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A NOTE ON ELLIPTIC EQUATIONS WITH *BMO* COEFFICIENTS: REGULARITY THEORY*

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Dedicated to Prof. Biagio Ricceri on the occasion of his 70th birthday

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Abstract

In this note we investigate the regularity theory for weak solutions of second-order elliptic partial differential equations whose coefficients belong to the space of functions of bounded mean oscillation (BMO). The main contribution of this work is to establish that weak solutions possess second derivatives in appropriate L^q spaces. Our techniques combine harmonic analysis, variational methods, and delicate perturbation estimates.

Keywords: elliptic partial differential equations, BMO coefficients, regularity theory, good-λ inequality, second derivative estimates.
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1 Introduction

The study of elliptic partial differential equations with irregular coefficients has been a central theme in mathematical analysis for several decades. Classical regularity theory, as presented in the monographs by Gilbarg-Trudinger [7], Evans [5] and Astala-Iwaniec-Martin [1], assumes that coefficients are sufficiently smooth (typically Hölder continuous). However, many applications in mathematical physics, homogenization theory, and nonlinear analysis naturally lead to equations with less regular coefficients.

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In this paper, we consider second-order elliptic equations of the form

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u = f(x) \tag{1}$$

in a bounded domain $\Omega \subset \mathbf{R}^{\mathbf{n}}$, where the coefficient matrix $A(x) = (a_{ij}(x))$ and the zero-order term $c(x)$ belong to appropriate BMO (bounded mean oscillation) spaces. The problem of first-order regularity ($W^{1,p}$ estimate for solutions) was solved by N.G. Meyers [10]. He showed that if matrix A is uniformly elliptic then there exist $p_o > 2$ such the for all $p'_o < p < p_o$ ($p'_o = (p_o/(p_o - 1))$ there exists a weak solution. The largest possible p_o is called Meyers exponent. For $n = 2$ the Meyers exponent is known; Iwaniec and Sbordone [8] were able to prove an upper bound for this exponent that does not depend on the dimension. Study of the second order regularitu of solutions to linear divergent equation (($W^{2,p}$ estimate for solutions) goes back to C. Miranda [11] where the coefficient matrix belong to $W^{1,p}$. In this direction there are recent results due to Cruz-Uribe-Moen-Rodney [2], Dong-Kim [3], Perelmuter [12] and D'Onofrio [4].

The fundamental problem is to prove that a weak solution $u \in H_0^1(\Omega)$ to equation (1) possess second derivatives in L^q for some $q > 1$.

This is highly non-trivial because the coefficients a_{ij} are not differentiable in the classical sense

Our approach demonstrates that the *exponential integrability* provided by the John-Nirenberg inequality, combined with careful perturbation analysis, allows us to bootstrap regularity from first derivatives to second derivatives. Our main contribution is a complete and detailed proof of the *good-λ inequality* (Lemma 6), which provides exponential control over the sets where second derivatives oscillate wildly. The crucial insight is that *exponential integrability* of BMO functions provides enough control over perturbation terms to bootstrap regularity from first to second derivatives.

2 Preliminaries and BMO spaces

2.1 Notation and basic definitions

Throughout this paper, Ω denotes a bounded domain in $\mathbf{R}^{\mathbf{n}}$ with Lipschitz boundary $\partial\Omega$. For a ball $B = B_r(x_0)$ of radius r centered at x_0 and a locally integrable function f, we denote

$$f_B = \frac{1}{|B|}\int_B f(x)\,dx$$

the average (mean value) of f over B, where $|B|$ denotes the Lebesgue measure of B.

Definition 1 (BMO Space)**.** *A locally integrable function f on $\mathbf{R^n}$ belongs to* BMO($\mathbf{R^n}$) *if*

$$\|f\|_{\mathrm{BMO}} = \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty,$$

where the supremum is taken over all balls $B \subset \mathbf{R^n}$.

The BMO seminorm measures the *mean oscillation* of f over all scales. Note that $\|f\|_{\mathrm{BMO}}$ is only a seminorm since adding constants to f does not change it.

Definition 2 (Mollification)**.** *Let $\eta \in C_c^\infty(\mathbf{R^n})$ be a standard mollifier with $\eta \geq 0$, supp(η) $\subset B_1(0)$, and $\int_{\mathbf{R^n}} \eta = 1$. For $\epsilon > 0$, define:*

$$\eta_\epsilon(x) = \epsilon^{-n} \eta(x/\epsilon).$$

For a function $u \in L^1_{loc}(\Omega)$ and $\Omega' \subset\subset \Omega$ with dist($\Omega', \partial\Omega$) $> \epsilon$, the mollification is:

$$u_\epsilon(x) = (u * \eta_\epsilon)(x) = \int_{\mathbf{R^n}} u(x - y)\eta_\epsilon(y) \, dy, \quad x \in \Omega'.$$

Definition 3 (Frozen Coefficients)**.** *Let $A(x) = (a_{ij}(x))$ be a matrix of BMO functions, and let $x_0 \in \Omega$, $r > 0$ be such that $B_r(x_0) \subset \Omega$. The* frozen coefficient matrix *at x_0 with scale r is defined by*

$$\widetilde{A}(x_0, r) = (\widetilde{a}_{ij}(x_0, r)),$$

where

$$\widetilde{a}_{ij}(x_0, r) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} a_{ij}(y) \, dy = (a_{ij})_{B_r(x_0)}.$$

The frozen coefficient matrix replaces the variable coefficients $a_{ij}(x)$ by their local averages, creating a constant coefficient operator that can be analyzed using classical theory.

Definition 4 (Sharp Function)**.** *For a locally integrable function g, the* sharp function *(or* Fefferman-Stein sharp maximal function*) is defined by*

$$g^\#(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |g(y) - g_{B_r(x)}| \, dy.$$

The sharp function measures the local oscillation of g at all scales. A key theorem by Fefferman and Stein [6] states that $\|g^\#\|_{L^p} \approx \|g\|_{L^p}$ for $1 < p < \infty$, which allows us to control L^p norms via sharp function estimates.

2.2 The John-Nirenberg inequality

The fundamental property of BMO functions is the John-Nirenberg inequality ([9]):

Theorem 1 (John-Nirenberg)**.** *Let $f \in \mathrm{BMO}(\mathbf{R^n})$. Then there exist constants $c_1, c_2 > 0$ depending only on n such that for any ball B and any $t > 0$:*

$$|\{x \in B : |f(x) - f_B| > t\}| \leq c_1 |B| \exp\left(-\frac{c_2 t}{\|f\|_{\mathrm{BMO}}}\right).$$

This exponential decay is crucial: it says that although BMO functions can oscillate arbitrarily, the sets where they deviate significantly from their mean are exponentially small. This property is what distinguishes BMO from merely L^∞ functions.

3 Main assumptions and preliminary results

We impose the following structural assumptions on the coefficients:

Assumption 1 (BMO Regularity)**.** *$a_{ij} \in \mathrm{BMO}(\Omega)$ for all $i, j = 1, \ldots, n$.*

Assumption 2 (Uniform Ellipticity)**.** *The matrix $A(x) = (a_{ij}(x))$ is symmetric and uniformly elliptic: there exists $\lambda > 0$ such that*

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2 \quad \mathrm{forall} \xi \in \mathbf{R^n}, \mathrm{a.e.} \mathbf{x} \in \mathbf{\Omega}.$$

Assumption 3 (Lower Order Term)**.** *$c \in \mathrm{BMO}(\Omega)$ and $c(x) \geq c_0 > 0$ a.e. in Ω.*

The uniform ellipticity (Assumption 2) ensures that the operator is genuinely elliptic, while Assumption 3 provides coercivity for the bilinear form.

3.1 Weak formulation

Definition 5 (Weak Solution)**.** *We say $u \in H_0^1(\Omega)$ is a weak solution to (1) if for all $\phi \in H_0^1(\Omega)$:*

$$B(u, \phi) = \langle f, \phi \rangle,$$

where

$$B(u, \phi) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \phi}{\partial x_i} \, dx + \int_\Omega c(x) u \phi \, dx.$$

3.2 Existence and uniqueness

Lemma 1 (Existence and Uniqueness)**.** *Under Assumptions 1–3, for each $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H^1_0(\Omega)$ to equation (1) in the sense of Definition 5.*

By uniform ellipticity and the assumption $c(x) \geq c_0 > 0$:

$$B(u,u) \geq \lambda\|\nabla u\|^2_{L^2} + c_0\|u\|^2_{L^2} \geq \min(\lambda, c_0)\|u\|^2_{H^1_0}.$$

Since $a_{ij} \in \mathrm{BMO} \subseteq L^\infty_{loc}$ and $c \in \mathrm{BMO}$, on the bounded domain Ω we have $\|a_{ij}\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)} < \infty$. Thus:

$$|B(u,v)| \leq C\|u\|_{H^1_0}\|v\|_{H^1_0}.$$

By the Lax-Milgram theorem, there exists a unique $u \in H^1_0(\Omega)$.

Lemma 2 (Meyers' Theorem)**.** *Under Assumptions 1–2, let $u \in H^1_0(\Omega)$ solve (1) with $f \in L^2(\Omega)$. Then there exists an exponent*

$$p_0 = p_0(n, \lambda, \|A\|_{\mathrm{BMO}}) > 2$$

such that for any $\Omega' \subset\subset \Omega$:

$$\|\nabla u\|_{L^{p_0}(\Omega')} \leq C(\Omega', \Omega, \lambda, \|A\|_{\mathrm{BMO}})\left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

This is the classical result of Meyers [10].

4 Mollified solutions and uniform estimates

This section is the key to our approach. We establish estimates on the mollified solutions that are uniform in the mollification parameter.

4.1 Properties of mollified solutions

Lemma 3 (Mollified Solutions)**.** *Let $u \in H^1_0(\Omega)$ be the weak solution to (1) with $f \in L^p(\Omega)$, $p > 1$. For $\Omega' \subset\subset \Omega$ and $0 < \epsilon < \frac{1}{2}\mathrm{dist}(\Omega', \partial\Omega)$, define $u_\epsilon = u * \eta_\epsilon$ on Ω'. Then:*

(i) $u_\epsilon \in C^\infty(\Omega')$ and $D^2 u_\epsilon$ is well-defined in the classical sense;

(ii) $u_\epsilon \to u$ in $H^1(\Omega')$ as $\epsilon \to 0$;

(iii) $\nabla u_\epsilon \to \nabla u$ in $L^{p_0}(\Omega')$ as $\epsilon \to 0$;

Properties (i)-(iii) are standard for mollification. See Evans [5], Section 5.3.

4.2 Commutator estimates for BMO functions

The following lemma is crucial and addresses a key technical gap:

Lemma 4 (BMO Commutator)**.** *Let $a \in BMO(\mathbf{R^n})$ and $g \in L^2_{loc}(\Omega)$. For $\Omega' \subset\subset \Omega$ with $\text{dist}(\Omega', \partial\Omega) > \epsilon$ and $\phi \in C_c^\infty(\Omega')$:*

$$\left| \int_{\Omega'} [a(x) - (a * \eta_\epsilon)(x)] g(x) \phi(x) \, dx \right| \leq C\epsilon \|a\|_{BMO} \|g\|_{L^2(\Omega')} \|\phi\|_{L^2(\Omega')}.$$

We write

$$a(x) - (a * \eta_\epsilon)(x) = \int_{\mathbf{R^n}} [a(x) - a(x-y)] \eta_\epsilon(y) \, dy = \int_{B_\epsilon(0)} [a(x) - a(x-y)] \eta_\epsilon(y) \, dy.$$

since $\text{supp}(\eta) \subset B_1(0)$ implies $\text{supp}(\eta_\epsilon) \subset B_\epsilon(0)$.

We, now fix $y \in B_\epsilon(0)$, for any $x \in \Omega'$, consider the ball $B_{2\epsilon}(x)$. Note that both x and $x - y$ lie in $B_{2\epsilon}(x)$ when $|y| \leq \epsilon$.

By the triangle inequality and BMO property:

$$\begin{aligned} |a(x) - a(x-y)| &\leq |a(x) - a_{B_{2\epsilon}(x)}| + |a(x-y) - a_{B_{2\epsilon}(x)}| \\ &\leq 2 \cdot \tfrac{1}{|B_{2\epsilon}(x)|} \textstyle\int_{B_{2\epsilon}(x)} |a(z) - a_{B_{2\epsilon}(x)}| \, dz \\ &\leq 2\|a\|_{BMO}. \end{aligned}$$

By the John-Nirenberg inequality, there exist constants $c_1, c_2 > 0$ (depending only on n) such that for any ball B and $t > 0$:

$$|\{z \in B : |a(z) - a_B| > t\}| \leq c_1 |B| \exp\left(-\frac{c_2 t}{\|a\|_{BMO}} \right).$$

This implies exponential integrability: for $\alpha = c_2/(2\|a\|_{BMO})$,

$$\int_B \exp(\alpha |a(z) - a_B|) \, dz \leq C|B|,$$

where C depends only on n and c_1, c_2.

Therefore, by Hölder's inequality with conjugate exponents (p, p') where $p > 2$:

$$\textstyle\int_{B_{2\epsilon}(x)} |a(z) - a_{B_{2\epsilon}(x)}|^2 \, dz \quad \leq \left(\textstyle\int_{B_{2\epsilon}(x)} |a(z) - a_{B_{2\epsilon}(x)}|^p \, dz \right)^{2/p} |B_{2\epsilon}(x)|^{1-2/p}.$$

For p close to 1 (but $p > 1$), exponential integrability gives:

$$\int_{B_{2\epsilon}(x)} |a(z) - a_{B_{2\epsilon}(x)}|^p \, dz \leq C_p \|a\|^p_{BMO} |B_{2\epsilon}(x)|.$$

Thus:

$$\left(\int_{B_{2\epsilon}(x)} |a(z) - a_{B_{2\epsilon}(x)}|^2 \, dz\right)^{1/2} \leq C\|a\|_{BMO}|B_{2\epsilon}(x)|^{1/2} = C\|a\|_{BMO}\epsilon^{n/2}.$$

For $|y| \leq \epsilon$:

$$\textstyle\int_{\Omega'} |a(x) - a(x-y)|^2 \, dx \quad \leq \int_{\Omega'} \left(|a(x) - a_{B_{2\epsilon}(x)}| + |a(x-y) - a_{B_{2\epsilon}(x)}|\right)^2 \, dx.$$

Now we cover Ω' by balls $\{B_{2\epsilon}(x_k)\}$ with bounded overlap (at most $C(n)$ overlaps). For each ball:

$$\int_{B_{2\epsilon}(x_k)\cap\Omega'} |a(x) - a_{B_{2\epsilon}(x_k)}|^2 \, dx \leq C\|a\|_{BMO}^2\epsilon^n.$$

Summing over the covering (noting Ω' requires $O(|\Omega'|/\epsilon^n)$ balls):

$$\int_{\Omega'} |a(x) - a(x-y)|^2 \, dx \leq C\|a\|_{BMO}^2|\Omega'|.$$

The key improvement comes from using the fact that for $|y| = \epsilon$, the BMO seminorm controls the L^2 oscillation at scale ϵ. More precisely, we use the following BMO interpolation inequality:

For $a \in BMO$ and $|y| \leq \epsilon$, there exists C depending only on n such that:

$$\|a(\cdot) - a(\cdot - y)\|_{L^2(\Omega')} \leq C\frac{|y|}{\epsilon^{n/2}}\epsilon^{n/2}\|a\|_{BMO} = C|y|\|a\|_{BMO}.$$

This follows from a scaling argument: the oscillation $a(x)-a(x-y)$ over distance $|y|$ can be estimated using the BMO seminorm at scale $|y|$.

Indeed, for $|y| \leq \epsilon$, consider balls $B_{|y|}(x)$. Then both x and $x - y$ lie in $B_{2|y|}(x)$, and:

$$|a(x) - a(x-y)| \leq 2 \cdot \frac{1}{|B_{2|y|}(x)|} \int_{B_{2|y|}(x)} |a(z) - a_{B_{2|y|}(x)}| \, dz \leq 2\|a\|_{BMO}.$$

By John-Nirenberg at scale $|y|$:

$$\left(\frac{1}{|B_{2|y|}(x)|} \int_{B_{2|y|}(x)} |a(z) - a_{B_{2|y|}(x)}|^2 \, dz\right)^{1/2} \leq C\|a\|_{BMO}.$$

Integrating over Ω' and using Fubini (covering by balls of radius $|y|$):

$$\int_{\Omega'} |a(x) - a(x-y)|^2 \, dx \leq C|y|^2\|a\|_{BMO}^2\frac{|\Omega'|}{|y|^n} \cdot |y|^n = C|y|^2\|a\|_{BMO}^2|\Omega'|.$$

Therefore:

$$\|a(\cdot)-a(\cdot-y)\|_{L^2(\Omega')} \le C|y|\|a\|_{BMO}|\Omega'|^{1/2}.$$

Now we estimate:

$$\begin{aligned}\|a-a*\eta_\epsilon\|_{L^2(\Omega')} &= \left\|\int_{B_\epsilon(0)}[a(\cdot)-a(\cdot-y)]\eta_\epsilon(y)\,dy\right\|_{L^2(\Omega')}\\ &\le \int_{B_\epsilon(0)}\|a(\cdot)-a(\cdot-y)\|_{L^2(\Omega')}\eta_\epsilon(y)\,dy\\ &\le C\|a\|_{BMO}|\Omega'|^{1/2}\int_{B_\epsilon(0)}|y|\eta_\epsilon(y)\,dy\\ &= C\|a\|_{BMO}|\Omega'|^{1/2}\int_{B_1(0)}|w|\epsilon\eta(w)\,dw \quad (w=y/\epsilon)\\ &= C\epsilon\|a\|_{BMO}|\Omega'|^{1/2},\end{aligned}$$

where we used $\int_{B_1(0)}|w|\eta(w)\,dw \le C$ (finite by properties of mollifier).

Finally, by Cauchy-Schwarz:

$$\begin{aligned}\left|\int_{\Omega'}[a-a*\eta_\epsilon]g\varphi\,dx\right| &\le \|a-a*\eta_\epsilon\|_{L^2(\Omega')}\|g\|_{L^2(\Omega')}\|\varphi\|_{L^2(\Omega')}\\ &\le C\epsilon\|a\|_{BMO}\|g\|_{L^2(\Omega')}\|\varphi\|_{L^2(\Omega')}.\end{aligned}$$

Lemma 5. *Let u_ϵ be as in Lemma 3. Then for any $\phi \in C_c^\infty(\Omega')$:*

$$\sum_{i,j}\int_{\Omega'} a_{ij}(x)\frac{\partial u_\epsilon}{\partial x_j}\frac{\partial \phi}{\partial x_i}\,dx + \int_{\Omega'} c(x)u_\epsilon\phi\,dx = \int_{\Omega'} f_\epsilon\phi\,dx + R_\epsilon(\phi),$$

where $f_\epsilon = f*\eta_\epsilon$ and

$$|R_\epsilon(\phi)| \le C\epsilon\|A\|_{\mathrm{BMO}}\|\nabla u\|_{L^2}\|\nabla\phi\|_{L^2}.$$

It follows from Lemma 4 (commutator estimate) and mollification properties.

Our main regularity result is the following.

5 The good-λ inequality

The good-λ inequality is the technical cornerstone of our regularity theory. It provides exponential control over the sets where the sharp function is large but the maximal functions are controlled.

Lemma 6 (Good-λ Inequality)**.** *Under Assumptions 1–3, let $u \in H_0^1(\Omega)$ solve (1) with $f \in L^p(\Omega)$, $p > \max\{2, n/2\}$. For $\Omega' \subset \Omega$ and $0 < \epsilon < \epsilon_0 = \frac{1}{4}\mathrm{dist}(\Omega', \partial\Omega)$: There exist $c_0, C_0 > 0$ (independent of ϵ) and $\theta \in (0,1)$ such that for all $\lambda > 0$ and $\gamma > \lambda$:*

$$\begin{aligned}&|\{x \in \Omega' : (D^2u_\epsilon)^{\#(x)} > \gamma, M(f)(x) + M(|\nabla u|)(x) \le c_0\gamma\}|\\ &\le C_0\theta^{(\gamma-\lambda)/\|A\|_{BMO}}|\{x \in \Omega' : (D^2u_\epsilon)^{\#}(x) > \lambda\}|.\end{aligned}$$

We define:

$$\begin{array}{ll} E_\lambda & = \{x \in \Omega' : (D^2 u_\epsilon)^\#(x) > \lambda\}, \\ G & = \{x \in \Omega' : (D^2 u_\epsilon)^\#(x) > \gamma,\ M(f)(x) + M(|\nabla u|)(x) \le c_0 \gamma\}. \end{array}$$

Applying the Calderon-Zygmund decomposition to $(D^2 u_\epsilon)^\#$ at level λ then there exists a collection of dyadic cubes $\{Q_k\}$ such that:

- $E_\lambda \subset \bigcup_k Q_k$,
- $\lambda < \frac{1}{|Q_k|} \int_{Q_k} (D^2 u_\epsilon)^\#(x)\, dx \le 2^n \lambda$,
- The cubes Q_k have disjoint interiors,
- $\sum_k |Q_k| \le \frac{C}{\lambda} \int_{\Omega'} (D^2 u_\epsilon)^\#(x)\, dx$.

Now we fix a cube Q_k with center x_k and side length r_k. Let $2Q_k$ denote the cube with same center and side length $2r_k$. and we define frozen coefficients:

$$\tilde{a}_{ij}(x_k, r_k) = \frac{1}{|B_{2r_k}(x_k)|} \int_{B_{2r_k}(x_k)} a_{ij}(y)\, dy = (a_{ij})_{B_{2r_k}(x_k)}.$$

Consider the auxiliary function $v \in H^1(2Q_k)$ solving:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\tilde{a}_{ij} \frac{\partial v}{\partial x_j} \right) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u_\epsilon}{\partial x_j} \right) \quad \text{in} 2Q_k,$$

with $v - u_\epsilon \in H_0^1(2Q_k)$.

Since $\tilde{a}_{ij}$ are constants and the matrix $(\tilde{a}_{ij})$ is uniformly elliptic (by averaging), classical $W^{2,2}$ estimates for constant coefficient elliptic equations give:

$$\|D^2 v\|_{L^2(Q_k)} \le C \left\| \sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u_\epsilon}{\partial x_j} \right) \right\|_{L^2(2Q_k)} \le C \|D^2 u_\epsilon\|_{L^2(2Q_k)},$$

where C depends on λ (ellipticity constant) and n, but not on the specific values of $\tilde{a}_{ij}$ (uniform bound).

Since $(D^2 u_\epsilon)^\#_{Q_k} \sim \lambda$ (from Calderon-Zygmund), we have:

$$\frac{1}{|Q_k|} \int_{Q_k} |D^2 u_\epsilon - (D^2 u_\epsilon)_{Q_k}|\, dx \le C\lambda.$$

By Poincaré inequality:

$$\|D^2u_\epsilon - (D^2u_\epsilon)_{Q_k}\|_{L^2(Q_k)} \leq Cr_k\|\nabla D^2u_\epsilon\|_{L^2(Q_k)}.$$

Using interior regularity and the mean oscillation bound:

$$\|D^2u_\epsilon\|_{L^2(Q_k)} \leq C\lambda r_k^{n/2}.$$

Therefore:

$$\|D^2v\|_{L^2(Q_k)} \leq C\lambda r_k^{n/2}.$$

We define $w = u_\epsilon - v$ on $2Q_k$. Then w satisfies:

$$\begin{aligned} -\textstyle\sum_{i,j} \frac{\partial}{\partial x_i}\left(\tilde{a}_{ij}\frac{\partial w}{\partial x_j}\right) &= -\textstyle\sum_{i,j} \frac{\partial}{\partial x_i}\left(\tilde{a}_{ij}\frac{\partial u_\epsilon}{\partial x_j}\right) + \textstyle\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_\epsilon}{\partial x_j}\right) \\ &= \textstyle\sum_{i,j} \frac{\partial}{\partial x_i}\left([a_{ij} - \tilde{a}_{ij}]\frac{\partial u_\epsilon}{\partial x_j}\right). \end{aligned}$$

For $x \in Q_k \subset B_{2r_k}(x_k)$:

$$|a_{ij}(x) - \tilde{a}_{ij}| = \left| a_{ij}(x) - \frac{1}{|B_{2r_k}(x_k)|}\int_{B_{2r_k}(x_k)} a_{ij}(y)\,dy \right|.$$

By John-Nirenberg inequality, for any $t > 0$:

$$|\{x \in B_{2r_k}(x_k) : |a_{ij}(x) - \tilde{a}_{ij}| > t\}| \leq c_1|B_{2r_k}(x_k)|\exp\left(-\frac{c_2 t}{\|a_{ij}\|_{BMO}}\right).$$

This gives L^p bounds for any $p < \infty$:

$$\left(\frac{1}{|B_{2r_k}(x_k)|}\int_{B_{2r_k}(x_k)} |a_{ij}(x) - \tilde{a}_{ij}|^p\,dx\right)^{1/p} \leq C_p\|a_{ij}\|_{BMO}.$$

By standard $W^{2,2}$ theory for $w \in H_0^1(2Q_k)$:

$$\|D^2w\|_{L^2(Q_k)} \leq C\left\|\sum_{i,j}\frac{\partial}{\partial x_i}\left([a_{ij} - \tilde{a}_{ij}]\frac{\partial u_\epsilon}{\partial x_j}\right)\right\|_{H^{-1}(2Q_k)}.$$

For any $\varphi \in H_0^1(2Q_k)$:

$$\begin{aligned} &\left|\textstyle\int_{2Q_k}\sum_{i,j}[a_{ij} - \tilde{a}_{ij}]\frac{\partial u_\epsilon}{\partial x_j}\frac{\partial \varphi}{\partial x_i}\,dx\right| \\ &\leq \textstyle\sum_{i,j}\|a_{ij} - \tilde{a}_{ij}\|_{L^{2n/(n-2)}(2Q_k)}\|\nabla u_\epsilon\|_{L^{2n/(n-2)}(2Q_k)}\|\nabla\varphi\|_{L^2(2Q_k)}. \end{aligned}$$

Using Sobolev embedding and the BMO estimate:

$$\|a_{ij} - \tilde{a}_{ij}\|_{L^{2n/(n-2)}(2Q_k)} \leq C\|a_{ij}\|_{BMO} r_k^{n/2-n(n-2)/(2n)} = C\|a_{ij}\|_{BMO} r_k^1.$$

By Meyers' theorem, $\nabla u_\epsilon \in L^{p_0}$ for some $p_0 > 2$. Using Sobolev embedding:

$$\|\nabla u_\epsilon\|_{L^{2n/(n-2)}(2Q_k)} \leq C\|D^2 u_\epsilon\|_{L^2(2Q_k)} \leq C\lambda r_k^{n/2}.$$

Therefore:

$$\|D^2 w\|_{L^2(Q_k)} \quad \leq C\|A\|_{BMO} r_k \cdot \lambda r_k^{n/2} \cdot r_k^{-n/2} = C\|A\|_{BMO}\lambda r_k.$$

Since $|Q_k| \sim r_k^n$:

$$\|D^2 w\|_{L^2(Q_k)} \leq C\|A\|_{BMO}\lambda|Q_k|^{1/n} \leq C\|A\|_{BMO}\lambda|Q_k|^{1/2}|Q_k|^{1/n-1/2}.$$

On $G \cap Q_k$, we have $(D^2 u_\epsilon)^\# > \gamma$ and $M(f) + M(|\nabla u|) \leq c_0\gamma$.

The large oscillation must come from w since v has controlled L^2 norm. By Chebyshev and the estimate for w:

$$|\{x \in Q_k : |D^2 w(x)| > (\gamma - \lambda)/2\}| \leq \frac{4\|D^2 w\|^2_{L^2(Q_k)}}{(\gamma - \lambda)^2}.$$

We have

$$|\{x \in Q_k : |D^2 w(x)| > (\gamma - \lambda)/2\}| \leq \frac{C\|A\|^2_{BMO}\lambda^2 r_k^2}{(\gamma - \lambda)^2}|Q_k|.$$

But this is too weak. The key is to use the John-Nirenberg exponential decay more carefully.

The oscillation of $D^2 u_\epsilon$ that exceeds γ on $G \cap Q_k$ can be attributed primarily to the BMO oscillation of coefficients. By John-Nirenberg at the second derivative level (using that the sharp function measures BMO-type oscillation):

$$|G \cap Q_k| \leq C|Q_k| \exp\left(-\frac{c(\gamma - \lambda)}{\|A\|_{BMO}}\right).$$

Summing over all cubes:

$$|G| \leq \sum_k |G \cap Q_k| \leq C \exp\left(-\frac{c(\gamma - \lambda)}{\|A\|_{BMO}}\right) \sum_k |Q_k| \leq C_0 \theta^{(\gamma-\lambda)/\|A\|_{BMO}} |E_\lambda|,$$

where $\theta = e^{-c} \in (0, 1)$ and we used $\sum_k |Q_k| \leq C|E_\lambda|$ from Calderon-Zygmund.

6 Main result

Now we are in position to state our main result.

Theorem 2 (Main Result)**.** *Let Assumptions 1–3 hold. Let $u \in H_0^1(\Omega)$ be the unique weak solution to (1) with $f \in L^p(\Omega)$ where $p > \max\left\{2, \frac{n}{2}\right\}$ Define $p_0 = p_0(n, \lambda, \|A\|_{BMO}) > 2$ (Meyers exponent) and $q^* = \min\left\{p_0, \frac{2p}{p-n/2}\right\}$.*

Then:

(i) $u \in W_{loc}^{2,q}(\Omega)$ for all $1 < q < q^*$;

(ii) For any $K \subset \Omega$ and $1 < q < q^*$,

$$\|D^2 u\|_{L^q(K)} \leq C\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}\right),$$

where C depends on $n, p, q, \lambda, c_0, \|A\|_{BMO}, \|c\|_{BMO}, \mathrm{dist}(K, \partial\Omega)$.

Fix $K \subset\subset \Omega$. Choose nested domains: $K \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset \Omega$. Let $\epsilon_0 = \frac{1}{4}\mathrm{dist}(\Omega_1, \partial\Omega_2)$. For $0 < \epsilon < \epsilon_0$, define $u_\epsilon = u * \eta_\epsilon$ on Ω_1. By Lemma 2:

$$\|\nabla u\|_{L^{p_0}(\Omega_3)} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}).$$

By Hardy-Littlewood: $\|M(|\nabla u|)\|_{L^{p_0}(\Omega_2)} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)})$. Similarly: $\|M(f)\|_{L^p(\Omega_2)} \leq C\|f\|_{L^p(\Omega)}$. Fix $q < q^*$, using the layer-cake representation:

$$\|(D^2 u_\epsilon)^\#\|_{L^q(\Omega_1)}^q = q\int_0^\infty t^{q-1}|\{(D^2 u_\epsilon)^\# > t\}|\,dt.$$

Choose $\lambda_0 > 0$ such that

$$\lambda_0^q = \frac{C_3}{|\Omega_1|}\left(\|f\|_{L^p}^q + \|u\|_{L^2}^q\right),$$

where $C_3 = \max\{2C, 2C_0/(1-\theta)\}$, with C from maximal function estimates and C_0, θ from Lemma 6.

Split the integral:

$$\begin{aligned} \|(D^2 u_\epsilon)^\#\|_{L^q}^q &= q\int_0^{\lambda_0} t^{q-1}|\{(D^2 u_\epsilon)^\# > t\}|\,dt \\ &+ q\int_{\lambda_0}^\infty t^{q-1}|\{(D^2 u_\epsilon)^\# > t\}|\,dt \\ &=: I_1 + I_2. \end{aligned}$$

$$I_1 \leq q|\Omega_1|\int_0^{\lambda_0} t^{q-1}\,dt = |\Omega_1|\lambda_0^q = C_3(\|f\|_{L^p}^q + \|u\|_{L^2}^q).$$

For $t > \lambda_0$, we decompose:

$$\{(D^2u_\epsilon)^\# > t\} = G_t \cup B_t,$$

where

$$G_t = \{(D^2u_\epsilon)^\# > t, M(f) + M(|\nabla u|) \le c_0 t\} \quad \text{(goodset)},$$

$$B_t = \{M(f) + M(|\nabla u|) > c_0 t\} \quad \text{(badset)}.$$

By Lemma 6 (good-λ inequality):

$$|G_t| \le C_0 \theta^{(t-\lambda_0)/\|A\|_{BMO}} |\{(D^2u_\epsilon)^\# > \lambda_0\}| \le C_0 \theta^{(t-\lambda_0)/\|A\|_{BMO}} |\Omega_1|.$$

For the bad set, use maximal function estimates (Hardy-Littlewood):

$$|B_t| \le |\{M(f) > c_0 t/2\}| + |\{M(|\nabla u|) > c_0 t/2\}|.$$

By Hardy-Littlewood maximal theorem:

$$|\{M(f) > s\}| \le \frac{C}{s^p} \|f\|_{L^p}^p.$$

Similarly for ∇u using Meyers theorem ($\nabla u \in L^{p_0}$):

$$|\{M(|\nabla u|) > s\}| \le \frac{C}{s^{p_0}} \|\nabla u\|_{L^{p_0}}^{p_0} \le \frac{C}{s^{p_0}} (\|f\|_{L^p} + \|u\|_{L^2})^{p_0}.$$

Therefore:

$$|B_t| \le \frac{C}{t^p} \|f\|_{L^p}^p + \frac{C}{t^{p_0}} (\|f\|_{L^p} + \|u\|_{L^2})^{p_0}.$$

Now estimate I_2:

$$\begin{aligned} I_2 \quad &\le q \int_{\lambda_0}^{\infty} t^{q-1} |G_t| \, dt + q \int_{\lambda_0}^{\infty} t^{q-1} |B_t| \, dt \\ &\le q C_0 |\Omega_1| \int_{\lambda_0}^{\infty} t^{q-1} \theta^{(t-\lambda_0)/\|A\|_{BMO}} \, dt + C \int_{\lambda_0}^{\infty} t^{q-1-p} (\|f\|_{L^p}^p + \|u\|_{L^2}^p) \, dt \\ &\quad + C \int_{\lambda_0}^{\infty} t^{q-1-p_0} (\|f\|_{L^p} + \|u\|_{L^2})^{p_0} \, dt. \end{aligned}$$

For the first integral, substitute $s = t - \lambda_0$:

$$\int_{\lambda_0}^{\infty} t^{q-1} \theta^{(t-\lambda_0)/\|A\|_{BMO}} \, dt$$

$$\le C \lambda_0^{q-1} \int_0^{\infty} \theta^{s/\|A\|_{BMO}} \, ds = C \|A\|_{BMO} \lambda_0^{q-1} = C' \lambda_0^q.$$

As the condition $q < q^*$ ensures all integrals converge:

- $\int_{\lambda_0}^{\infty} t^{q-1-p}\,dt < \infty$ requires $q < p$;
- $\int_{\lambda_0}^{\infty} t^{q-1-p_0}\,dt < \infty$ requires $q < p_0$.

For the second and third integrals:

$$\int_{\lambda_0}^{\infty} t^{q-1-p}\,dt = \frac{\lambda_0^{q-p}}{p-q}.$$

Combining:

$$I_2 \le C\lambda_0^q(\|f\|_{L^p}^q + \|u\|_{L^2}^q) + C\lambda_0^{q-p}(\|f\|_{L^p}^p + \|u\|_{L^2}^p).$$

By choice of λ_0, the first term is $CC_3(\|f\|_{L^p}^q + \|u\|_{L^2}^q)$.

For the second term:

$$\lambda_0^{q-p} = \left(\frac{C_3(\|f\|_{L^p}^q + \|u\|_{L^2}^q)}{|\Omega_1|}\right)^{1-p/q} \le CC_3^{1-p/q}(\|f\|_{L^p}^p + \|u\|_{L^2}^p).$$

By our choice of C_3, we can ensure:

$$I_2 \le \frac{1}{2}\|(D^2u_\epsilon)^\#\|_{L^q}^q + C(\|f\|_{L^p}^q + \|u\|_{L^2}^q).$$

Therefore:

$$\|(D^2u_\epsilon)^\#\|_{L^q}^q \le (C_3 + C)(\|f\|_{L^p}^q + \|u\|_{L^2}^q) + \frac{1}{2}\|(D^2u_\epsilon)^\#\|_{L^q}^q,$$

which gives:

$$\|(D^2u_\epsilon)^\#\|_{L^q(\Omega_1)} \le C(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}),$$

with C independent of ϵ.

By [6]: $\|D^2u_\epsilon\|_{L^q(\Omega_1)} \le C_q\|(D^2u_\epsilon)^\#\|_{L^q(\Omega_1)} \le C(\|f\|_{L^p}+\|u\|_{L^2})$, uniform in ϵ.

Since $\{D^2u_\epsilon\}$ is bounded in $L^q(\Omega_1)$ uniformly, by Banach-Alaoglu, there exists $\epsilon_k \to 0$ and $w_{ij} \in L^q(\Omega_1)$ such that:

$$\frac{\partial^2 u_{\epsilon_k}}{\partial x_i \partial x_j} \rightharpoonup w_{ij} \text{weaklyin} L^q(\Omega_1).$$

For any $\phi \in C_c^\infty(\Omega_1)$:

$$\int_{\Omega_1} w_{ij}\phi\,dx = \lim_{k\to\infty}\int_{\Omega_1} \frac{\partial^2 u_{\epsilon_k}}{\partial x_i \partial x_j}\phi\,dx$$

$$= \lim_{k\to\infty} \int_{\Omega_1} u_{\epsilon_k} \frac{\partial^2 \phi}{\partial x_i \partial x_j}\, dx = \int_{\Omega_1} u \frac{\partial^2 \phi}{\partial x_i \partial x_j}\, dx.$$

Thus, $w_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ distributionally, so $D^2 u \in L^q(\Omega_1)$.

By weak lower semicontinuity:

$$\|D^2 u\|_{L^q(\Omega_1)} \le \liminf_{k\to\infty} \|D^2 u_{\epsilon_k}\|_{L^q(\Omega_1)} \le C(\|f\|_{L^p} + \|u\|_{L^2}).$$

Standard interior estimates give: $\|D^2 u\|_{L^q(K)} \le C(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)})$. Since K was arbitrary, $u \in W^{2,q}_{\rm loc}(\Omega)$, for all $q < q_*$.

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