+)$. Let $f \in L^\infty(\mathbb{C}_+)$. Now if $z = Ms$ and $w = M\overline{a}$, then

$$\begin{aligned}(Df)(s) &= \int_{\mathbb{C}_+} f(w)|b_{\overline{w}}(s)|^2 d\widetilde{A}(w)\\ &= \int_{\mathbb{C}_+} f(w)|Wk_a(s)|^2 d\widetilde{A}(w)\\ &= \tfrac{1}{\pi}\int_{\mathbb{C}_+} f(w)|k_a(Ms)|^2|M'(s)|^2 d\widetilde{A}(w)\\ &= \tfrac{|M'(s)|^2}{\pi}\int_{\mathbb{C}_+}(f\circ M)(\overline{a})|k_a(Ms)|^2 d\widetilde{A}(M\overline{a})\\ &= |M'(s)|^2\int_{\mathbb{D}}(f\circ M)(a)|k_{\overline{a}}(Ms)|^2|M'(a)|^2 dA(a).\end{aligned}$$

Hence

$$\begin{aligned}(D1)(s) &= \int_{\mathbb{C}_+}|b_{\overline{w}}(s)|^2 d\widetilde{A}(w)\\ &= |M'(s)|^2\int_{\mathbb{D}}|k_{\overline{a}}(Ms)|^2|M'(a)|^2 dA(a)\\ &\le \|M'\|_\infty^4\int_{\mathbb{D}}|k_{\overline{a}}(z)|^2 dA(a)\\ &\le 2^4\int_{\mathbb{D}}|k_{\overline{a}}(z)|^2 dA(a).\end{aligned}$$

Now

$$\begin{aligned}\int_{\mathbb{D}}|k_{\overline{a}}(z)|^2 dA(a) &= \int_{\mathbb{D}}\frac{(1-|a|^2)^2}{|1-az|^4}dA(a)\\ &= \int_0^1(1-r^2)^2\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-\overline{z}re^{it}|^4}dt2rdr\\ &= \int_0^1(1-r^2)^2\sum_{n=0}^\infty(n+1)^2r^{2n}|z|^{2n}2rdr\end{aligned}$$

since

$$\begin{aligned}\int_0^{2\pi}\frac{1}{|1-\overline{z}re^{it}|^4}dt &= \tfrac{1+|z|^2r^2}{(1-|z|^2r^2)^3}\\ &= \sum_{n=0}^\infty(n+1)^2r^{2n}|z|^{2n},\end{aligned}$$

for $z\in\mathbb{D}$ and $r\in(0,1)$. Thus

$$\begin{aligned}|(D1)(s)| &\le 2^4\int_0^1\sum_{n=0}^\infty(n+1)^2(1-t)^2t^n|z|^{2n}dt\\ &= 2^4\sum_{n=0}^\infty\frac{2(n+1)}{(n+2)(n+3)}|z|^{2n}\end{aligned}$$

where $s = Mz$. As $|z| \to 1$, this expression behaves (asymptotically) like $-2^4 \log(1-|z|^2)$, hence $D1 \notin L^\infty(\mathbb{C}_+)$, so $D \equiv E^d$ cannot be a bounded operator on $L^\infty(\mathbb{C}_+)$. □

Lemma 3.1. *The integral operator D given by (3) is a contraction on $L^1(\mathbb{C}_+, d\widetilde{A})$ which maps $L^\infty(\mathbb{C}_+)$ boundedly into $L^p(\mathbb{C}_+, d\widetilde{A})$ for $1 \le p < \infty$.*

Proof. For arbitrary $f \in L^1(\mathbb{C}_+, d\widetilde{A})$, by Fubini's theorem [6] it follows that

$$\begin{aligned}\int_{\mathbb{C}_+} |(Df)(s)| d\widetilde{A}(s) &\le \int_{\mathbb{C}_+}\int_{\mathbb{C}_+} |b_{\overline{w}}(s)|^2 |f(w)| d\widetilde{A}(w) d\widetilde{A}(s) \\ &= \int_{\mathbb{C}_+} |f(w)| \int_{\mathbb{C}_+} |b_{\overline{w}}(s)|^2 d\widetilde{A}(s) d\widetilde{A}(w) \\ &= \int_{\mathbb{C}_+} |f(w)| \langle b_{\overline{w}}, b_{\overline{w}}\rangle d\widetilde{A}(w) \\ &= \int_{\mathbb{C}_+} |f(w)| d\widetilde{A}(w),\end{aligned}$$

so D is a contraction on $L^1(\mathbb{C}_+, d\widetilde{A})$. If $f \in L^\infty(\mathbb{C}_+)$, then

$$|(Df)(s)| \le \|f\|_\infty \int_{\mathbb{C}_+} |b_{\overline{w}}(s)|^2 d\widetilde{A}(w) = \|f\|_\infty |(D1)(s)|.$$

Hence, to prove the second assertion of the lemma, it suffices to check that $D1$ belongs to $L^p(\mathbb{C}_+, d\widetilde{A})$ for each $p \in [1,\infty)$. We have already observed that $(D1)(s)$ behaves like $-2^4 \log(1-|Ms|^2)$ as $|Ms| \to 1$, so it is enough to show that $\log(1-|z|^2) \in L^p(\mathbb{D}, dA)$ for all $p \in [1,\infty)$. Now, $\int_{\mathbb{D}} |\log(1-|z|^2)|^p dA(z) = \int_0^1 |\log(1-r^2)|^p 2r dr = \int_0^1 |\log(1-t)|^p dt = \int_0^1 |\log t|^p dt$, and, changing the variable to $y = -\log t$, this reduces to $\int_0^\infty y^p e^{-y} dy = \Gamma(p+1) < \infty$. □

On $\mathbb{D}$, the only measure left invariant by all Möbius transformations $e^{i\theta}\phi_a(z)$, $\theta \in \mathbb{R}$ is the pseudo-hyperbolic measure $d\eta(z) = \frac{dA(z)}{(1-|z|^2)^2} = K(z,z)dA(z)$. The invariance may be verified by direct computation. It turns out that the Berezin transform behaves well [7] with respect to the invariant measures. Consider the Fourier-Helgason transform [5], [3] on the disk. It maps a function $f(z)$ on the disk into a function $\widehat{f}(t,b)$ of $t \in \mathbb{R}$ and b on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In fact

$$\widehat{f}(t,b) = \int_{\mathbb{D}} f(x) e_{t,b}(x) d\eta(x)$$

where $e_{t,b}(x) = \left(\frac{1-|x|^2}{|b-x|^2}\right)^{\frac{1}{2}+it}, x \in \mathbb{D}, t \in \mathbb{R}$ and $b \in \mathbb{T}$. On $L^2(\mathbb{D}, d\eta)$ with respect to the invariant measure, the Berezin transform is a Fourier multiplier with respect to the Fourier-Helgason transform; the multiplier function being $(t^2 + \frac{1}{4})\frac{\pi}{\cosh(\pi t)}$, $t \in \mathbb{R}$. That is, for the Berezin transform B one has $\widehat{(Bf)}(t, b) = m(t)\widehat{f}(t, b)$ where $m(t) = (t^2 + \frac{1}{4})\frac{\pi}{\cosh(\pi t)}$. For more details see [5] and [2].

Lemma 3.2. *The following is true for the Berezin-type map E as an operator on $L^2(\mathbb{C}_+, d\mu)$ where $d\mu(w) = |B(\overline{w}, w)|d\widetilde{A}(w), w \in \mathbb{C}_+$.*

(i) *E is positive.*

(ii) *E^n converges to 0 in SOT.*

(iii) *E^n converges to 0 in norm.*

Proof. We shall first show that $f \in L^2(\mathbb{D}, d\eta)$ if and only if $f \circ M \in L^2(\mathbb{C}_+, d\mu)$. So let $f \in L^2(\mathbb{D}, d\eta)$ where $d\eta(z) = \frac{dA(z)}{(1-|z|^2)^2} = K(z, z)dA(z)$. For $w = M\overline{a}$,

$$\begin{aligned}
\int_{\mathbb{D}} |f(\overline{a})|^2 K(\overline{a}, \overline{a})dA(\overline{a}) &= \int_{\mathbb{D}} |f(\overline{a})|^2 |B(\overline{w}, w)\frac{K(\overline{a}, \overline{a})}{B(\overline{w}, w)}|dA(\overline{a}) \\
&= \int_{\mathbb{D}} |f(\overline{a})|^2 |B(\overline{w}, w)|\frac{|K(\overline{a}, \overline{a})|}{|B(\overline{w}, w)|}dA(\overline{a}) \\
&= 4\pi \int_{\mathbb{D}} |f(\overline{a})|^2 |B(\overline{w}, w)|\frac{1}{|1+a|^4}dA(\overline{a}) \\
&= 4\pi \int_{\mathbb{D}} |f(\overline{a})|^2 |B(Ma, M\overline{a})|\frac{1}{|1+a|^4}dA(\overline{a}) \\
&= \pi \int_{\mathbb{D}} |(f \circ M)(M\overline{a})|^2 |M'(\overline{a})|^2 |B(\overline{w}, w)|dA(\overline{a}) \\
&= \int_{\mathbb{C}_+} |(f \circ M)(w)|^2 |B(\overline{w}, w)|d\widetilde{A}(w).
\end{aligned}$$

Now we proceed to prove that the Berezin transform B is positive as an operator on $L^2(\mathbb{D}, d\eta)$. Observe that the function $m(t)$ has a maximum at $t = 0$ with value $\frac{\pi}{4}$. By spectral theorem, B has thus norm $\frac{\pi}{4}$, which is strictly less than 1. Using the Fourier-Helgason transform, one has (by Plancherel theorem, which also holds [5] for this transform) $\langle Bf, f\rangle = \langle\widehat{(Bf)}, \widehat{f}\rangle = \int_{\mathbb{R}}\int_{\mathbb{T}} m(t)|\widehat{f}(t, b)|^2 dtdb \geq 0$ since the multiplier function $m(t) = (t^2 + \frac{1}{4})\frac{\pi}{\cosh(\pi t)}$ is positive. Thus the operator B is positive. This also gives the spectral decomposition of B. Let $E(\lambda)$ be the resolution of identity for the self-adjoint operator B. Then $\|B^n f\|^2 = \int_{[0, \frac{\pi}{4}]} |\lambda^n|^2 d\langle E(\lambda)f, f\rangle$.

According to the Lebesgue monotone convergence theorem, this tends to $\|(I-E(1-))f\|^2 = \|P_{\ker(B-I)}f\|^2$ where $P_{\ker(B-I)}$ is the orthogonal projection from $L^2_a(\mathbb{D})$ onto $\ker(B-I)$.

Further notice that if $f \in L^2(\mathbb{D}, d\eta)$ is harmonic then $f \equiv 0$. To see this, let

$$M(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt.$$

This is a nonnegative and nondecreasing function of r. Now,

$$\|f\|^2_{L^2(\mathbb{D},d\eta)} = \int_0^1 M(r)\frac{2r}{(1-r^2)^2}dr < \infty.$$

So $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$, whence $f \equiv 0$. Thus it follows that $\ker(I-B) = \{0\}$ since 1 is not in the spectrum of B and hence, $\|B^n f\|$ tends to zero. In fact, even $\|B^n\|$ tends to zero as $\|B\| < 1$. Since $(E^n g)(M\overline{a}) = B^n(g \circ M)(a), a \in \mathbb{D}$ for $g \in L^2(\mathbb{C}_+, d\mu)$, the result follows. □

4 Harmonic and subharmonic functions

In this section we show that if $f \in L^1(\mathbb{C}_+, d\widetilde{A})$ is harmonic, then $\widetilde{f} = Jf$ where $(Jf)(w) = f(\overline{w})$ and if $f \in L^2(\mathbb{C}_+, d\mu)$ and $\widetilde{f} = Jf$ then $f \equiv 0$. Further if $f \in L^1(\mathbb{C}_+, d\widetilde{A})$ is a real-valued subharmonic function on $\mathbb{C}_+$ which admits an integrable harmonic majorant then the functions $E^n f$ are subharmonic for all $n \in \mathbb{N}$ and $E^n f \to Ju$ where u is the least harmonic majorant of f.

Theorem 2. *If a function $f \in L^1(\mathbb{C}_+, d\widetilde{A})$ is harmonic, then $\widetilde{f} = Jf$ where $(Jf)(w) = f(\overline{w})$. If $f \in L^2(\mathbb{C}_+, d\mu)$ and $\widetilde{f} = Jf$ then $f \equiv 0$.*

Proof. Notice that if $w = M\overline{a}$,

$$\begin{aligned} \widetilde{f}(w) &= \langle f b_{\overline{w}}, b_{\overline{w}}\rangle \\ &= \langle fWk_a, Wk_a\rangle \\ &= B(f \circ M)(a). \end{aligned}$$

Let $f = Wg = \frac{(-1)}{\sqrt{\pi}}(g \circ M)M', g \in L^1(\mathbb{D}, dA)$. Now f is harmonic implies $(g \circ M)M'$ is harmonic and therefore $g(M' \circ M)$ is harmonic. Thus $\frac{g}{M'}$ and

$\frac{g}{M'} \circ \phi_a$ are harmonic. Hence

$$\begin{aligned}
\widetilde{f}(w) &= B(f \circ M)(a) \\
&= \tfrac{(-1)}{\sqrt{\pi}} B[((g \circ M)M') \circ M](a) \\
&= \tfrac{(-1)}{\sqrt{\pi}} B(g(M' \circ M))(a) \\
&= \tfrac{(-1)}{\sqrt{\pi}}(-1)\sqrt{\pi}\left(\tfrac{(f \circ M)M'}{M'}\right)(a) \\
&= (f \circ M)(a) = f(Ma) = f(\overline{w}) \\
&= (Jf)(w)
\end{aligned}$$

since $(M' \circ M)M' = 1$ and $|k_a(\phi_a(z))||k_a(z)| = 1$.

Suppose $f \in L^2(\mathbb{C}_+, d\mu)$ is harmonic. Then $f \circ M \in L^2(\mathbb{D}, d\eta)$. Now since $\widetilde{f}(w) = (Jf)(w) = f(\overline{w})$ for all $w \in \mathbb{C}_+$, hence $B(f \circ M)(a) = (f \circ M)(a)$ for all $a \in \mathbb{D}$. Therefore $f \circ M$ is harmonic in $\mathbb{D}$ and $f \circ M \in L^2(\mathbb{D}, d\eta) \subset L^2(\mathbb{D}, dA)$. Thus f is harmonic in $\mathbb{C}_+$.

But the only harmonic function in $L^2(\mathbb{C}_+, d\mu)$ is constant zero. To see this, let $f \in L^2(\mathbb{C}_+, d\mu)$ is harmonic. Then $g = f \circ M \in L^2(\mathbb{D}, d\eta)$ is harmonic. Let $M(r) = \frac{1}{2\pi}\displaystyle\int_0^{2\pi} |g(re^{it})|^2 dt$. This is a nonnegative and non-decreasing function of r and

$$\|g\|^2_{L^2(\mathbb{D},d\eta)} = \int_0^1 M(r)\frac{2r}{(1-r^2)^2}dr < \infty.$$

Thus $M(r)$ must tend to zero as $r \to 1$. Thus $M(r) \equiv 0$. Hence $g \equiv 0$. That is, $f \circ M \equiv 0$. This implies $f \equiv 0$. □

Theorem 3. *Assume that $f \in L^1(\mathbb{C}_+, d\widetilde{A})$ is a real-valued subharmonic function on $\mathbb{C}_+$ which admits an integrable harmonic majorant (i.e., there exists a function $v \in L^1(\mathbb{C}_+, d\widetilde{A})$ harmonoic on $\mathbb{C}_+$ and such that $v(s) \geq f(s)$ for all $s \in \mathbb{C}_+$.) Then $E^n f \to Ju$ where u is the least harmonic majorant of f.*

Proof. Let $f \in L^1(\mathbb{C}_+, d\widetilde{A})$ is a real-valued subharmonic function on $\mathbb{C}_+$. Then $f \circ M \in L^1(\mathbb{D}, dA)$ is real-valued subharmonic function on $\mathbb{D}$. If f admits an integrable harmonic majorant v then $f \circ M$ admits an integrable harmonic majorant $v \circ M$ on $\mathbb{D}$ and if u is the least harmonic majorant of f on $\mathbb{C}_+$ then $u \circ M$ is the least harmonic majorant of $f \circ M$ on $\mathbb{D}$.

According to a theorem of Frostman ([3]), there exists a positive Borel measure κ on $\mathbb{D}$ such that

$$f(s) = u(s) + \frac{1}{4}\int_{\mathbb{D}} \ln|\phi_{Ms}(z_1)|^2 d\kappa(z_1)$$

for all $s \in \mathbb{C}_+$. Let $g(z_1) = \ln|z_1|^2$. Since $|\phi_{Ms}(z_1)| = |\phi_{z_1}(Ms)|$, we have

$$f(s) \quad = \quad u(s) + \tfrac{1}{4}\int_{\mathbb{D}} (g \circ \phi_{z_1})(Ms)d\kappa(z_1).$$

Hence

$$(Ef \circ M)(\bar{z}) = (Eu \circ M)(\bar{z}) + \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{Ms}z|^4} \frac{1}{4}\int_{\mathbb{D}} (g \circ \phi_{z_1})(Ms)d\kappa(z_1)dA(Ms). \tag{4}$$

Since $f \le u \le v$ and $f, v \in L^1(\mathbb{C}_+, d\widetilde{A})$, we have $u \in L^1(\mathbb{C}_+, d\widetilde{A})$. Further, from Theorem 2 , it follows that $(Eu \circ M) = (Ju \circ M)$. Since $(g \circ \phi_{z_1})(Ms)$ takes nonpositive values, we may interchange the order of integration in (4), to obtain

$$(Ef \circ M)(\bar{z}) = (Eu \circ M)(\bar{z}) + \frac{1}{4}\int_{\mathbb{D}} B(g \circ \phi_{z_1})(z)d\kappa(z_1). \tag{5}$$

Proceeding by induction, we obtain

$$(E^n f \circ M)(\bar{z}) = (Eu \circ M)(\bar{z}) + \frac{1}{4}\int_{\mathbb{D}} B^n(g \circ \phi_{z_1})(z)d\kappa(z_1).$$

We shall now show that

$$B^n(g \circ \phi_{z_1})(z) \to 0$$

as $n \to \infty$, for all $z_1, z \in \mathbb{D}$. Notice that $B^n(g \circ \phi_{z_1}) = (B^n g) \circ \phi_{z_1}$. Thus it is suffices to prove that $B^n g \to 0$. It is not difficult to show that $(Bg)(z_1) = |z_1|^2 - 1$. Thus $g \le Bg$. This implies $B^k g \le B^{k+1} g$ for all $k \in \mathbb{N}$. So

$$g \le Bg \le B^2 g \le B^3 g \le \cdots \le 0.$$

Hence the limit $\lim_{n\to\infty} (B^n g)(z_1) = \Psi(z_1)$ must exist and $\Psi \le 0$. From Lebesgue monotone convergence theorem, it follows that

$$\begin{aligned}
(B\Psi)(z) &= \int_{\mathbb{D}} (\Psi \circ \phi_z)(w)dA(w) \\
&= \int \lim_{n\to\infty} (B^n g)(\phi_z(w))dA(w) \\
&= \lim_{n\to\infty} \int B^n(g \circ \phi_z)(w)dA(w) \\
&= \lim_{n\to\infty} B^n(\int (g \circ \phi_z)(w)dA(w)) \\
&= \lim_{n\to\infty} (B^{n+1} g)(z) = \Psi(z).
\end{aligned}$$

We claim that $\Psi \equiv 0$. Assume the contrary. Because $|z_1|^2 - 1 = (Bg)(z_1) \leq \Psi(z_1) \leq 0$, we have $\lim_{|z_1| \to 1} \Psi(z_1) = 0$. Consequently, Ψ must attain its infimum at some point $z_2 \in \mathbb{D}$– suppose (replacing Ψ by $\Psi \circ \phi_{z_2}$ otherwise) that $z_2 = 0$. Then $\Psi(0) = (B\Psi)(0) = \int_{\mathbb{D}} \Psi(z_1) dA(z_1) > \inf_{z_1 \in \mathbb{D}} \Psi(z_1) \int_{\mathbb{D}} dA(z_1) = \Psi(0)$. This is a contradiction. Hence $\Psi \equiv 0$.

Because κ is a positive measure, we may apply the Lebesgue monotone convergence theorem to conclude that $(E^n f) \circ M \to (Ju) \circ M$ as $n \to \infty$. Thus $(E^n f) \to Ju$ as $n \to \infty$. □

Remark 4.1: If f is real-valued, subharmonic and $f \in L^2(\mathbb{C}_+, d\mu)$, we may proceed a little more quickly. The subharmonicity of f implies that $f \circ M$ is real-valued, subharmonic and $h_1 = f \circ M \in L^2(\mathbb{D}, d\eta)$ and for $a \in \mathbb{D}$,

$$(Bh_1)(a) = \int_{\mathbb{D}} (h_1 \circ \phi_a)(z)) dA(z) \geq h_1(\phi_a(0)) = h_1(a),$$

that is, $Bh_1 \geq h_1$. Further, B commutes with $\triangle_h$, where $\triangle_h := (1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}}$, the Laplace-Beltrami operator on $\mathbb{D}$. Hence $\triangle_h B h_1 = B(\triangle_h h_1) \geq 0$ since $\triangle_h h_1 \geq 0$; in other words, Bh_1 is also subharmonic. Proceeding by induction, we obtain a nondecreasing sequence $\{B^k h_1\}_{k \in \mathbb{N}}$ of subharmonic functions. Their limit Ψ is either identically $+\infty$, or is a subharmonic function satisfying $B\Psi = \Psi$. Since $\Psi \in L^2(\mathbb{D}, d\eta)$, the former case cannot occur; further,

$$\Psi(0) = (B\Psi)(0) = \int_{\mathbb{D}} \Psi(z) dA(z),$$

and so Ψ is actually harmonic; hence, it is a harmonic majorant of $h_1 = f \circ M$. If Υ is another harmonic majorant of h_1, then $h_1 \leq \Upsilon$ implies $B^n h_1 \leq B^n \Upsilon = \Upsilon$, whence also $\Psi \leq \Upsilon$; consequently, Ψ is the least harmonic majorant of $h_1 = f \circ M$. This in turn implies $\Psi \circ M$ is the least harmonic majorant of f and $E^n f \to J(\Psi \circ M)$ as $n \to \infty$.

Theorem 4. *Assume $f \in L^1(\mathbb{C}_+, d\widetilde{A})$ is real-valued subharmonic function on $\mathbb{C}_+$ which admits an integrable harmonic majorant v. Then the functions $E^n f$ are subharmonic for all $n \in \mathbb{N}$.*

Proof. Let $0 < R < 1$. From [3], it follows that

$$f(s) = u(s) + \frac{1}{4} \int_{\mathbb{D}} \ln |\phi_{Ms}(z_1)|^2 d\kappa(z_1)$$

for all $s \in \mathbb{C}_+$. Since $|\phi_{Ms}(z_1)| = |\phi_{z_1}(Ms)|$, hence

$$f(s) = u(s) + \frac{1}{4} \int_{\mathbb{D}} (g \circ \phi_{z_1})(Ms) d\kappa(z_1)$$

where $g(z) = \ln|z|^2$. Thus

$$(Ef\circ M)(\bar{z}) = (Eu\circ M)(\bar{z}) + \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{Ms}z|^4} \frac{1}{4} \int_{\mathbb{D}} (g\circ\phi_{z_1})(Ms)d\kappa(z_1)dA(Ms).$$

Hence it follows that

$$\frac{1}{2\pi}\int_0^{2\pi}(Ef\circ M)(Re^{-it})dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(u\circ M)(Re^{it})dt + \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{4}\int_{\mathbb{D}} B(g\circ\phi_{z_1})(Re^{it})d\kappa(z_1)dt.$$

Since the second integrand is nonpositive, we may interchange the order of integration; consequently,

$$\frac{1}{2\pi}\int_0^{2\pi}(Ef\circ M)(Re^{-it})dt$$

$$= (u\circ M)(0) + \frac{1}{4}\int_{\mathbb{D}}\left(\frac{1}{2\pi}\int_0^{2\pi} B(g\circ\phi_{z_1})(Re^{it})dt\right)d\kappa(z_1).$$

It is not difficult to check that if $g(x) = \ln|x|^2$, then $(Bg)(x) = |x|^2 - 1$ and therefore $B(g\circ\phi_{z_1})(z) = |\phi_{z_1}(z)|^2 - 1$ is a subharmonic function of z. This implies

$$\frac{1}{2\pi}\int_0^{2\pi} B(g\circ\phi_{z_1})(Re^{it})dt \geq B(g\circ\phi_{z_1})(0).$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi}(Ef\circ M)(Re^{-it})dt \geq (u\circ M)(0) + \frac{1}{4}\int_{\mathbb{D}} B(g\circ\phi_{z_1})(0)d\kappa(z_1) = (Ef\circ M)(0)$$

for every $R \in (0,1)$. Similarly one can show that

$$\frac{1}{2\pi}\int_0^{2\pi}(E(f\circ t_a)\circ M)(Re^{-it})dt \geq (E(f\circ t_a)\circ M)(0)$$

for $0 < R < 1$. Thus $B(f\circ M)$ satisfies the sub-mean value property, and therefore is subharmonic on $\mathbb{D}$. Hence $Ef\circ M$ is subharmonic on $\mathbb{D}$ and therefore Ef is subharmonic on $\mathbb{C}_+$. Since $f \leq v$, hence $f\circ M \leq v\circ M$. Therefore $B(f\circ M) \leq B(v\circ M) = v\circ M$. Thus $B(f\circ M)$ also has an integrable harmonic majorant and $Ef\circ M$ also has an integrable harmonic majorant . Consequently, we may proceed by induction, and the theorem follows. □

Remark 4.2: There is no nonzero harmonic function in $L^2(\mathbb{C}_+, d\mu)$, but there are plenty of subharmonic functions. The functions $E^n g, n \in \mathbb{N}$ where $g(s) = \ln|Ms|^2 = \ln|\frac{1-s}{1+s}|^2, s \in \mathbb{C}_+$ serve as an example. This can be verified as follows:

$$\begin{aligned}
\int_{\mathbb{C}_+} |g(w)|^2 |B(\overline{w}, w)| d\widetilde{A}(w) &= \int_{\mathbb{D}} |(g \circ M)(Mw)|^2 K(Mw, Mw) dA(Mw) \\
&= \int_{\mathbb{D}} |(g \circ M)(z)|^2 K(z, z) dA(z) \\
&= \int_{\mathbb{D}} (\ln|z|^2)^2 \frac{dA(z)}{(1-|z|^2)^2} \\
&= \int_0^1 \left(\frac{\ln t}{1-t}\right)^2 dt \\
&= \int_0^1 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} t^{m+n} \ln^2 t dt \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{2}{(m+n+1)^3} \\
&= \sum_{k=0}^{\infty} \frac{2}{(k+1)^2} \\
&= \frac{\pi^2}{3} < +\infty.
\end{aligned}$$

Hence $g \in L^2(\mathbb{C}_+, d\mu)$ and g is subharmonic, and therefore by Theorem 4, $E^n g$ is subharmonic for all $n \in \mathbb{N}$.

Given a bounded real-valued subharmonic function on $\mathbb{D}$, the boundary values of its least harmonic majorant can be described explicitly. In fact if Φ is a bounded real-valued subharmonic function on $\mathbb{D}$, define Φ on $\mathbb{T}$ (the unit circle in $\mathbb{C}$) by $\Phi(e^{i\theta}) = \lim_{r\to 1} \sup \Phi(re^{i\theta})$, and let Ψ be the Poisson extension of $\Phi|_{\mathbb{T}}$ into the interior of $\mathbb{D}$. Then Ψ is the least harmonic majorant of Φ. The following is also valid.

Theorem 5. *Suppose ϕ is a bounded real-valued subharmonic function on $\mathbb{C}_+$. Define ϕ on $i\mathbb{R}$ by*

$$\phi(iy) = \limsup_{x\to 0} \phi(x+iy), x > 0,$$

and let ψ be the Poisson extension of $\phi|_{i\mathbb{R}}$ into $\mathbb{C}_+$. Then ψ is the least harmonic majorant of ϕ.

Proof. Let ϕ be a bounded real-valued subharmonic function on $\mathbb{C}_+$. Then $\phi \circ M$ is a real-valued subharmonic function on $\mathbb{D}$. Let

$$(\phi \circ M)(\epsilon) = \limsup_{r\to 1} (\phi \circ M)(r\epsilon), \epsilon \in \mathbb{T},$$

and let $\psi \circ M$ be the Poisson extension of $\phi \circ M|_{\mathbb{T}}$ into $\mathbb{D}$. Then it is not difficult to see that $\psi \circ M$ is the least harmonic majorant of $\phi \circ M$. To see this, let $u \circ M$ be the least harmonic majorant of $\phi \circ M$. Except for ϵ on a set of measure(arc-length measure) zero, we have

$$\lim_{r\to 1}(u \circ M)(r\epsilon) \geq \limsup_{r\to 1}(\phi \circ M)(r\epsilon) = (\phi \circ M)(\epsilon) = \lim_{r\to 1}(\psi \circ M)(r\epsilon).$$

It follows thus that the bounded harmonic function $(u - \psi) \circ M = u \circ M - \psi \circ M$ has nonnegative radial limits almost everywhere on $\mathbb{T}$; hence, $u \circ M \geq \psi \circ M$ on $\mathbb{D}$. We shall now show that $u \circ M \leq \psi \circ M$. Because subharmonicity and harmonicity is invariant under Mobius transformations, it suffices to show that $(\psi \circ M)(0) \geq (u \circ M)(0)$. Without loss of generality, we shall assume that $\phi \circ M \leq 0$, hence also $u \circ M \leq 0$ and $\psi \circ M \leq 0$. Applying the Fatou's lemma to the function $t \to (\phi \circ M)(re^{it})$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \limsup_{r\to 1}(\phi \circ M)(re^{it})dt \geq \limsup_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}(\phi \circ M)(re^{it})dt. \tag{6}$$

The left hand side of (6) is equal to $(\psi \circ M)(0)$ and in the right hand side, we can replace limsup by either lim or by sup and the right hand side equals $(u \circ M)(0)$. Thus $\psi \circ M$ is the least harmonic majorant of $\phi \circ M$. Hence ψ is the least harmonic majorant of ϕ as claimed. □

Let

$$V(\mathbb{C}_+) = \{f \in L^\infty(\mathbb{C}_+) : \operatorname{ess}\lim_{x\to 0} f(x+iy) = 0\}$$

and

$$V(\mathbb{D}) = \{f \in L^\infty(\mathbb{D}) : \ \operatorname{ess}\lim_{|z|\to 1} f(z) = 0\}.$$

Theorem 6. *If $f \in V(\mathbb{C}_+)$, then $E^n f$ converges uniformly to* 0.

Proof. Let $f \in V(\mathbb{C}_+)$. Then $\operatorname{ess}\lim_{\mathrm{Re}s\to 0} f(s) = 0$. That is, $\operatorname{ess}\lim_{|z|\to 1}(f \circ M)(z) = 0$. Hence $f \circ M \in V(\mathbb{D})$. We shall now show that $B^n(f \circ M) \to 0$ uniformly. That will imply $(E^n f)\circ M \to 0$ uniformly and therefore $E^n f \to 0$ uniformly. Let $g = f \circ M \in V(\mathbb{D})$. Without loss of generality, we shall assume $g \leq 0$ since B is linear. As $\mathcal{D}(\mathbb{D})$ (the set of all infinitely differentiable functions on $\mathbb{D}$ whose support is a compact subset of $\mathbb{D}$) is dense in $V(\mathbb{D})$ and B is a contraction on $L^\infty(\mathbb{D})$, we shall consider only $g \in \mathcal{D}(\mathbb{D})$. So assume $g \leq 0$ and support of g is contained in $\{z \in \mathbb{D} : |z| < R\}, 0 < R < 1$. Define the function G on $[0, 1]$ as follows:

$$G(t) = -\|g\|_\infty, \quad \text{if} \ \ 0 \leq t \leq R,$$

$$G(1) = 0 \text{ and } G(t) \text{ is linear on } [R, 1],$$

and set $\Phi(z) = G(|z|), z \in \overline{\mathbb{D}}$. The function Φ is subharmonic, its least harmonic majorant being constant zero. By Theorem 3, $B^n\Phi \to 0$, since $\Phi = 0$ on $\partial\mathbb{D} = \mathbb{T}$. By Dini's theorem, $\{B^n\Phi\}$ converges uniformly to 0. But $\Phi \leq g \leq 0$, hence $B^k\Phi \leq B^k g \leq 0$ and so $\{B^n(f \circ M)\} = \{B^n g\}$ converges uniformly to 0 as well. Thus $E^n f \to 0$ uniformly on $\mathbb{C}_+$. □

Corollary 1. *Suppose* $f \in C(\mathbb{C}_+ \cup i\mathbb{R})$. *Then* $\{E^n f\}$ *converges uniformly to* Jh, *where* h *is the harmonic function whose boundary values coincides with* $f|_{i\mathbb{R}}$.

Proof. Because $f \in C(\mathbb{C}_+ \cup i\mathbb{R})$, hence $f \circ M \in C(\overline{\mathbb{D}})$ and $f \circ M|_{\mathbb{T}} \in C(\mathbb{T})$. Let $h \circ M$ be the harmonic extension of $f \circ M$ into $\mathbb{D}$. Then $h \circ M \in C(\overline{\mathbb{D}})$ and $f \circ M - h \circ M \in V(\mathbb{D})$ and $B^n(f \circ M - h \circ M) \to 0$ uniformly. That is, $B^n((f - h) \circ M) \to 0$ uniformly. But $B(h \circ M) = h \circ M$. Hence $\{B^n(f \circ M)\}$ converges uniformly to $h \circ M$. This implies $(E^n f) \circ M \to J(h \circ M) = (Jh) \circ M$. Thus $\{E^n f\}$ converges uniformly to Jh. □

References

[1] Conway, J. B., *A Course in Functional Analysis*, Graduate Texts in Mathematics **96**, Springer, New York, 1996.

[2] Elliott, S., and Wynn A., *Composition operators on weighted Bergman spaces of a half plane*, Proc. Edinb. Math. Soc. **54**(2011), 373-379.

[3] Hayman, W. K., Kennedy, P. B., *"Subharmonic functions"*, *vol. 1*, Academic Press, 1976.

[4] Hedenmalm, H., Korenblum, B. and Zhu, K. *"Theory of Bergman spaces"*, Springer-Verlag, New York, 2000.

[5] Helgason, S., *"Groups and geometric analysis"*, Academic Press, Orlando, 1984.

[6] Rudin, W., *Principles of Mathematical Analysis*, International Series in Pure and Applied Mathematics, 3rd edition, McGraw-Hill, New York, 1976.

[7] Zhu, K., *Operator Theory in Fuction Spaces*, Monographs and textbooks in pure and applied Mathematics **139**, Marcel Dekker, New York, 1990.