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WEAK SOLUTIONS FOR QUASILINEAR BIHARMONIC SYSTEMS*

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Dedicated to Prof. Biagio Ricceri on the occasion of his 70th birthday

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Abstract

We investigate weak solutions of a quasilinear (p, p)-biharmonic system with variational structure. Using the principal eigenvalue of the associated system and the linking theorem of Brezis and Nirenberg, we establish the existence of at least two nontrivial weak solutions for the eigenvalue parameter λ in a closed right neighborhood of zero. Our results apply in both resonant and nonresonant cases, depending on the asymptotic behavior of the nonlinear term. These findings extend earlier work on scalar biharmonic equations and systems, and cover several important special cases arising from different choices of the coefficients.

Keywords: biharmonic systems, principal eigenvalues, variational methods, weak solutions, linking theorem.

MSC: 35P30, 35J48.

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1 Introduction

We study the existence of weak solutions to the quasilinear (p, p)-biharmonic system

$$\begin{cases} \Delta\big(\mu(x)|\Delta u|^{p-2}\Delta u\big) \\ \quad = \lambda a(x)|u|^{p-2}u + \lambda c(x)|u|^{\alpha-1}|v|^{\beta+1}u + \frac{1}{\alpha+1}F_u(x,u,v) & \text{in } \Omega, \\ \Delta\big(\nu(x)|\Delta v|^{p-2}\Delta v\big) \\ \quad = \lambda b(x)|v|^{p-2}v + \lambda c(x)|u|^{\alpha+1}|v|^{\beta-1}v + \frac{1}{\beta+1}F_v(x,u,v) & \text{in } \Omega, \\ u = \Delta u = 0, \quad v = \Delta v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where λ is a nonnegative eigenvalue parameter, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary, $p > 1$ is a constant, $\mu, \nu \in C(\overline{\Omega})$ with $\mu, \nu > 0$ on $\overline{\Omega}$, α, β, a, b, c satisfy the following conditions:

(A1) α and β are nonnegative constants such that $\alpha + \beta + 2 = p$;

(A2) $a, b, c : \Omega \to \mathbb{R}$ are nonnegative measurable functions with

$$a, b, c \in L^\infty(\Omega) \quad \text{and} \quad a^2 + b^2 + c^2 \not\equiv 0 \;\text{ in }\; \Omega.$$

The function F belongs to the class $\mathcal{A}$ of functions $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$, which are measurable in Ω, continuously differentiable in $\mathbb{R}^2$ for a.e. $x \in \Omega$, $F(x, 0, 0) = 0$, and satisfy the following growth conditions:

$$|F_s(x,s,t)| \le C\left(1 + |s|^{\overline{p}-1} + |t|^{\frac{\overline{q}(\overline{p}-1)}{\overline{p}}}\right), \tag{2}$$

$$|F_t(x,s,t)| \le C\left(1 + |t|^{\overline{q}-1} + |s|^{\frac{\overline{p}(\overline{q}-1)}{\overline{q}}}\right), \tag{3}$$

where $\overline{p}, \overline{q} \in \left[1, \frac{Np}{N-2p}\right)$ if $p < \frac{N}{2}$ and $\overline{p}, \overline{q} \in [1, \infty)$ if $p \ge \frac{N}{2}$, and $F_s(x, s, t)$ and $F_t(x, s, t)$ denote the partial derivatives of $F(x, s, t)$ with respect to s and t, respectively.

In this paper, we employ the linking theorem of Brezis and Nirenberg (see Lemma 4 in Section 4) to establish the existence of at least two nontrivial weak solutions of system (1). Sufficient conditions are provided to ensure that system (1) admits at least two nontrivial weak solutions for λ in a closed right neighborhood of 0. Our existence result applies to both resonant and nonresonant cases by comparing the behavior of the nonlinearity F with the principal eigenvalue of the associated biharmonic system

$$\begin{cases} \Delta\big(\mu(x)|\Delta u|^{p-2}\Delta u\big) \\ \quad = \lambda a(x)|u|^{p-2}u + \lambda c(x)|u|^{\alpha-1}|v|^{\beta+1}u, & \text{in } \Omega, \\ \Delta\big(\nu(x)|\Delta v|^{p-2}\Delta v\big) \\ \quad = \lambda b(x)|v|^{p-2}v + \lambda c(x)|u|^{\alpha+1}|v|^{\beta-1}v, & \text{in } \Omega, \\ u = \Delta u = 0, \quad v = \Delta v = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

For a detailed discussion of the resonant and nonresonant cases, see Remarks 3 and 4 in Section 3. For additional background on linking theorems and related applications, see, for example, [4, 5, 18–20].

In recent years, biharmonic problems have been the focus of extensive research. Regarding problems with principal eigenvalues, among many others, we refer to [2, 3, 6, 24] for scalar cases and to [7, 14–16] for systems. For biharmonic problems involving non-discrete eigenvalues, see, for instance, [8, 11–13, 18] in the scalar setting and [1, 9, 10, 17, 21, 22] for systems.

In the scalar case, Drábek and Ótanik [6] proved in 2001 that the p-biharmonic boundary value problem

$$\begin{cases} \Delta\big(|\Delta u|^{p-2}\Delta u\big) = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a principal eigenvalue λ_1, which is both simple and isolated. Later, Benedikt and Drábek [2] obtained two-sided estimates for λ_1 and used them to study its asymptotic behavior as $p \to \infty$. When Ω is a ball centered at the origin, they [3] further analyzed the asymptotics of λ_1 as $p \to 1^+$.

In the case of systems, in 2019, Leadi and Toyou [16] investigated the existence and simplicity of the principal eigenvalue for the (p, q)-biharmonic system

$$\begin{cases} \Delta\big(|\Delta u|^{p-2}\Delta u\big) = \lambda a(x)|u|^{p-2}u + c(x)|u|^{\alpha-1}|v|^{\beta+1}u & \text{in } \Omega, \\ \Delta\big(|\Delta v|^{q-2}\Delta v\big) = \lambda b(x)|v|^{q-2}v + c(x)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega, \\ u = \Delta u = 0, \quad v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases}$$

More recently, Kong and Nichols [14] and Kong, Nichols, and Wang [15] studied the principal eigenvalue of system (4) (in fact, its generalizations) along with the associated eigenfunctions. Some of their results, relevant to the present work, are summarized in Lemma 1 and Remark 2 below.

The studies in [14, 15] provide the theoretical foundation for our analysis, as the behavior of the principal eigenvalue and its associated eigenfunctions plays a crucial role in establishing the existence of nontrivial solutions for system (1).

We emphasize that, depending on the values of λ, α, β, a, b, and c, system (1) encompasses several forms. For example, when $\lambda = \alpha = \beta = 0$, we have

$$\begin{cases} \Delta\big(\mu(x)|\Delta u|^{p-2}\Delta u\big) = F_u(x,u,v) & \text{in } \Omega, \\ \Delta\big(\nu(x)|\Delta v|^{p-2}\Delta v\big) = F_v(x,u,v) & \text{in } \Omega, \\ u = \Delta u = 0, \ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

When $c(x) \equiv 0$ in Ω, the system takes the form

$$\begin{cases} \Delta\big(\mu(x)|\Delta u|^{p-2}\Delta u\big) = \lambda a(x)|u|^{p-2}u + F_u(x,u,v) & \text{in } \Omega, \\ \Delta\big(\nu(x)|\Delta v|^{p-2}\Delta v\big) = \lambda b(x)|v|^{p-2}v + F_v(x,u,v) & \text{in } \Omega, \\ u = \Delta u = 0, \ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases}$$

When $a(x) = b(x) \equiv 0$ in Ω, it becomes

$$\begin{cases} \Delta\big(\mu(x)|\Delta u|^{p-2}\Delta u\big) = \lambda c(x)|u|^{\alpha-1}|v|^{\beta+1}u + \frac{1}{\alpha+1}F_u(x,u,v) & \text{in } \Omega, \\ \Delta\big(\nu(x)|\Delta v|^{p-2}\Delta v\big) = \lambda c(x)|u|^{\alpha+1}|v|^{\beta-1}v + \frac{1}{\beta+1}F_v(x,u,v) & \text{in } \Omega, \\ u = \Delta u = 0, \ v = \Delta v = 0 & \text{on } \partial\Omega. \end{cases}$$

Analogous systems arise when $a(x) = c(x) \equiv 0$ or $b(x) = c(x) \equiv 0$ in Ω. Our existence theorem applies to all these cases.

The remainder of the paper is organized as follows. Section 2 presents some results on the eigenvalue problem (4), Section 3 states the main theorem, and its proof is provided in Section 4.

2 An eigenvalue problem

Throughout this paper, for any $r \in (1,\infty)$, we denote the norm of the space $L^r(\Omega)$ by

$$\|u\|_r = \left(\int_\Omega |u|^r \, dx\right)^{1/r}, \quad u \in L^r(\Omega).$$

We define the space X as

$$X = \Big(W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)\Big) \times \Big(W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)\Big). \tag{6}$$

Then, X is a separable and reflexive Banach space equipped with the standard norm

$$\|(u,v)\|_X = \|\Delta u\|_p + \|\Delta v\|_p, \quad (u,v) \in X.$$

Let the functionals $\Phi, \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u,v) = \frac{\alpha+1}{p}\int_\Omega \mu(x)|\Delta u|^p\,dx + \frac{\beta+1}{p}\int_\Omega \nu(x)|\Delta v|^p\,dx \tag{7}$$

and

$$\begin{aligned}\Psi(u,v) = {}&\frac{\alpha+1}{p}\int_\Omega a(x)|u|^p\,dx + \frac{\beta+1}{p}\int_\Omega b(x)|v|^p\,dx \\ &+ \int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta+1}\,dx,\end{aligned} \tag{8}$$

where $(u,v) \in X$.

Under the assumptions (A1) and (A2), it is straightforward to verify that Φ and Ψ are well-defined and belong to $C^1(X,\mathbb{R})$. Moreover, for all $(u,v),(\phi,\psi) \in X$, their Fréchet derivatives satisfy

$$\begin{aligned}\langle \Phi'(u,v),(\phi,\psi)\rangle = {}&(\alpha+1)\int_\Omega \mu(x)|\Delta u|^{p-2}\Delta u\,\Delta\phi\,dx \\ &+ (\beta+1)\int_\Omega \nu(x)|\Delta v|^{p-2}\Delta v\,\Delta\psi\,dx\end{aligned} \tag{9}$$

and

$$\begin{aligned}\langle \Psi'(u,v),(\phi,\psi)\rangle = {}&(\alpha+1)\int_\Omega a(x)|u|^{p-2}u\,\phi\,dx \\ &+ (\alpha+1)\int_\Omega c(x)|u|^{\alpha-1}|v|^{\beta+1}u\,\phi\,dx \\ &+ (\beta+1)\int_\Omega b(x)|v|^{p-2}v\,\psi\,dx \\ &+ (\beta+1)\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta-1}v\,\psi\,dx,\end{aligned} \tag{10}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and its dual space X^*.

Lemma 1. *Assume that (A1) and (A2) hold. Then system* (4) *admits an eigenpair*

$$\big(\lambda_1(a,b,c),(u_1,v_1)\big),$$

where $\lambda_1(a,b,c) > 0$ is the principal eigenvalue and u_1, v_1 are nonnegative in Ω. Moreover,

$$\lambda_1(a,b,c) = \Phi(u_1,v_1) = \inf_{(u,v)\in M}\Phi(u,v), \tag{11}$$

where

$$M = \{(u,v) \in X \,:\, \Psi(u,v) = 1\}. \tag{12}$$

Remark 1. Lemma 1 is a direct consequence of [14, Theorem 2.3] in the case $c \not\equiv 0$ and $\mu = \nu \equiv 1$ in Ω, and of [15, Corollary 3.1] in the case $a \not\equiv 0$ and $b \not\equiv 0$ in Ω. The other situations, where $a^2 + b^2 + c^2 \not\equiv 0$ in Ω, can be treated in a similar way.

Remark 2. To further understand the principal eigenvalue $\lambda_1(a, b, c)$ defined in (11) and its associated eigenfunction (u_1, v_1), we present below a summary of several special cases corresponding to different values a, b, and c:

Case 1: Fully coupled system, $c(x) \not\equiv 0$.

- The corresponding eigenfunction (u_1, v_1) has both components nontrivial and of fixed sign. In our setting, we take u_1 and v_1 to be positive throughout Ω.

Case 2: Decoupled system, $c(x) \equiv 0$.

- System (4) reduces to two independent eigenvalue problems:

$$\Delta(\mu(x)|\Delta u|^{p-2}\Delta u) = \lambda a(x)|u|^{p-2}u, \quad u = \Delta u = 0, \tag{13}$$

$$\Delta(\nu(x)|\Delta v|^{p-2}\Delta v) = \lambda b(x)|v|^{p-2}v, \quad v = \Delta v = 0. \tag{14}$$

  Let

$$\lambda_{1,u} := \inf_{u \neq 0} \frac{\int_\Omega \mu(x)|\Delta u|^p}{\int_\Omega a(x)|u|^p} \quad \text{and} \quad \lambda_{1,v} := \inf_{v \neq 0} \frac{\int_\Omega \nu(x)|\Delta v|^p}{\int_\Omega b(x)|v|^p}.$$

  Then, it can be shown that

$$\lambda_1(a, b, 0) = \min\{\lambda_{1,u}, \lambda_{1,v}\}.$$

  Moreover, if $\mu(x) = \nu(x)$ and $a(x) = b(x)$, then $\lambda_{1,u} = \lambda_{1,v}$, hence

$$\lambda_1(a, b, 0) = \lambda_{1,u} = \lambda_{1,v}.$$

- The eigenfunction corresponding to $\lambda_1(a, b, 0)$ for the decoupled system (4):
  - ◇ $(u_1, 0)$ if $\lambda_{1,u} < \lambda_{1,v}$, where u_1 solves (13) and is positive in Ω.
  - ◇ $(0, v_1)$ if $\lambda_{1,v} < \lambda_{1,u}$, where v_1 solves (14) and is positive in Ω.
  - ◇ Any $(\gamma u_1, \delta v_1)$ with $(\gamma, \delta) \neq (0, 0)$ if $\lambda_{1,v} = \lambda_{1,u}$. In this context, we take $\gamma = \delta = 1$.

Case 3: One equation active.

- If $a \not\equiv 0$, $b \equiv 0$, $c \equiv 0$, then

$$\lambda_1(a,0,0) = \inf_{u \neq 0} \frac{\int_\Omega \mu(x)|\Delta u|^p}{\int_\Omega a(x)|u|^p},$$

  with eigenfunctions of the form $(u_1, 0)$.

- Similarly, if $a \equiv 0$, $b \not\equiv 0$, $c \equiv 0$, then

$$\lambda_1(0,b,0) = \inf_{v \neq 0} \frac{\int_\Omega \nu(x)|\Delta v|^p}{\int_\Omega b(x)|v|^p},$$

  with eigenfunctions of the form $(0, v_1)$.

In the sequel, we write λ_1 instead of $\lambda_1(a,b,c)$ when the dependence on a, b, and c is clear from the context.

Lemma 2. *Assume that (A1) and (A2) hold. If $(u_1, v_1) \in M$ and $(u_2, v_2) \in M$ are both minimizers of Φ on M, then they are linearly dependent, that is, there exists $\rho \in \mathbb{R} \setminus \{0\}$ such that $(u_2, v_2) = \rho\,(u_1, v_1)$. In other words, any two minimizers of Φ on M are scalar multiples of one another.*

Proof. Since $(u_1, v_1), (u_2, v_2) \in M$ are both minimizers of Φ, we have

$$\lambda_1 = \Phi(u_1, v_1) = \Phi(u_2, v_2).$$

Suppose (u_1, v_1) and (u_2, v_2) are not scalar multiples. Define the convex combination

$$(u_t, v_t) := (1-t)(u_1, v_1) + t(u_2, v_2), \quad t \in [0,1].$$

Because Φ is strictly convex,

$$\Phi(u_t, v_t) < (1-t)\Phi(u_1, v_1) + t\Phi(u_2, v_2) = \lambda_1 \quad \text{for all } t \in (0,1). \tag{15}$$

Define the function

$$G(t,s) := \Psi\big((u_t, v_t) + s(w,z)\big) - 1$$

for some fixed $(w, z) \in X$, where s is a scalar parameter. Observe that

$$G(0,0) = \Psi(u_1, v_1) - 1 = 0$$

since $(u_1, v_1) \in M$. The partial derivative of G with respect to s at $(0,0)$ is

$$D_s G(0,0) = D\Psi(u_1, v_1)[(w,z)].$$

If (w,z) is chosen such that

$$\langle \Psi'(u_1, v_1), (w,z) \rangle \neq 0,$$

then by the Implicit Function Theorem, there exist a neighborhood I of 0 and a unique continuously differentiable function

$$s = s(t), \quad t \in I,$$

such that $G(t, s(t)) = 0$, i.e.,

$$\Psi\big((u_t, v_t) + s(t)(w,z)\big) = 1.$$

Hence the curve

$$(\tilde{u}_t, \tilde{v}_t) := (u_t, v_t) + s(t)(w,z)$$

lies entirely on the constraint set M for t near 0. By the continuity of Φ and (15), for small $t \neq 0$, we have

$$\Phi(\tilde{u}_t, \tilde{v}_t) = \Phi\big((u_t, v_t) + s(t)(w,z)\big) < \lambda_1.$$

This contradicts the fact that λ_1 is the minimum of Φ on M. The contradiction shows (u_1, v_1) and (u_2, v_2) must be scalar multiples. The lemma is thus proved. □

Lemma 3. *Assume that (A1) and (A2) hold. Let (u_1, v_1) be given as in Lemma 1. Set $V = \mathrm{span}\{(u_1, v_1)\}$ and choose a closed subspace $W \subset X$ such that $X = V \oplus W$. Define*

$$\widehat{\lambda} = \inf_{(u,v) \in W \setminus \{0\}} \frac{\Phi(u,v)}{\Psi(u,v)}. \tag{16}$$

Then $\widehat{\lambda} > \lambda_1$.

Proof. Assume, by contradiction, that $\widehat{\lambda} = \lambda_1$. Then there exists a sequence $(u_n, v_n) \subset W$ with $\Psi(u_n, v_n) = 1$ (we may normalize each element so the denominator equals 1) and $\Phi(u_n, v_n) \downarrow \widehat{\lambda} = \lambda_1$. The sequence $\{(u_n, v_n)\}$ is bounded in X, hence (up to a subsequence) $(u_n, v_n) \rightharpoonup (u_\infty, v_\infty)$ weakly in

X and $(u_n, v_n) \to (u_\infty, v_\infty)$ strongly in $L^p(\Omega)\times L^p(\Omega)$. The constraint passes to the limit: $\Psi(u_\infty, v_\infty) = 1$. By weak lower semicontinuity, we obtain

$$\Phi(u_\infty, v_\infty) \le \liminf_{n\to\infty} \Phi(u_n, v_n) = \lambda_1,$$

so $\Phi(u_\infty, v_\infty) = \lambda_1$. Therefore $(u_\infty, v_\infty) \in M$ attains the global infimum λ_1. By Lemma 2, it follows that (u_∞, v_∞) must be a scalar multiple of (u_1, v_1), hence $(u_\infty, v_\infty) \in V$.

But each (u_n, v_n) lies in W, and W is closed and hence weakly closed, so the weak limit $(u_\infty, v_\infty) \in W$. Thus, $(u_\infty, v_\infty) \in V \cap W$. Since $X = V \oplus W$ is a direct sum, we have $V \cap W = \{0\}$. Therefore $(u_\infty, v_\infty) = (0, 0)$. This contradicts the fact that $\Psi(u_\infty, v_\infty) = 1$. This contradiction shows $\widehat{\lambda} > \lambda_1$, as claimed. □

3 Existence of weak solutions

For a.e. $x \in \Omega$ and all $(s,t) \in \mathbb{R}^2$, since $F \in \mathcal{A}$, from (2) and (3), we have

$$\begin{aligned}
&|F(x,s,t)| \\
&= \left| F(x,0,0) + \int_0^1 \frac{\partial F(x,\tau s,\tau t)}{\partial \tau}\, d\tau \right| \\
&\le \int_0^1 \big(|s|\, |F_s(x,\tau s,\tau t)| + |t|\, |F_t(x,\tau s,\tau t)| \big)\, d\tau \\
&\le C \int_0^1 \left(|s| + |s|^{\bar p}\tau^{\bar p-1} + |\tau t|^{\frac{\bar q(\bar p-1)}{\bar p}} |s| + |t| + |t|^{\bar q}\tau^{\bar q-1} + |\tau s|^{\frac{\bar p(\bar q-1)}{\bar q}} |t| \right) d\tau \\
&\le C \left(|s| + |t| + |s|^{\bar p} + |t|^{\bar q} + |t|^{\frac{\bar q(\bar p-1)}{\bar p}} |s| + |s|^{\frac{\bar p(\bar q-1)}{\bar q}} |t| \right).
\end{aligned}$$

This, together with Young's inequality, implies that

$$|F(x,s,t)| \le C_1 \left(|s| + |t| + |s|^{\bar p} + |t|^{\bar q} \right), \tag{17}$$

where C_1 is a positive constant independent of $(s,t) \in \mathbb{R}^2$.

Define the functional $N : X \to \mathbb{R}$ by

$$N(u,v) = \int_\Omega F(x,u,v)\, dx. \tag{18}$$

Then, it can be shown that N is well-defined and belongs to $C^1(X,\mathbb{R})$. Moreover, for all $(u,v), (\phi,\psi) \in X$, its Fréchet derivative is given by

$$\langle N'(u,v), (\phi,\psi) \rangle = \int_\Omega \Big[F_u(x,u,v)\, \phi + F_v(x,u,v)\, \psi \Big]\, dx. \tag{19}$$

Definition 1. *We say that $(u, v) \in X$ is a weak solution of system* (1) *if it satisfies*

$$\begin{aligned}&\int_\Omega \mu(x)|\Delta u|^{p-2}\Delta u\,\Delta\phi\,dx\\ &= \lambda\int_\Omega a(x)|u|^{p-2}u\,\phi\,dx + \lambda\int_\Omega c(x)|u|^{\alpha-1}|v|^{\beta+1}u\,\phi\,dx\\ &\quad+\frac{1}{\alpha+1}\int_\Omega F_u(x,u,v)\,\phi\,dx\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega \nu(x)|\Delta v|^{p-2}\Delta v\,\Delta\psi\,dx\\ &= \lambda\int_\Omega b(x)|v|^{p-2}v\,\psi\,dx + \lambda\int_\Omega c(x)|u|^{\alpha+1}|v|^{\beta-1}v\,\psi\,dx\\ &\quad+\frac{1}{\beta+1}\int_\Omega F_v(x,u,v)\,\psi\,dx\end{aligned}$$

for all $(\phi, \psi) \in X$.

Let the functional $I : X \to \mathbb{R}$ be defined by

$$I(u, v) = \Phi(u, v) - \lambda\Psi(u, v) - N(u, v), \tag{20}$$

where Φ and Ψ are defined by (7) and (8), respectively. Then, using (9), (10), and (19), we deduce that (u, v) is a critical point of I if and only if it is a weak solution of system (1).

Henceforth, let λ_1 be as in Lemma 1, and let $\widehat{\lambda}$ be as in Lemma 3.

We make the following assumptions:

(H1) **(Local growth near the origin)** There exist $r > 0$ and $\bar{\lambda} \in (\lambda_1, \widehat{\lambda})$ such that, for a.e. $x \in \Omega$ and all $s, t \in [0, r]$, we have

$$\lambda_1 K(x, s, t) \le F(x, s, t) \le \bar{\lambda}K(x, s, t),$$

where

$$K(x, s, t) = \frac{\alpha+1}{p}a(x)|s|^p + \frac{\beta+1}{p}b(x)|t|^p + c(x)|s|^{\alpha+1}|t|^{\beta+1}. \tag{21}$$

(H2) **(Subcritical growth at infinity)** As $|s| + |t| \to \infty$, we require

$$\limsup_{|s|+|t|\to\infty}\ \operatorname*{ess\,sup}_{x\in\Omega} Q(x, s, t) < \lambda_1 + \bar{\lambda} - \widehat{\lambda}, \tag{22}$$

where

$$Q(x,s,t) = \frac{F(x,s,t)}{K(x,s,t)} \quad \text{for a.e. } x \in \Omega \text{ and all } (s,t) \in \mathbb{R}^2.$$

(H3) **(Raywise asymptotics)** For a.e. $x \in \Omega$ and all (s,t) with $K(x,s,t) \neq 0$, the following limits hold:

$$\lim_{\rho\to\infty} Q(x,\rho s,\rho t) = \lambda_1 + \bar{\lambda} - \widehat{\lambda} \tag{23}$$

and

$$\lim_{\rho\to\infty}\left(\rho\frac{dF(x,\rho s,\rho t)}{d\rho} - pF(x,\rho s,\rho t)\right) = \infty. \tag{24}$$

We now state the main theorem of this paper.

Theorem 1. *Suppose that (H1) holds and either (H2) or (H3) is satisfied. Then, for each* $\lambda \in \big[0,\, \widehat{\lambda}-\bar{\lambda}\big]$*, system* (1) *admits at least two nontrivial weak solutions.*

Remark 3. Under assumption (H1), system (5) (i.e., system (1) with $\lambda = 0$) is resonant at the origin if

$$\lim_{(s,t)\to(0^+,0^+)} Q(x,s,t) = \lambda_1.$$

Otherwise, if this limit is strictly above λ_1, the system is nonresonant near zero.

Remark 4. Assumption (H2) corresponds to a subcritical and uniformly nonresonant regime: $Q(x,s,t)$ remains strictly below the critical threshold $\lambda_1 + \bar{\lambda} - \widehat{\lambda}$ at infinity, and the energy functional has standard variational geometry for all $\lambda \in [0, \widehat{\lambda} - \bar{\lambda}]$.

In contrast, (H3) describes an asymptotically critical regime along rays, with Q converging to the critical threshold and the additional superlinearity condition (24) ensuring sufficient growth. Here, the problem can become resonant as λ approaches $\widehat{\lambda} - \bar{\lambda}$, but compactness is preserved, guaranteeing multiple nontrivial solutions.

Thus, the main existence conclusion holds in both cases, but the asymptotic behavior and resonance properties of the nonlinearity differ.

Intuitively, (H2) keeps the nonlinearity safely below the critical level (nonresonant), while (H3) lets it approach the critical level along rays (resonant), but the extra superlinearity ensures solutions still exist.

4 Proof of Theorem 1

Recall that a functional I, defined on a real Banach space X, is said to satisfy the Palais–Smale condition (PS condition, for short) if every sequence $\{w_n\} \subset X$, such that

$$I(w_n) \text{ is bounded} \quad \text{and} \quad I'(w_n) \to 0 \text{ in } X^*,$$

admits a strongly convergent subsequence in X. Such a sequence $\{w_n\}$ is called a PS sequence.

Lemma 4. ([4, Theorem 4]) *Let X be a real Banach space with a direct sum decomposition $X = V \oplus W$ with $\dim V < \infty$. Suppose that $I \in C^1(X, \mathbb{R})$ satisfies the PS condition and is bounded below, $I(0) = 0$, and $\inf_{w \in X} I(w) < 0$. Assume also that I has a local linking at 0, that is, for some $\rho > 0$,*

$$I(w) \le 0 \quad \text{for } w \in V \text{ with } ||w|| \le \rho,$$
$$I(w) \ge 0 \quad \text{for } w \in W \text{ with } ||w|| \le \rho.$$

Then, I has at least two nontrivial critical points.

In what follows, let X be defined by (6) and let V and W be given as in Lemma 3. We also denote by I the functional given in (20). Then, $X = V \oplus W$ with $\dim V = 1 < \infty$.

Lemma 5. *Assume that (H1) holds. Then, for each $\lambda \in \big[0, \widehat{\lambda} - \bar{\lambda}\big]$, I has a local linking at 0 with respect to the decomposition $X = V \oplus W$.*

Proof. Since V is finite-dimensional, all norms on V are equivalent. Hence, there exists a constant $C_2 > 0$ such that

$$\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \le C_2 \|(u, v)\|_X \quad \text{for all } (u, v) \in V.$$

Taking $\rho > 0$ sufficiently small ensures that, for all $(u, v) \in V$ with $\|(u, v)\|_X \le \rho$, we have

$$|u(x)| \le r \text{ and } |v(x)| \le r \text{ for all } x \in \Omega,$$

where $r > 0$ is as in assumption (H1).

Below, we fix $\lambda \in \big[0, \widehat{\lambda} - \bar{\lambda}\big]$. Let $(u, v) \in V$ be such that $\|(u, v)\|_X \le \rho$. Note that $\alpha + \beta + 2 = p$ by (A1). Then, in view of (8) and (21), we have

$$\begin{aligned}\Phi(u, v) &= \lambda_1 \Psi(u, v) \\ &= \lambda_1 \int_\Omega K(x, u, v)\, dx.\end{aligned}$$

Thus, from (18), (20) and (H1), it follows that

$$
\begin{aligned}
I(u,v) &= \lambda_1 \int_\Omega K(x,u,v)\,dx - \lambda\Psi(u,v) - \int_\Omega F(x,u,v)\,dx \\
&\le \int_\Omega \big[\, \lambda_1 K(x,u,v) - F(x,u,v) \,\big]\, dx \\
&\le 0.
\end{aligned}
$$

Now, let $(u,v) \in W$ be such that $\|(u,v)\|_X \le \rho$. Define

$$S = \{x \in \Omega \,|\, |u(x)| \le r \text{ and } |v(x)| \le r\}$$

and denote by S^c the complement of S in Ω. From (16), we have

$$\Phi(u,v) \ge \widehat{\lambda}\Psi(u,v).$$

Then, from (20) and (H1), we derive that

$$
\begin{aligned}
I(u,v) &= \Phi(u,v) - \big(\lambda - \bar{\lambda}\big)\,\Psi(u,v) - N(u,v) + \bar{\lambda}\Psi(u,v) \\
&\ge \left(1 - \frac{\lambda - \widehat{\lambda}}{\widehat{\lambda}}\right)\Phi(u,v) - \int_S \left[F(x,u,v) - \bar{\lambda}K(x,u,v)\right]\,dx \\
&\quad - \int_{S^c} \left[F(x,u,v) - \bar{\lambda}K(x,u,v)\right]\,dx \\
&\ge \left(1 - \frac{\lambda - \widehat{\lambda}}{\widehat{\lambda}}\right)\Phi(u,v) - \int_{S^c} \left[F(x,u,v) - \bar{\lambda}K(x,u,v)\right]\,dx. \qquad (25)
\end{aligned}
$$

Note from (17) and (21) that there exist constants $C_3 > 0$ and $w \in \left(p, \frac{Np}{N-2p}\right]$ if $p < \frac{N}{2}$, and $w \in (p,\infty)$ if $p \ge \frac{N}{2}$, such that

$$F(x,u,v - \bar{\lambda}K(x,u,v)) \le C_3\left(|u|^w + |v|^w\right) \quad \text{for all} \quad (u,v) \in S^c.$$

Thus, in view of the fact that

$$1 - \frac{\lambda - \widehat{\lambda}}{\widehat{\lambda}} \ge 0,$$

from (7) and (25), it follows that

$$\begin{aligned} I(u,v) &\geq \left(1 - \frac{\lambda - \widehat{\lambda}}{\widehat{\lambda}}\right) \Big(\frac{\alpha+1}{p}\, \underline{\mu}\, \|\Delta u\|_p^p + \frac{\beta+1}{p}\, \underline{\nu}\, \|\Delta v\|_p^p \Big) \\ &\quad - C_3 \int_{S^c} \big(|u|^w + |v|^w \big)\, dx \\ &\geq \left(1 - \frac{\lambda - \widehat{\lambda}}{\widehat{\lambda}}\right) \Big(\frac{\alpha+1}{p}\, \underline{\mu}\, \|\Delta u\|_p^p + \frac{\beta+1}{p}\, \underline{\nu}\, \|\Delta v\|_p^p \Big) \\ &\quad - C_3 \big(\|u\|_w^w + \|v\|_w^w \big) \\ &\geq \left(1 - \frac{\lambda - \widehat{\lambda}}{\widehat{\lambda}}\right) \Big(\frac{\alpha+1}{p}\, \underline{\mu}\, \|\Delta u\|_p^p + \frac{\beta+1}{p}\, \underline{\nu}\, \|\Delta v\|_p^p \Big) \\ &\quad - C_4 \big(\|\Delta u\|_p^w + \|\Delta v\|_p^w \big), \end{aligned}$$

where

$$\underline{\mu} = \min_{x \in \overline{\Omega}} |\mu(x)| > 0, \quad \underline{\nu} = \min_{x \in \overline{\Omega}} |\nu(x)| > 0, \tag{26}$$

and C_4 is a positive constant independent of (u, v).

Since $w > p$, we see that $I(u, v) \geq 0$ for $\rho > 0$ sufficiently small. Hence, we have proved that I has a local linking at 0. This completes the proof of the lemma. □

Lemma 6. *Any bounded sequence $\{(u_n, v_n)\} \subset X$ such that $I'(u_n, v_n) \to 0$ in X^* has a convergent subsequence.*

Proof. Let $\{(u_n, v_n)\} \subset X$ be bounded and satisfy $I'(u_n, v_n) \to 0$ in X^*. Then, by the reflexivity of X, passing to a subsequence if necessary (which we do not relabel), we may assume that

$$(u_n, v_n) \rightharpoonup (u, v) \quad \text{in } X.$$

We shall show the convergence is actually strong in X. We need the standard pointwise monotonicity estimate (see, for example (2.2) in [23]): For every $k > 1$, there exists a constant $D_k > 0$ (depending only on k) such that for all vectors $\xi, \eta \in \mathbb{R}^m$ (in our application $m = 1$) we have

$$\big(|\xi|^{k-2}\xi - |\eta|^{k-2}\eta\big) \cdot (\xi - \eta) \geq D_k \cdot \begin{cases} |\xi - \eta|^k, & k \geq 2, \\ (|\xi| + |\eta|)^{k-2} |\xi - \eta|^2, & 1 < k < 2. \end{cases} \tag{27}$$

For any index n, we can write the identity

$$\begin{aligned} &\left\langle \Phi'(u_n, v_n) - \Phi'(u, v),\, (u_n - u, v_n - v)\right\rangle \\ &\quad = \left\langle I'(u_n, v_n) - I'(u, v),\, (u_n - u, v_n - v)\right\rangle \\ &\qquad\quad + \lambda\left\langle \Psi'(u_n, v_n) - \Psi'(u, v),\, (u_n - u, v_n - v)\right\rangle \\ &\qquad\quad + \left\langle N'(u_n, v_n) - N'(u, v),\, (u_n - u, v_n - v)\right\rangle. \end{aligned} \tag{28}$$

Since $I'(u_n, v_n) \to 0$ in X^* and $(u_n - u, v_n - v)$ is uniformly bounded in X, we have

$$\left\langle I'(u_n, v_n) - I'(u, v),\, (u_n - u, v_n - v)\right\rangle \to 0 \quad \text{as } n \to \infty. \tag{29}$$

Under our standing hypotheses the maps

$$\Psi' : X \to X^*, \qquad N' : X \to X^*$$

are compact (Nemytskii operators built from $a, b, c \in L^\infty(\Omega)$ and from the F-data ($F \in \mathcal{A}$)). Concretely:

• Boundedness of $\{(u_n, v_n)\}$ in X implies, by the Rellich–Kondrachov theorem, that (up to a subsequence) $u_n \to u$ strongly in $L^r(\Omega)$ and $v_n \to v$ strongly in $L^s(\Omega)$ for every exponent r, s strictly below the Sobolev critical exponents associated to $W^{2,p}(\Omega)$.

• The pointwise (or strong $L^r(\Omega)$) convergence together with the growth assumptions on the nonlinearities yields, by dominated convergence and standard continuity of Nemytskii maps, that each of the coefficient-type terms

$$\begin{gathered} a(x)|u_n|^{p-2}u_n, \quad b(x)|v_n|^{p-2}v_n, \\ c(x)|u_n|^{\alpha-1}u_n|v_n|^{\beta+1}, \quad c(x)|u_n|^{\alpha+1}|v_n|^{\beta-1}v_n \end{gathered}$$

converges strongly in the appropriate Lebesgue spaces which embed continuously into X^* via the standard duality pairings with ϕ, ψ. Therefore, $\Psi'(u_n, v_n)$ has a strongly convergent subsequence in X^*. The same reasoning applies to $N'(u_n, v_n)$ (using the growth/regularity of F_u, F_v).

Hence, by passing to a further subsequence if necessary, we may assume

$$\Psi'(u_n, v_n) \to T \quad \text{in } X^* \quad \text{and} \quad N'(u_n, v_n) \to S \quad \text{in } X^*$$

for some $T, S \in X^*$. Then, since $(u_n - u, v_n - v)$ is uniformly bounded in X, it follows that

$$\left\langle \Psi'(u_n, v_n) - \Psi'(u, v),\, (u_n - u, v_n - v)\right\rangle \to 0 \tag{30}$$

and

$$\big\langle N'(u_n, v_n) - N'(u, v),\, (u_n - u, v_n - v)\big\rangle \to 0 \tag{31}$$

as $n \to \infty$.

Now, from (28)–(31) we obtain that

$$\big\langle \Phi'(u_n, v_n) - \Phi'(u, v),\, (u_n - u, v_n - v)\big\rangle \to 0 \qquad \text{as } n \to \infty. \tag{32}$$

From (9), we see that

$$\begin{aligned}
&\big\langle \Phi'(u_n, v_n) - \Phi'(u, v),\, (u_n - u, v_n - v)\big\rangle \\
&\quad = (\alpha+1)\int_\Omega \mu(x)\big(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u\big)\cdot(\Delta u_n - \Delta u)\,dx \\
&\qquad + (\beta+1)\int_\Omega \nu(x)\big(|\Delta v_n|^{p-2}\Delta v_n - |\Delta v|^{p-2}\Delta v\big)\cdot(\Delta v_n - \Delta v)\,dx.
\end{aligned}$$

Therefore, by (27), (32), and the fact that $\nu \in C(\overline{\Omega})$ with $\mu, \nu > 0$ on $\overline{\Omega}$, we deduce that

$$\int_\Omega \Theta_p(\Delta u_n, \Delta u)\,dx \to 0 \quad \text{and} \quad \int_\Omega \Theta_p(\Delta v_n, \Delta v)\,dx \to 0 \tag{33}$$

as $n \to \infty$, where

$$\Theta_p(\xi, \eta) = \begin{cases} |\xi - \eta|^p, & p \geq 2, \\ (|\xi| + |\eta|)^{p-2}|\xi - \eta|^2, & 1 < p < 2. \end{cases}$$

We analyze two cases:

Case A: $p \geq 2$. Then $\Theta_p(\Delta u_n, \Delta u) = |\Delta u_n - \Delta u|^p$. From (33), we see that $\{\Delta u_n\}$ converges strongly in $L^p(\Omega)$ to Δu. Thus $\Delta u_n \to \Delta u$ strongly in $L^p(\Omega)$. The same argument gives $\Delta v_n \to \Delta v$ in $L^p(\Omega)$.

Case B: $1 < p < 2$. In this case, $\Theta_p(\xi, \eta) = (|\xi| + |\eta|)^{p-2}|\xi - \eta|^2$. From (33), we know that

$$A_n := \int_\Omega (|\Delta u_n| + |\Delta u|)^{p-2}|\Delta u_n - \Delta u|^2\,dx \to 0 \quad \text{as } n \to \infty.$$

We claim that this implies $\Delta u_n \to \Delta u$ strongly in $L^p(\Omega)$. To see this, write

$$|\Delta u_n - \Delta u|^p = \big((|\Delta u_n| + |\Delta u|)^{p-2}|\Delta u_n - \Delta u|^2\big)^{p/2} \cdot (|\Delta u_n| + |\Delta u|)^{\frac{p(2-p)}{2}}.$$

Use Hölder's inequality with the conjugate exponents $q = \frac{2}{p}$ and $q' = \frac{2}{2-p}$ (note $q > 1$ because $p < 2$) to get

$$\|\Delta u_n - \Delta u\|_{L^p}^p \leq \Big(\int_\Omega (|\Delta u_n| + |\Delta u|)^{p-2} |\Delta u_n - \Delta u|^2 \, dx \Big)^{p/2} \cdot \Big(\int_\Omega (|\Delta u_n| + |\Delta u|)^p \, dx \Big)^{(2-p)/2}.$$

The second factor is uniformly bounded because $\{\Delta u_n\}$ is bounded in L^p. Hence, there exists $C_5 > 0$ such that

$$\|\Delta u_n - \Delta u\|_{L^p}^p \leq C_5 \, A_n^{p/2}.$$

Since $A_n \to 0$, we have $\|\Delta u_n - \Delta u\|_{L^p} \to 0$. This proves strong convergence also in the subquadratic case $1 < p < 2$.

Finally, since the norm on X is equivalent to the sum of the L^p-norm of Δu and the L^p-norm of Δv, the strong convergence of $\Delta u_n \to \Delta u$ in $L^p(\Omega)$ and $\Delta v_n \to \Delta v$ in $L^p(\Omega)$ yields

$$(u_n, v_n) \to (u, v) \quad \text{in } X.$$

This completes the proof of the lemma. □

Lemma 7. *Assume that either (H2) or (H3) holds. Then, for any* $\lambda \in \big[0, \widehat{\lambda} - \bar{\lambda}\big]$, *the following assertions are true:*

(a) I is coercive on X, that is, $I(u,v) \to \infty$ as $\|(u,v)\|_X \to \infty$;

(b) I satisfies the PS condition.

Proof. (a) We first assume that (H2) holds. From (17) and (22), there exist $\varepsilon > 0$ and $C_6 > 0$ such that

$$F(x,s,t) \leq \big(\lambda_1 + \bar{\lambda} - \widehat{\lambda} - \varepsilon\big) \, K(x,s,t) + C_6$$

for a.e. $x \in \Omega$ and all $(s,t) \in \mathbb{R}^2$.

Also, in view of (11), it is clear that

$$\Phi(u,v) \geq \lambda_1 \Psi(u,v) \quad \text{for all } (u,v) \in X. \tag{34}$$

Thus, for any $(u,v) \in X$, from (8), (20), and (21), we obtain

$$\begin{aligned}
&I(u,v)\\
&\geq \Phi(u,v) - \frac{\lambda}{\lambda_1}\Phi(u,v) - \left(\lambda_1 + \bar{\lambda} - \widehat{\lambda} - \varepsilon\right)\int_\Omega K(x,u,v)\,dx - C_4|\Omega|\\
&= \Phi(u,v) - \frac{\lambda}{\lambda_1}\Phi(u,v) - \left(\lambda_1 + \bar{\lambda} - \widehat{\lambda} - \varepsilon\right)\Psi(u,v) - C_6|\Omega|\\
&\geq \left(1 - \frac{\lambda + \lambda_1 + \bar{\lambda} - \widehat{\lambda} - \varepsilon}{\lambda_1}\right)\Phi(u,v) - C_6|\Omega|\\
&\geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right)\Phi(u,v) - C_6|\Omega|,
\end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure of a set.

Then, by (7), we have

$$I(u,v) \geq \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right)\left(\frac{\alpha+1}{p}\,\underline{\mu}\,\|\Delta u\|_p^p + \frac{\beta+1}{p}\,\underline{\nu}\,\|\Delta v\|_p^p\right) - C_6|\Omega|,$$

where $\underline{\mu}$ and $\underline{\nu}$ are defined in (26). This shows that $I(u,v) \to \infty$ as $\|(u,v)\|_X \to \infty$.

Next, we assume that (H3) holds. Suppose, by contradiction, that J is not coercive. Then there exist a sequence $\{(u_n,v_n)\} \subset X$ and a constant $C_7 > 0$ such that

$$\|(u_n,v_n)\|_X \to \infty \quad \text{and} \quad J(u_n,v_n) \leq C_7.$$

Define the normalized sequence

$$(\tilde{u}_n,\tilde{v}_n) = \frac{(u_n,v_n)}{\|(u_n,v_n)\|_X}, \quad \|(\tilde{u}_n,\tilde{v}_n)\|_X = 1.$$

By reflexivity and compact embeddings, up to a subsequence, we have

$$(\tilde{u}_n,\tilde{v}_n) \rightharpoonup (\tilde{u}_0,\tilde{v}_0) \quad \text{in } X,$$

$$(\tilde{u}_n,\tilde{v}_n) \to (\tilde{u}_0,\tilde{v}_0) \quad \text{in } L^p(\Omega) \times L^p(\Omega),$$
$$(\tilde{u}_n,\tilde{v}_n)(x) \to (\tilde{u}_0,\tilde{v}_0)(x) \quad \text{for a.e. } x \in \Omega.$$

For any $\rho \geq 0$, write

$$F(x,\rho\tilde{u}_n,\rho\tilde{v}_n) = \left(\lambda_1 + \bar{\lambda} - \widehat{\lambda}\right)K(x,\rho\tilde{u}_n,\rho\tilde{v}_n) + G(x,\rho\tilde{u}_n,\rho\tilde{v}_n), \tag{35}$$

where K is defined by (21) and

$$G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)=F(x,\rho\tilde{u}_n,\rho\tilde{v}_n)-\left(\lambda_1+\bar{\lambda}-\widehat{\lambda}\right)K(x,\rho\tilde{u}_n,\rho\tilde{v}_n).$$

Then, we have

$$\begin{aligned}&\rho\frac{dF(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{d\rho}-pF(x,\rho\tilde{u}_n,\rho\tilde{v}_n)\\&=\rho\frac{dG(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{d\rho}-pG(x,\rho\tilde{u}_n,\rho\tilde{v}_n).\end{aligned}\tag{36}$$

Moreover, for a.e. $x\in\Omega$, in view of (23), (24), and (36), we derive that

$$\lim_{\rho\to\infty}\frac{G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{K(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}=0\tag{37}$$

and

$$\lim_{\rho\to\infty}\left(\rho\frac{dG(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{d\rho}-pG(x,\rho\tilde{u}_n,\rho\tilde{v}_n)\right)=\infty.\tag{38}$$

Note from (A1) and (21) that

$$\begin{aligned}\frac{d}{d\rho}\left(\frac{G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{K(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}\right)&=\frac{d}{d\rho}\left(\frac{G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{\rho^pK(x,\tilde{u}_n,\tilde{v}_n)}\right)\\&=\frac{\rho^p\frac{d}{d\rho}G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)-p\rho^{p-1}G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{\rho^{2p}K(x,\tilde{u}_n,\tilde{v}_n)}\\&=\frac{\rho\frac{d}{d\rho}G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)-pG(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{\rho^{p+1}K(x,\tilde{u}_n,\tilde{v}_n)}.\end{aligned}$$

For any $M>0$, (38) implies that there exists $R_M>0$ such that for all $\rho\geq R_M$,

$$\rho\frac{dG(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{d\rho}-pG(x,\rho\tilde{u}_n,\rho\tilde{v}_n)\geq M.$$

Then, we have

$$\frac{d}{d\rho}\left(\frac{G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{K(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}\right)\geq\frac{M}{\rho^{p+1}K(x,\tilde{u}_n,\tilde{v}_n)}.$$

Integrating the above inequality over the interval $[\rho,T]\subset[R_M,\infty)$ yields that

$$\frac{G(x,T\tilde{u}_n,T\tilde{v}_n)}{K(x,T\tilde{u}_n,T\tilde{v}_n)}-\frac{G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{K(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}\geq\frac{M}{pK(x,\tilde{u}_n,\tilde{v}_n)}\left(\frac{1}{\rho^p}-\frac{1}{T^p}\right).$$

Let $T \to \infty$. Using (37), we obtain that

$$\frac{G(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}{K(x,\rho\tilde{u}_n,\rho\tilde{v}_n)} \leq -\frac{M}{p\rho^p K(x,\tilde{u}_n,\tilde{v}_n)} = -\frac{M}{pK(x,\rho\tilde{u}_n,\rho\tilde{v}_n)}.$$

Thus,

$$G(x,\rho\tilde{u}_n,\rho\tilde{v}_n) \leq -\frac{M}{p} \quad \text{for all } \rho \geq R_M.$$

By the arbitrariness of $M > 0$, we have

$$\lim_{\rho\to\infty} G(x,\rho\tilde{u}_n,\rho\tilde{v}_n) = -\infty \tag{39}$$

Choosing $\rho = \|(u_n,v_n)\|_X$ in (35) and (39), we obtain that

$$F(x,u_n,v_n) = \left(\lambda_1 + \bar{\lambda} - \widehat{\lambda}\right) K(x,u_n,v_n) + G(x,u_n,v_n)$$

and

$$\lim_{\rho\to\infty} G(x,u_n,v_n) = -\infty.$$

Hence, in view of (34) and the fact that $\lambda \in [0, \widehat{\lambda} - \bar{\lambda}]$, we see thst

$$\begin{aligned}
I(u_n,v_n) &= \Phi(u_n,v_n) - \lambda\Psi(u_n,v_n) - N(u_n,v_n)\\
&\geq \Phi(u_n,v_n) - \lambda\Psi(u_n,v_n) - \left(\lambda_1 + \bar{\lambda} - \widehat{\lambda}\right)\int_\Omega K(x,u_n,v_n)\,dx\\
&\qquad - \int_\Omega G(x,u_n,v_n)\,dx\\
&= \Phi(u_n,v_n) - \left(\lambda + \lambda_1 + \bar{\lambda} - \widehat{\lambda}\right)\Psi(u_n,v_n) - \int_\Omega G(x,u_n,v_n)\,dx\\
&\geq \Phi(u_n,v_n) - \lambda_1\Psi(u_n,v_n) - \int_\Omega G(x,u_n,v_n)\,dx\\
&\geq -\int_\Omega G(x,u_n,v_n)\,dx \to \infty \quad \text{as } n \to \infty.
\end{aligned}$$

This contradicts $I(u_n,v_n) \leq C_7$. Therefore, I is coercive.

(b) The conclusion follows directly from part (a) and Lemma 6. This completes the proof of the lemma. □

Proof of Theorem 1. Let $\lambda \in [0, \widehat{\lambda} - \bar{\lambda}]$. By Lemma 5, I has a local linking at 0 with respect to the decomposition $X = V \oplus W$, where $\dim V = 1 < \infty$. Lemma 7 ensures that I satisfies the Palais–Smale condition and is bounded below. From (20), we have $I(0) = 0$.

We consider two cases:

- If $\inf_{(u,v)\in X} I(u,v) < 0$, then by Lemma 4, I has at least two nontrivial critical points, which correspond to nontrivial weak solutions of (1).
- If $\inf_{(u,v)\in X} I(u,v) \geq 0$, then $I(u,v) = 0$ for all $(u,v) \in V$ with $\|(u,v)\|_X \leq \rho$, where $\rho > 0$ is as in the proof of Lemma 5. In this case, I has infinitely many critical points, so (1) has infinitely many weak solutions.

This completes the proof. □

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