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EXISTENCE AND LOCATION OF SOLUTIONS FOR A GENERAL ELLIPTIC SYSTEM WITH INTRINSIC OPERATORS*

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Dedicated to Prof. Biagio Ricceri on the occasion of his 70th birthday

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Abstract

The existence and location of solutions are established for an elliptic system with full gradient dependence and intrinsic operators. The abstract results are applied to a system with convolution products.

Keywords: elliptic system, intrinsic operator, gradient dependence, convolution product.

MSC: 35J57, 35J92, 47H05.

1 Introduction

The object of this article is to study the following quasilinear elliptic system

$$\begin{cases} -\Delta_{p_1} u_1 = f_1\left(x, B_1u_1, B_2u_2, \nabla\left(B_1u_1\right), \nabla\left(B_2u_2\right)\right) & \text{in } \Omega \\ -\Delta_{p_2} u_2 = f_2\left(x, B_1u_1, B_2u_2, \nabla\left(B_1u_1\right), \nabla\left(B_2u_2\right)\right) & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

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on a nonempty bounded open set Ω in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. The equations in (P) are driven by the negative p_i−Laplacians $-\Delta_{p_i} : W_0^{1,p_i}(\Omega) \to W^{-1,p_i'}(\Omega)$ for $i = 1, 2$, where $p_i \in (1, +\infty)$ and $p_i' = p_i/(p_i - 1)$. These operators are given by

$$\langle -\Delta_{p_i} u, v\rangle = \int_\Omega |\nabla u(x)|^{p_i-2} \nabla u(x) \cdot \nabla v(x) dx \quad \text{for all } u, v \in W_0^{1,p_i}(\Omega).$$

System (P) also contains the continuous operators called intrinsic $B_i : W_0^{1,p_i}(\Omega) \to W^{1,p_i}(\Omega)$ for $i = 1, 2$ as well as the Carathéodory functions $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ meaning that $f_i(\cdot, s_1, s_2, \xi_1, \xi_2)$ is measurable for all $(s_1, s_2, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ and $f_i(x, \cdot, \cdot, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$, with $i = 1, 2$. Such nonlinearities depending on the solution and its gradient are often called convection terms. Here the problem is more involved due to the composition with the intrinsic operators.

The counterpart for a single equation is presented in [9]. Here, in the system setting, the arguments are more complex because the variables interact and thus cannot be separated. The first paper that studied problems with intrinsic operators is [10]. For other recent developments in this direction we address at [6, 8, 13, 14]. Regarding problems exhibiting full dependence on the gradient of the solution we refer to [2, 4, 5, 12]. Our application concerns equations with convolutions. For this topic we recommend [7, 11, 15].

A weak solution to system (P) is any pair $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ such that

$$f_i\left(x, B_1u_1, B_2u_2, \nabla\left(B_1u_1\right), \nabla\left(B_2u_2\right)\right) v_i \in L^1(\Omega), \quad i = 1, 2,$$

and

$$\int_\Omega |\nabla u_1|^{p_1-2} \nabla u_1 \cdot \nabla v_1 dx = \int_\Omega f_1\left(x, B_1u_1, B_2u_2, \nabla\left(B_1u_1\right), \nabla\left(B_2u_2\right)\right) v_1 dx,$$

$$\int_\Omega |\nabla u_2|^{p_2-2} \nabla u_2 \cdot \nabla v_2 dx = \int_\Omega f_2\left(x, B_1u_1, B_2u_2, \nabla\left(B_1u_1\right), \nabla\left(B_2u_2\right)\right) v_2 dx$$

for all $(v_1, v_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$.

Due to the Dirichlet boundary condition the underlying space for problem (P) is the product space $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ endowed with the norm

$$\|(u_1, u_2)\| = \|u_1\|_{W_0^{1,p_1}} + \|u_2\|_{W_0^{1,p_2}},$$

where $\|\cdot\|_{W_0^{1,p_i}} = \left(\int_\Omega |\nabla u|^{p_i} dx\right)^{\frac{1}{p_i}}$ for $i = 1, 2$. The corresponding norm on the dual space $W^{-1,p_1'}(\Omega) \times W^{-1,p_2'}(\Omega)$ is denoted $\|(\cdot,\cdot)\|_*$. Recall that

$W_0^{1,p_i}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,p_i}(\Omega)$ endowed with the norm $\|\cdot\|_{W^{1,p_i}} = (\int_\Omega |\nabla u|^{p_i} dx)^{\frac{1}{p_i}} + (\int_\Omega |u|^{p_i} dx)^{\frac{1}{p_i}}$. In the sequel we suppose that $N > p_i$ for $i = 1, 2$. The other cases are simpler and can be handled in the same way. In this situation, the Sobolev critical exponents are $p_i^* = \frac{Np_i}{N-p_i}$ for $i = 1, 2$. In order to simplify the presentation, for any real number $r \in (1, +\infty)$, we denote $r' = \dfrac{r}{r-1}$ (the Hölder conjugate of r). For $r \in [1, +\infty)$, we will denote by $\|\cdot\|_r$ the usual norm in $L^r(\Omega)$.

We formulate the assumptions on the Carathéodory functions $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and the operators $B_i : W_0^{1,p_i}(\Omega) \to W^{1,p_i}(\Omega)$ with $i = 1, 2$.

(H1) There exist functions $\sigma_i \in L^{r_i'}(\Omega)$ with $r_i \in (1, p_i^*)$, constants $c_i \geq 0$, $d_i \geq 0$ for $i = 1, 2$, and constants $\alpha_1 \in \left[0, \min\left\{p_1^* - 1, \frac{p_1^*}{(p_2^*)'}\right\}\right)$, $\alpha_2 \in \left[0, \min\left\{p_2^* - 1, \frac{p_2^*}{(p_1^*)'}\right\}\right)$, $\beta_1 \in \left[0, \min\left\{\frac{p_1}{(p_1^*)'}, \frac{p_1}{(p_2^*)'}\right\}\right)$ and $\beta_2 \in \left[0, \min\left\{\frac{p_2}{(p_2^*)'}, \frac{p_2}{(p_1^*)'}\right\}\right)$ such that

$$|f_i(x, s_1, s_2, \xi_1, \xi_2)| \leq \sigma_i(x) + c_1|s_1|^{\alpha_1} + c_2|s_2|^{\alpha_2} + d_1|\xi_1|^{\beta_1} + d_2|\xi_2|^{\beta_2}$$

for a.e. $x \in \Omega$ and all $(s_1, s_2, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, $i = 1, 2$.

(H2) The maps $B_i : W_0^{1,p_i}(\Omega) \to W^{1,p_i}(\Omega)$ are continuous and there exist nonnegative constants K_1 and K_2 such that

$$\|B_i u\|_{p_i} + \|\,|\nabla(B_i u)|\,\|_{p_i} \leq K_1 \|u\|_{W_0^{1,p_i}} + K_2$$

for all $u \in W_0^{1,p_i}(\Omega)$ and $i = 1, 2$.

(H3) There exist functions $\zeta_i \in L^1(\Omega)$ and constants $a_i \geq 0$, $b_i \geq 0$ for $i = 1, 2$ with

$$1 > 2^{p_i}(a_i + b_i)K_1^{p_i} \tag{1}$$

such that

$$f_i(x, s_1, s_2, \xi_1, \xi_2)s_i \leq \zeta_i(x) + a_1|s_1|^{p_1} + a_2|s_2|^{p_2} + b_1|\xi_1|^{p_1} + b_2|\xi_2|^{p_2}$$

for a.e. $x \in \Omega$ and all $(s_1, s_2, \xi_1, \xi_2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$.

Our existence result of weak solutions to problem (P) reads as follows.

Theorem 1. *Assume that conditions (H1)-(H3) are fulfilled. Then problem* (P) *admits at least one weak solution.*

The proof of Theorem 1 is given in Section 4.

Next we consider a refinement of problem (P) targeting the location of solutions: given the constants $R_1 > 0$ and $R_2 > 0$, find a weak solution $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ of system (P) satisfying

$$\|u_i\|_{W_0^{1,p_i}} \leq R_i, \quad i = 1, 2. \tag{2}$$

The interest for this problem is that it offers an a priori bound for the solution.

Our result in this direction is as follows.

Theorem 2. *Fix* $R_1 > 0$ *and* $R_2 > 0$ *and assume that conditions (H1)-(H3) are verified together with*

$$\left(\frac{\|\zeta_1\|_1 + \|\zeta_2\|_1 + 2^{p_1}(a_1 + b_1)K_2^{p_1} + 2^{p_2}(a_2 + b_2)K_2^{p_2}}{1 - 2^{p_1}(a_1 + b_1)K_1^{p_1}} \right)^{\frac{1}{p_1}} \leq R_1$$

and

$$\left(\frac{\|\zeta_1\|_1 + \|\zeta_2\|_1 + 2^{p_1}(a_1 + b_1)K_2^{p_1} + 2^{p_2}(a_2 + b_2)K_2^{p_2}}{1 - 2^{p_2}(a_1 + b_1)K_1^{p_2}} \right)^{\frac{1}{p_2}} \leq R_2.$$

Then problem (P) *admits at least one weak solution* $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ *for which the bounds* (2) *hold.*

The proof of Theorem 2 is done in Section 4.

2 Preliminaries

The Sobolev embedding theorem ensures that the space $W_0^{1,p_i}(\Omega)$ is continuously embedded in $L^\tau(\Omega)$ whenever $\tau \in [1, p_i^*]$ for $i = 1, 2$. Therefore there exists a constant $\theta_{i,\tau} > 0$ such that

$$\|u\|_\tau \leq \theta_{i,\tau} \|u\|_{W_0^{1,p_i}}, \quad \forall u \in W_0^{1,p_i}(\Omega). \tag{3}$$

By the Rellich-Kondrachov theorem, the embedding of $W_0^{1,p_i}(\Omega)$ into $L^\tau(\Omega)$ is compact for $\tau \in [1,\, p_i^*)$.

We quote from [3] the following basic result.

Proposition 1. *The negative p_i-Laplacian $-\Delta_{p_i} : W_0^{1,p_i}(\Omega) \to W^{-1,p_i'}(\Omega)$ for $i = 1, 2$ is continuous, bounded (in the sense that it maps bounded sets into bounded sets), maximal monotone, strictly monotone (so, pseudomonotone) and satisfies the $(S_+)-$property, that is, any sequence $\{u_n\} \subset W_0^{1,p_i}(\Omega)$ for which $u_n \rightharpoonup u$ in $W_0^{1,p_i}(\Omega)$ and $\limsup_{n\to+\infty}\langle -\Delta_{p_i} u_n, u_n - u\rangle \le 0$ fulfills $u_n \to u$ in $W_0^{1,p_i}(\Omega)$.*

From Proposition 1 we can establish the $(S_+)-$property of the operator $(-\Delta_{p_1}, -\Delta_{p_2})$ on the product space $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$. For the proof we refer to [4, Lemma 4].

Proposition 2. *If $(u_{1n}, u_{2n}) \rightharpoonup (u_1, u_2)$ in $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ and*

$$\limsup_{n\to+\infty}\langle(-\Delta_{p_1} u_{1n}, -\Delta_{p_2} u_{2n}), (u_{1n} - u_1, u_{2n} - u_2)\rangle \le 0,$$

then $(u_{1n}, u_{2n}) \to (u_1, u_2)$ in $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$.

In our approach we need the following basic theorem.

Theorem 3 (see [3, Theorem 2.99])**.** *Let X be a real reflexive Banach space and let $A : X \to X^*$ be a bounded, coercive and pseudomonotone operator. Then for every $b \in X^*$ the equation $Ax = b$ has at least one solution $x \in X$.*

Our application makes use of the convolution product, a notion that we now recall in our context. The convolution of $\rho \in L^1(\mathbb{R}^N)$ and $u \in W_0^{1,p}(\Omega)$, $1 < p < +\infty$, is defined as

$$(\rho * u)(x) = \int_{\mathbb{R}^N} \rho(y)u(x - y)dy \quad \text{for } x \in \Omega,$$

where $u \in W_0^{1,p}(\Omega) \subset W^{1,p}(\mathbb{R}^N)$ is extended by 0 on $\mathbb{R}^N \setminus \Omega$.

Young's theorem for convolution provides the estimate

$$\|\rho * u\|_{L^p(\mathbb{R}^N)} \le \|\rho\|_{L^1(\mathbb{R}^N)}\|u\|_{L^p(\mathbb{R}^N)} = \|\rho\|_{L^1(\mathbb{R}^N)}\|u\|_p \tag{4}$$

(see [1, Theorem 4.15]). The weak partial derivatives of $\rho * u$ are given by

$$\frac{\partial}{\partial x_i}(\rho * u) = \rho * \frac{\partial u}{\partial x_i}, \quad i = 1, \dots, N, \tag{5}$$

so the gradient of $\rho * u$ is equal to

$$\nabla(\rho * u) = \left(\rho * \frac{\partial u}{\partial x_1}, \dots, \rho * \frac{\partial u}{\partial x_N}\right).$$

From (4) and (5) we obtain

$$\| |\nabla(\rho * u)| \|_p^p \leq \int_{\mathbb{R}^N} |\nabla(\rho * u)|^p dx = \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} \left(\rho * \frac{\partial u}{\partial x_i} \right)^2 \right)^{\frac{p}{2}} dx$$
$$\leq \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} \left| \rho * \frac{\partial u}{\partial x_i} \right| \right)^p dx \leq N^p \|\rho\|^p_{L^1(\mathbb{R}^N)} \| |\nabla u| \|^p_{L^p(\mathbb{R}^N)}$$
$$= N^p \|\rho\|^p_{L^1(\mathbb{R}^N)} \| |\nabla u| \|^p_p.$$

We are thus led to

$$\begin{aligned} \|\rho * u\|_{W^{1,p}} &= \|\rho * u\|_p + \| |\nabla(\rho * u)| \|_p \\ &\leq \|\rho\|_{L^1(\mathbb{R}^N)} \left(\lambda_{1,p}^{-\frac{1}{p}} + N \right) . \| |\nabla u| \|_p \end{aligned} \tag{6}$$

In (6) the notation $\lambda_{1,p}$ stands for the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$, that is,

$$\lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\| |\nabla u| \|_p^p}{\|u\|_p^p}.$$

3 Properties of the associated Nemytskii operators

The right-hand sides of the equations in system (P) are expressed through the operator $A : W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega) \to W^{-1,p_1'}(\Omega) \times W^{-1,p_2'}(\Omega)$ described by

$$\langle A(u_1, u_2), (v_1, v_2) \rangle = A^{(1)}(u_1, u_2)(v_1) + A^{(2)}(u_1, u_2)(v_2), \tag{7}$$

where

$$A^{(i)}(u_1, u_2)(v_i) = \int_{\Omega} f_i \left(x, B_1 u_1, B_2 u_2, \nabla(B_1 u_1), \nabla(B_2 u_2) \right) v_i dx$$

for all $(u_1, u_2), (v_1, v_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$, with $i = 1, 2$. The next proposition shows in particular that the operator A is well defined.

Proposition 3. *Assume that condition (H1) holds. The following bounds hold*

$$\begin{aligned} & \|A^{(1)}(u_1, u_2)\|_{W^{-1,p_1'}} \\ \leq \quad & \|\sigma_1\|_{r_1'} \theta_{1,r_1} + c_1 \theta_{1,\frac{p_1^*}{p_1^* - \alpha_1}} \|B_1 u_1\|_{p_1^*}^{\alpha_1} + c_2 \theta_{1,\frac{p_2^*}{p_2^* - \alpha_2}} \|B_2 u_2\|_{p_2^*}^{\alpha_2} \\ & + d_1 \theta_{1,\frac{p_1}{p_1 - \beta_1}} \| |\nabla(B_1 u_1)| \|_{p_1}^{\beta_1} + d_2 \theta_{1,\frac{p_2}{p_2 - \beta_2}} \| |\nabla(B_2 u_2)| \|_{p_2}^{\beta_2} \end{aligned} \tag{8}$$

and

$$\begin{aligned} & \|A^{(2)}(u_1,u_2)\|_{W^{-1,p_2'}} \\ \leq\ & \|\sigma_2\|_{r_2'}\theta_{2,r_2} + c_1\theta_{2,\frac{p_1^*}{p_1^*-\alpha_1}}\|B_1u_1\|_{p_1^*}^{\alpha_1} + c_2\theta_{2,\frac{p_2^*}{p_2^*-\alpha_2}}\|B_2u_2\|_{p_2^*}^{\alpha_2} \\ & + d_1\theta_{2,\frac{p_1}{p_1-\beta_1}}\|\,|\nabla(B_1u_1)|\,\|_{p_1}^{\beta_1} + d_2\theta_{2,\frac{p_2}{p_2-\beta_2}}\|\,|\nabla(B_2u_2)|\,\|_{p_2}^{\beta_2} \end{aligned} \tag{9}$$

for all $(u_1,u_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$.

Proof. By hypothesis (H1) and using (3) in conjunction with Hölder's inequality we find that

$$\begin{aligned} & \left|\int_\Omega f_1\left(x,B_1u_1,B_2u_2,\nabla\left(B_1u_1\right),\nabla\left(B_2u_2\right)\right)v_1dx\right| \\ \leq\ & \int_\Omega \Big(|\sigma_1| + c_1|B_1u_1|^{\alpha_1} + c_2|B_2u_2|^{\alpha_2} \\ & + d_1|\nabla(B_1u_1)|^{\beta_1} + d_2|\nabla(B_2u_2)|^{\beta_2}\Big)|v_1|dx \\ \leq\ & \|\sigma_1\|_{r_1'}\|v_1\|_{r_1} + c_1\|B_1u_1\|_{p_1^*}^{\alpha_1}\|v_1\|_{\frac{p_1^*}{p_1^*-\alpha_1}} + c_2\|B_2u_2\|_{p_2^*}^{\alpha_2}\|v_1\|_{\frac{p_2^*}{p_2^*-\alpha_2}} \\ & + d_1\|\,|\nabla(B_1u_1)|\,\|_{p_1}^{\beta_1}\|v_1\|_{\frac{p_1}{p_1-\beta_1}} + d_2\|\,|\nabla(B_2u_2)|\,\|_{p_2}^{\beta_2}\|v_1\|_{\frac{p_2}{p_2-\beta_2}} \\ \leq\ & \left(\|\sigma_1\|_{r_1'}\theta_{1,r_1} + c_1\theta_{1,\frac{p_1^*}{p_1^*-\alpha_1}}\|B_1u_1\|_{p_1^*}^{\alpha_1} + c_2\theta_{1,\frac{p_2^*}{p_2^*-\alpha_2}}\|B_2u_2\|_{p_2^*}^{\alpha_2}\right)\|v_1\|_{W_0^{1,p_1}} \\ & + \left(d_1\theta_{1,\frac{p_1}{p_1-\beta_1}}\|\,|\nabla(B_1u_1)|\,\|_{p_1}^{\beta_1} + d_2\theta_{1,\frac{p_2}{p_2-\beta_2}}\|\,|\nabla(B_2u_2)|\,\|_{p_2}^{\beta_2}\right)\|v_1\|_{W_0^{1,p_1}} \end{aligned}$$

for all $u_1,v_1\in W_0^{1,p_1}(\Omega)$ and $u_2\in W_0^{1,p_2}(\Omega)$. Likewise we have

$$\begin{aligned} & \left|\int_\Omega f_2\left(x,B_1u_1,B_2u_2,\nabla\left(B_1u_1\right),\nabla\left(B_2u_2\right)\right)v_2dx\right| \\ \leq\ & \int_\Omega \Big(|\sigma_2| + c_1|B_1u_1|^{\alpha_1} + c_2|B_2u_2|^{\alpha_2} \\ & + d_1|\nabla(B_1u_1)|^{\beta_1} + d_2|\nabla(B_2u_2)|^{\beta_2}\Big)|v_2|dx \\ \leq\ & \|\sigma_1\|_{r_2'}\|v_2\|_{r_2} + c_1\|B_1u_1\|_{p_1^*}^{\alpha_1}\|v_2\|_{\frac{p_1^*}{p_1^*-\alpha_1}} + c_2\|B_2u_2\|_{p_2^*}^{\alpha_2}\|v_2\|_{\frac{p_2^*}{p_2^*-\alpha_2}} \\ & + d_1\|\,|\nabla(B_1u_1)|\,\|_{p_1}^{\beta_1}\|v_1\|_{\frac{p_1}{p_1-\beta_1}} + d_2\|\,|\nabla(B_2u_2)|\,\|_{p_2}^{\beta_2}\|v_1\|_{\frac{p_2}{p_2-\beta_2}} \\ \leq\ & \left(\|\sigma_2\|_{r_2'}\theta_{2,r_2} + c_1\theta_{2,\frac{p_1^*}{p_1^*-\alpha_1}}\|B_1u_1\|_{p_1^*}^{\alpha_1} + c_2\theta_{2,\frac{p_2^*}{p_2^*-\alpha_2}}\|B_2u_2\|_{p_2^*}^{\alpha_2}\right)\|v_2\|_{W_0^{1,p_2}} \\ & + \left(d_1\theta_{2,\frac{p_1}{p_1-\beta_1}}\|\,|\nabla(B_1u_1)|\,\|_{p_1}^{\beta_1} + d_2\theta_{2,\frac{p_2}{p_2-\beta_2}}\|\,|\nabla(B_2u_2)|\,\|_{p_2}^{\beta_2}\right)\|v_2\|_{W_0^{1,p_2}} \end{aligned}$$

for all $u_1 \in W_0^{1,p_1}(\Omega)$ and $u_2, v_2 \in W_0^{1,p_2}(\Omega)$. These estimates lead directly to the bounds (8) and (9). □

Corollary 1. *Under the hypotheses of Proposition 3 and assuming moreover that the operators B_1 and B_2 are bounded, the operator A defined in (7) is bounded.*

Proof. This follows readily from Proposition 3 and since the operators B_1 and B_2 are bounded. □

Corollary 2. *Under the hypotheses of Proposition 3 and assuming moreover that the operators B_1 and B_2 are bounded, if $(u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2)$ in $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ then we have*

$$\lim_{n\to\infty} \langle A(u_{1,n}, u_{2,n}), (u_{1,n} - u_1, u_{2,n} - u_2)\rangle = 0. \tag{10}$$

Proof. As in the proof of Proposition 3, for $i = 1, 2$ we note that

$$\begin{aligned} &\left| \int_\Omega f_i\left(x, B_1 u_{1,n}, B_2 u_{2,n}, \nabla\left(B_1 u_{1,n}\right), \nabla\left(B_2 u_{2,n}\right)\right)(u_{i,n} - u_i) dx \right| \\ \leq \;& \int_\Omega \Big(|\sigma_i| + c_1 \left|B_1 u_{1,n}\right|^{\alpha_i} + c_2 \left|B_2 u_{2,n}\right|^{\alpha_2} \\ &+ d_1 \left|\nabla(B_1 u_{1,n})\right|^{\beta_1} + d_2 \left|\nabla(B_2 u_{2,n})\right|^{\beta_2} \Big) \left|u_{i,n} - u_i\right| dx. \end{aligned}$$

Since $r_i \in (1, p_i^*)$ and $\alpha_i < p_i^* - 1$ for $i = 1, 2$, the Rellich-Kondrachov embedding theorem implies

$$\int_\Omega |\sigma_i(x)||u_{i,n} - u_i|\, dx \leq \|\sigma_1\|_{r_i'} \|u_{i,n} - u_i\|_{r_i} \to 0 \text{ as } n \to +\infty$$

and

$$\int_\Omega \left|B_i u_{i,n}\right|^{\alpha_i} \left|u_{i,n} - u_i\right| dx \leq \left\|B_i u_{i,n}\right\|_{p_i^*}^{\alpha_i} \left\|u_{i,n} - u_i\right\|_{\frac{p_i^*}{p_i^* - \alpha_i}} \to 0 \text{ as } n \to +\infty,$$

where the boundedness of the operators B_i has been used, too. Since $\alpha_1 < \frac{p_1^*}{(p_2^*)'}$ and $\alpha_2 < \frac{p_2^*}{(p_1^*)'}$, through the Rellich-Kondrachov embedding theorem and the boundedness of the operators B_i for $i = 1, 2$ we arrive at

$$\int_\Omega \left|B_2 u_{2,n}\right|^{\alpha_2} \left|u_{1,n} - u_1\right| dx \leq \left\|B_2 u_{2,n}\right\|_{p_2^*}^{\alpha_2} \left\|u_{1,n} - u_1\right\|_{\frac{p_2^*}{p_2^* - \alpha_2}} \to 0 \text{ as } n \to +\infty$$

and

$$\int_\Omega |B_1u_{1,n}|^{\alpha_1}\,|u_{2,n}-u_2|\,dx \le \|B_1u_{1,n}\|_{p_1^*}^{\alpha_1}\,\|u_{2,n}-u_2\|_{\frac{p_1^*}{p_1^*-\alpha_1}} \to 0 \text{ as } n\to+\infty.$$

We pass to the terms involving the gradient. The Rellich-Kondrachov embedding theorem can be applied thanks to $\beta_1 < \min\left\{\frac{p_1}{(p_1^*)'}, \frac{p_1}{(p_2^*)'}\right\}$ and $\beta_2 < \min\left\{\frac{p_2}{(p_2^*)'}, \frac{p_2}{(p_1^*)'}\right\}$ which ensures

$$\int_\Omega |\nabla(B_iu_{i,n})|^{\beta_i}|u_{i,n}-u_1|dx \le \|\,|\nabla(B_iu_{i,n})|\,\|_{p_i}^{\beta_i}\|u_{i,n}-u_i\|_{\frac{p_i}{p_i-\beta_i}} \to 0$$

as $n\to+\infty$, $i=1,2$,

$$\int_\Omega |\nabla(B_2u_{2,n})|^{\beta_2}|u_{1,n}-u_1|dx \le \|\,|\nabla(B_2u_{2,n})|\,\|_{p_2}^{\beta_2}\|u_{1,n}-u_1\|_{\frac{p_2}{p_2-\beta_2}} \to 0$$

as $n\to+\infty$,

$$\int_\Omega |\nabla(B_1u_{1,n})|^{\beta_1}|u_{2,n}-u_2|dx \le \|\,|\nabla(B_1u_{1,n})|\,\|_{p_1}^{\beta_1}\,\|u_{2,n}-u_2\|_{\frac{p_1}{p_1-\beta_1}} \to 0$$

as $n\to+\infty$. Taking into account (7), the proof of (10) is achieved. □

4 Existence of solutions

Our approach relies on the operator

$$(-\Delta_{p_1}, -\Delta_{p_2}) - A : W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega) \to W^{-1,p_1'}(\Omega)\times W^{-1,p_2'}(\Omega),$$

with A introduced in (7).

Lemma 1. *Assume that condition (H1) is fulfilled and the operators B_1 and B_2 are bounded. Then the operator $(-\Delta_{p_1}, -\Delta_{p_2}) - A$, with A defined by (7), is bounded and pseudomonotone.*

Proof. The definition of the negative p_i−Laplacian $-\Delta_{p_i}$, for $i=1,2$, reveals that the operator $(-\Delta_{p_1}, -\Delta_{p_2})$ is bounded. As known from Corollary 1, the operator A is bounded. Hence it turns out that the operator $(-\Delta_{p_1}, -\Delta_{p_2})-A$ is bounded.

In order to establish that the operator $(-\Delta_{p_1}, -\Delta_{p_2})-A$ is pseudomonotone, we have to prove for each sequence $(u_{1,n}, u_{2,n}) \subset W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ satisfying

$$(u_{1,n}, u_{2,n}) \rightharpoonup (u_1,u_2) \text{ in } W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$$

and

$$\limsup_{n\to\infty}\langle((-\Delta_{p_1},-\Delta_{p_2})-A)\,(u_{1,n},u_{2,n})\,,(u_{1,n}-u_1u_{2,n}-u_2)\rangle\leq 0, \quad (11)$$

that it holds

$$\begin{aligned}&\liminf_{n\to\infty}\langle((-\Delta_{p_1},-\Delta_{p_2})-A)\,(u_{1,n},u_{2,n})\,,(u_{1,n}-v_1,u_{2,n}-v_2)\rangle\\ &\geq\langle((-\Delta_{p_1},-\Delta_{p_2})-A)\,(u_1,u_2)\,,(u_1-v_1,u_2-v_2)\rangle \quad (12)\end{aligned}$$

for all $(v_1,v_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$.

Due to (10), we see that (11) results in

$$\limsup_{n\to\infty}\left(\langle-\Delta_{p_1}u_{1,n},u_{1,n}-u_1\rangle+\langle-\Delta_{p_2}u_{2,n},u_{2,n}-u_2\rangle\right)\leq 0.$$

Then Proposition 2 implies that $(u_{1n},u_{2n})\to(u_1,u_2)$ in $W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$. From here we can conclude that (12) is true, thus completing the proof. □

Lemma 2. *Assume that conditions (H1)-(H3) are fulfilled. Then the operator* $(-\Delta_{p_1},-\Delta_{p_2})-A$*, with* A *defined by* (7)*, is coercive, which means that*

$$\lim_{\|(u_1,u_2)\|\to\infty}\frac{\langle((-\Delta_{p_1},-\Delta_{p_2})-A)\,(u_1,u_2)\,,(u_1,u_2)\rangle}{\|\,(u_1,u_2)\,\|}=+\infty. \quad (13)$$

Proof. By hypotheses (H2)-(H3) and a well-known convexity inequality we get

$$\begin{aligned}&\int_\Omega f_i\,(x,B_1u_1,B_2u_2,\nabla(B_1u_1),\nabla(B_2u_2))\,u_i dx\\ \leq\;&\|\zeta_i\|_1+a_1\|B_1u_1\|_{p_1}^{p_1}+a_2\|B_2u_2\|_{p_2}^{p_2}\\ &+b_1\|\,|\nabla(B_1u_1)|\,\|_{p_1}^{p_1}+b_2\|\,|\nabla(B_2u_2)|\,\|_{p_2}^{p_2}\\ \leq\;&\|\zeta_i\|_1+(a_1+b_1)(K_1\|\,|\nabla u_1|\,\|_{p_1}+K_2)^{p_1}\\ &+(a_2+b_2)(K_1\|\,|\nabla u_2|\,\|_{p_2}+K_2)^{p_2}\\ \leq\;&\|\zeta_i\|_1+2^{p_1-1}(a_1+b_1)(K_1^{p_1}\|\,|\nabla u_1|\,\|_{p_1}^{p_1}+K_2^{p_1})\\ &+2^{p_2-1}(a_2+b_2)(K_1^{p_2}\|\,|\nabla u_2|\,\|_{p_2}^{p_2}+K_2^{p_2})\end{aligned}$$

for all $(u_1,u_2)\in W_0^{1,p_1}(\Omega)\times W_0^{1,p_2}(\Omega)$ and $i=1,2$. It follows that

$$\begin{aligned}&\langle((-\Delta_{p_1},-\Delta_{p_2})-A)(u_1,u_2),(u_1,u_2)\rangle\\ \geq\;&(1-2^{p_1}(a_1+b_1)K_1^{p_1})\|\,|\nabla u_1|\,\|_{p_1}^{p_1}+(1-2^{p_2}(a_2+b_2)K_1^{p_2})\|\,|\nabla u_2|\,\|_{p_2}^{p_2}\\ &-\|\zeta_1\|_1-\|\zeta_2\|_1-2^{p_1}(a_1+b_1)K_2^{p_1}-2^{p_2}(a_2+b_2)K_2^{p_2}.\end{aligned}$$

Therefore the operator $(-\Delta_{p_1},-\Delta_{p_2})-A$ satisfies (13) because of (1) and $p_i>1$, $i=1,2$. □

Now we are able to provide the proof of Theorem 1.

Proof of Theorem 1. The proof is carried out by applying Theorem 3 to the operator $(-\Delta_{p_1}, -\Delta_{p_2}) - A : W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega) \to W^{-1,p_1'}(\Omega) \times W^{-1,p_2'}(\Omega)$, with A defined in (7), observing that the resolution of system (P) is equivalent to solving the operator equation

$$((-\Delta_{p_1}, -\Delta_{p_2}) - A)(u_1, u_2) = 0. \tag{14}$$

Lemma 1 ensures that the operator $(-\Delta_{p_1}, -\Delta_{p_2}) - A$ is bounded and pseudomonotone. Lemma 2 shows that the operator $(-\Delta_{p_1}, -\Delta_{p_2}) - A$ is coercive. Hence the assumptions required to apply Theorem 3 for the operator $(-\Delta_{p_1}, -\Delta_{p_2}) - A$ on the space $W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ are fulfilled. Consequently, there exists $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ solving equation (14). Accordingly, (u_1, u_2) is a weak solution to problem (P), which completes the proof of Theorem 1.

Let us pass to the proof of Theorem 2

Proof of Theorem 2. Theorem 2 provides us with a solution $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ to system (P). Then (14) and the estimate in the proof of Lemma 2 lead to

$$\begin{aligned}
&(1 - 2^{p_1}(a_1 + b_1)K_1^{p_1})\| \, |\nabla u_i\|_{p_i}^{p_i} + (1 - 2^{p_2}(a_2 + b_2)K_1^{p_2})\| \, |\nabla u_2| \, \|_{p_2}^{p_2} \\
&\leq \|\zeta_1\|_1 + \|\zeta_2\|_1 + 2^{p_1}(a_1 + b_1)K_2^{p_1} + 2^{p_2}(a_2 + b_2)K_2^{p_2}.
\end{aligned}$$

whence the desired conclusion is achieved.

5 Application to systems with convolutions

We apply our theoretic results to the following system involving convolution products

$$\begin{cases} -\Delta_{p_1} u_1 = \frac{1}{1+(\rho * u_1)^2 + (\rho * u_2)^2}(1 + |\nabla(\rho * u_2)|^\gamma) & \text{in } \Omega \\ -\Delta_{p_2} u_2 = \frac{1}{1+(\rho * u_1)^2 + (\rho * u_2)^2}(1 + |\nabla(\rho * u_1)|^\gamma) & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{Q}$$

where $p_1, p_2 \in (1, +\infty)$, $\rho \in L^1(\Omega)$ and $\gamma > 0$.

Our result on problem (Q) is formulated as follows.

Theorem 4. *Assume that* $\gamma \in \left[0, \min\left\{\frac{p_1}{(p_1^*)'}, \frac{p_1}{(p_2^*)'}, \frac{p_2}{(p_2^*)'}, \frac{p_2}{(p_1^*)'}\right\}\right)$ *and*

$$\|\rho\|_{L^1(\mathbb{R}^N)} < 2^{-1}\left(\max\{\lambda_{1,p_1}^{-\frac{1}{p_1}}, \lambda_{1,p_2}^{-\frac{1}{p_2}}\} + N\right)^{-1}. \tag{15}$$

Then problem (Q) *admits at least one weak solution* $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$.

In addition, there is the estimate

$$\|u_i\|_{W_0^{1,p_i}} \le (4|\Omega|)^{\frac{1}{p_i}}\left(1 - \left(2\|\rho\|_{L^1(\mathbb{R}^N)}\left(\max\{\lambda_{1,p_1}^{-\frac{1}{p_1}}, \lambda_{1,p_2}^{-\frac{1}{p_2}}\} + N\right)\right)^{p_i}\right)^{-\frac{1}{p_i}}, \tag{16}$$

$i = 1, 2$.

Proof. The proof proceeds by applying Theorem 2. To this end we introduce the functions $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, 2$, by

$$f_1(x, s_1, s_2, \xi_1, \xi_2) = \frac{1}{1 + s_1^2 + s_2^2}(1 + |\xi_2|^\gamma),$$
$$f_2(x, s_1, s_2, \xi_1, \xi_2) = \frac{1}{1 + s_1^2 + s_2^2}(1 + |\xi_1|^\gamma),$$

and the maps $B_i : W_0^{1,p_i}(\Omega) \to W^{1,p_i}(\Omega)$, $i = 1, 2$, by

$$B_i(v) = \rho * v, \quad v \in W_0^{1,p_i}(\Omega).$$

The assumption on γ entails that hypothesis (H1) is satisfied. By (6) we infer that

$$\|\rho * v\|_{W^{1,p_i}} \le \|\rho\|_{L^1(\mathbb{R}^N)}\left(\max\{\lambda_{1,p_1}^{-\frac{1}{p_1}}, \lambda_{1,p_2}^{-\frac{1}{p_2}}\} + N\right)\|v\|_{W_0^{1,p_i}}$$

for all $v \in W_0^{1,p_i}(\Omega)$, $i = 1, 2$. Consequently, hypothesis (H2) holds with the constants $K_1 = \|\rho\|_{L^1(\mathbb{R}^N)}\left(\max\{\lambda_{1,p_1}^{-\frac{1}{p_1}}, \lambda_{1,p_2}^{-\frac{1}{p_2}}\} + N\right)$ and $K_2 = 0$. Since we have

$$f_1(x, s_1, s_2, \xi_1, \xi_2)s_1 = \frac{s_1}{1 + s_1^2 + s_2^2}(1 + |\xi_2|^\gamma) \le 2 + |\xi_2|^{p_2}$$

and

$$f_2(x, s_1, s_2, \xi_1, \xi_2)s_2 = \frac{s_2}{1 + s_1^2 + s_2^2}(1 + |\xi_2|^\gamma) \le 2 + |\xi_1|^{p_1},$$

and thanks to (15), hypothesis (H3) is verified, too. According to Theorem 1 we conclude that system (Q) possesses a weak solution $(u_1, u_2) \in W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$. The bound of the solution described in (16) is a direct consequence of Theorem 2. The proof is complete. ☐

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