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TWO - FUNCTION EXTENSIONS OF SOME MINIMAX THEOREMS OF RICCERI*

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Dedicated to Prof. Biagio Ricceri on the occasion of his 70th birthday

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Abstract

In this paper, we extend two topological minimax theorems due to Ricceri to the case of two functions.

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1 Introduction and statement of the main results

Let X, Y be two non-empty sets and let φ be a real-valued function on $X \times Y$. Set

$$\varphi_* = \sup_{y \in Y} \inf_{x \in X} \varphi(x, y)$$

and

$$\varphi^* = \inf_{x \in X} \sup_{y \in Y} \varphi(x, y).$$

It is clear that

$$\varphi_* \leq \varphi^*.$$

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This is called the trivial minimax inequality. The opposite inequality

$$\varphi^* \leq \varphi_*$$

is called non-trivial minimax inequality and, of course, it is equivalent to the minimax equality

$$\varphi_* = \varphi^*. \tag{1}$$

Starting from the pioneristic work of von Neumann (see [8]), many results ensuring (1) were established. For an introductory bibliography see, for example, the classical survey of Simons (see [6]).

Now, let $f, g : X \times Y \to \mathbb{R}$, with $f(x, y) \leq g(x, y)$ for every $x \in X, y \in Y$. We call non-trivial minimax inequality involving f, g the following

$$f^* \leq g_*. \tag{2}$$

So, if $f = g$, (2) is equivalent to (1). For a given minimax theorem for one function φ, it is a common fact to see whether it is possible to find a two-function version of it. The most natural way to obtain this is to divide the hypotheses in φ to f and g. For example, the two-function version of the most classical Fan-Sion's theorem (see [7]) (Theorem A below) has been obtained by Simons (see [5], Th. 1.4) (Theorem B below).

Theorem A. *Let X be a nonempty compact convex subset of a topological vector space, Y a nonempty convex subset of a topological vector space, and let $\varphi : X \times Y \to \mathbb{R}$ be quasi-convex and lower semicontinuous in X, and quasi-concave and upper semicontinuous in Y.*

Then, (1) holds.

Theorem B. *Let X be a nonempty compact convex subset of a topological vector space, Y a nonempty convex subset of a topological vector space, let $f : X \times Y \to \mathbb{R}$ be quasi-concave in Y and lower semicontinuous in X, and let $g : X \times Y \to \mathbb{R}$ be upper semicontinuous in Y and quasi-convex in X, with $f \leq g$ on $X \times Y$.*

Then, (2) holds.

First of all, for the reader's convenience, we recall that, if U is a topological space, a function $h : U \to [-\infty, +\infty[$ is said to be relatively inf-compact (see [4]) if, for each $r \in \mathbb{R}$, there exists a compact set $K \subseteq U$ such that $h^{-1}(]-\infty, r[\subseteq K$. We have the following.

Proposition 1. *Let U be a topological space and $h : U \to [-\infty, +\infty[$ a lower semicontinuous and relatively inf-compact function.*

Then, there exists a minimum global point for h.

Proof. Let $\gamma > \inf_U h$, then there exists a compact set $K : \{x \in U : h(x) < \gamma\} \subseteq K$. Let x^* be a global minimum point of h on K. If $x \in U \setminus K$, one has $h(x) \geq \gamma \geq h(x^*)$, so we can conclude that x^* is a minimum global point of h on U.

In [4], Ricceri proved the following results:

Theorem C. *Let X be a topological space, $I \subseteq \mathbb{R}$ an interval and $\Psi : X \times I \to \mathbb{R}$ a continuous function satisfying the following conditions:*

a) for each $\lambda \in I$, the set of all global minima of the function $\Psi(\cdot, \lambda)$ is connected

b) there exists a non-decreasing sequence of compact intervals, $\{I_n\}$, with $I = \cup_{n\in\mathbb{N}} I_n$ such that, for every $n \in \mathbb{N}$, the following conditions are satisfied:

i) the function $\inf_{\lambda\in I_n} \Psi(\cdot, \lambda)$ is relatively inf-compact;

ii) for each $x \in X$, the set of all global maxima of the restriction of the function $\Psi(x, \cdot)$ to I_n is connected.

Under such hypotheses, the function Ψ verifies the condition (1).

Theorem D. *Let X be a topological space, $I \subseteq \mathbb{R}$ a compact interval and $\Psi : X \times I \to \mathbb{R}$ an upper semicontinuous function satisfying the following conditions:*

c) for all $\lambda \in I$, the function $\Psi(\cdot, \lambda)$ is continuous

d) there exists a set $D \subseteq I$, dense in I, such that, for every $\lambda \in D$, the function $\Psi(\cdot, \lambda)$ is inf-connected

e) for each $x \in X$, the set of all global maxima of the function $\Psi(x, \cdot)$ is connected

Under such hypotheses, the function Ψ verifies condition (1).

The aim of the present paper is to establish Theorems 1 and 2 below that represent the extensions of Theorems C and D, respectively, to two functions.

Theorem 1. *Let X be a topological space, $I \subseteq \mathbb{R}$ an interval, and $f, g : X \times I \to \mathbb{R}$ two functions satisfying the following conditions:*

H1) for every $(x, \lambda) \in X \times I$ one has $f(x, \lambda) \leq g(x, \lambda)$;

H2) the function f is upper semicontinuous in $X \times I$ and, for every $\lambda \in I$, $f(\cdot, \lambda)$ is continuous;

H3) the function g is lower semicontinuous in $X \times I$ and, for every $x \in X$, $g(x, \cdot)$ is continuous;

H4) for every $\lambda \in I$, the set $\{y \in X : g(y, \lambda) = \inf_{x\in X} g(x, \lambda)\}$ is connected;

H5) there exists a non-decreasing sequence of compact intervals, $\{I_n\}$*, with* $I = \cup_{n\in\mathbb{N}} I_n$ *such that, for every* $n \in \mathbb{N}$*, the following conditions are satisfied:*

j) the function $x \to \inf_{\lambda\in I_n} g(x,\lambda)$ *is relatively inf-compact;*

jj) for every $x \in X$*, the set* $\{\mu \in I_n : f(x,\mu) = \sup_{\lambda\in I_n} f(x,\lambda)\}$ *is connected.*

Under such hypotheses, (2) holds.

To realize that when $f = g = \Psi$ Theorem 1 gives Theorem C, one has to observe that conditions H3), H4) of Theorem 1 immediately imply conditions a), b) of Theorem C.

Theorem 2. *Let* X *be a topological space,* $I \subseteq \mathbb{R}$ *a compact real interval and* $f, g : X \times I \to \mathbb{R}$ *two functions satisfying the following conditions:*

K1) for every $(x,\lambda) \in X \times I$ *one has* $f(x,\lambda) \leq g(x,\lambda)$*;*

K2) the functions f *is upper semicontinuous in* $X \times I$*, and, for every* $\lambda \in I$*, the function* $f(\cdot,\lambda)$ *is continuous in* X*;*

K3) for every $x \in X$*, the function* $g(x,\cdot)$ *is upper semicontinuous;*

K4) for each $x \in X$*, the set* $\{\mu \in I_n : f(x,\mu) = \sup_{\lambda\in I} f(x,\lambda)\}$ *is connected;*

K5) there exist a dense subset D *of* I *such that, for every* $\lambda \in D$ *and for every* $k \in \mathbb{R}$*, the set* $\{x \in X : g(x,\lambda) < k\}$ *is connected.*

Under such hypotheses, (2) holds.

To realize that when $f = g = \Psi$ Theorem 2 gives Theorem D, one has to observe that conditions K2), K3), K4), K5) of Theorem 2 immediately imply the conditions c), d), e) of Theorem D.

Finally, we recall two results, Theorem E end F below, that will be used to prove our theorems.

Theorem E. ([4], Th. A) *Let* X *be a topological space,* $I \subseteq \mathbb{R}$ *a compact interval and* $S \subseteq X \times I$*. Assume that* S *is connected and its projection on* I *is the whole of* I*. Then, for every upper semicontinuous multifunction* $\Phi : X \to 2^I$*, with non-empty, closed and connected values, the graph of* Φ *intersects* S*.*

Theorem F. ([4], Prop. 1.1) *Let* X, Y *be two topological spaces and let* $f : X \times Y \to \mathbb{R}$ *be a lower semicontinuous function such that* $f(x,\cdot)$ *is continuous for all* $x \in X$*. Moreover, assume that, for each* $y \in Y$*, there exists a neighborhood* V *of* y *such that the function* $\inf_{v\in V} f(\cdot,v)$ *is relatively inf-compact.*

Then, the multifunction $F : Y \to X$ defined by

$$F(x) = \{u \in X : f(u, y) = \inf_{x \in X} f(x, y)\}$$

is upper semicontinuous.

2 Proof of Theorem 1

For every $n \in \mathbb{N}$, let us set

$$g_*(n) = \sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda)$$

and

$$f^*(n) = \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda).$$

Fixed $n \in \mathbb{N}$, let us prove that

$$f^*(n) \leq g_*(n). \tag{3}$$

Let us define a multifunction $G : I_n \to 2^X$ by setting, for $\lambda \in I_n$,

$$G(\lambda) = \{u \in X : g(u, \lambda) = \inf_{x \in X} g(x, \lambda)\}.$$

Observe that, for each $\lambda \in I_n$, the function $g(\cdot, \lambda)$ is relatively inf-compact. In fact, fixed $\lambda \in I_n$ and $r \in \mathbb{R}$, consider the sets $A = \{x \in X : g(x, \lambda) < r\}$ and $B = \{x \in X : \inf_{\mu \in I_n} g(x, \mu) < r\}$ and observe that $A \subseteq B$. Thanks to H5), there exists a compact set K such that $B \subseteq K$, so, $A \subseteq K$. So, $g(\cdot, \lambda)$ is relatively inf-compact, hence, taking into account H3) and Proposition 1, it admits a global minimum point: so, the values of multifunction G are non-empty. Moreover, they are compact and connected thanks to H4). From Theorem F it follows that G is upper semicontinuous: so, its graph is connected (see [1]) . Hence, the set

$$S = gr(G^-) = \{(u, \lambda) \in X \times I_n : g(u, \lambda) = \inf_{x \in X} g(x, \lambda)\}$$

is also connected. Observe that, since all values of G are non-empty, the projection of S on I_n is the whole I_n.

Now, let us define a multifunction $F : X \to 2^{I_n}$ by setting, for $x \in X$,

$$F(x) = \{\mu \in I_n : f(x, \mu) = \sup_{\lambda \in I_n} f(x, \lambda)\}.$$

The values of F are evidently non-empty, and they are compact and connected thanks to jj). From Theorem F it follows that F is upper semicontinuous.

Hence, from Theorem E, there exists $(\bar{x}, \bar{\lambda}) \in S : \bar{\lambda} \in F(\bar{x})$, that is

$$g(\bar{x}, \bar{\lambda}) = \inf_{x \in X} g(x, \bar{\lambda})$$

and

$$f(\bar{x}, \bar{\lambda}) = \sup_{\lambda \in I_n} f(\bar{x}, \lambda).$$

So, we have

$$f^*(n) = \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) \leq \sup_{\lambda \in I_n} f(\bar{x}, \lambda) = f(\bar{x}, \bar{\lambda}) \leq g(\bar{x}, \bar{\lambda})$$

$$= \inf_{x \in X} g(x, \bar{\lambda}) \leq \sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda) = g_*(n).$$

So, (3) is proved. Now, to conclude the proof, arguing by contradiction, suppose that there exists $r : g_* < r < f^*$ and, for every $n \in \mathbb{N}$, let us put

$$C_n = \{x \in X : \sup_{\lambda \in I_n} f(x, \lambda) < r\}.$$

The sets C_n are non-empty: otherwise, one would have, for some $n \in \mathbb{N}$, $r < \sup_{\lambda \in I_n} f(x, \lambda)$ for every $x \in X$, and so

$$r \leq \inf_{x \in X} \sup_{\lambda \in I_n} f(x, \lambda) = f^*(n) \leq g_*(n)$$

$$= \sup_{\lambda \in I_n} \inf_{x \in X} g(x, \lambda) \leq g_* < r.$$

Since the sequence $\{I_n\}$ is increasing, the sequence $\{C_n\}$ is decreasing. Summarizing, taking into account H2), $\{C_n\}$ is a decreasing sequence of non-empty closed and compact sets. So, there exists $x^* \in \cap_{n \in \mathbb{N}} C_n$. Then, for every $n \in \mathbb{N}$, one has $f(x^*, \lambda) \leq r$ for every $\lambda \in I_n$ and, since $I = \cup_{n \in \mathbb{N}} I_n$, one has $f(x^*, \lambda) \leq r$ for every $\lambda \in I$, so $\sup_{\lambda \in I} f(x^*, \lambda) \leq r$ and, finally, $f^* \leq r$, that is absurd.

3 Proof of Theorem 2

Arguing by contradiction, let us fix $r \in \mathbb{R}$ satisfying $g_* < r < f^*$: so, for every $\lambda \in I$, there exists $x \in X$ such that $g(x, \lambda) < r$. Then, the multifunction $G : I \to 2^X$ defined by

$$G(\lambda) = \{x \in X : g(x, \lambda) < r\}$$

has non-empty values; and, thanks to K3), G is lower semicontinuous; moreover, for every $\lambda \in D$, $G(\lambda)$ is connected by K5). Then, by Proposition 5.7 of [3], its graph is connected. Hence, the set

$$S = gr(G^-) = \{(x, \lambda) \in X \times I : g(x, \lambda) < r\}$$

is also connected.

Let us introduce the multifunction $\Phi : X \to 2^I$ by putting

$$\Phi(x) = \{\mu \in I : f(x, \mu) = \sup_{\lambda \in I} f(x, \lambda)\}.$$

The values of Φ are non-empty and closed by K2) and connected by K4). From K2) again, it follows that Φ is upper semicontinuous thanks to Theorem F. So, by Theorem E, there exists $(\bar{x}, \bar{\lambda}) : \bar{\lambda} \in \Phi(\bar{x})$ and so

$$g(\bar{x}, \bar{\lambda}) < r < f^* < \sup_{\lambda \in I} f(\bar{x}, \lambda) = f(\bar{x}, \bar{\lambda}),$$

that is absurd.

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