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FIXED POINT THEORY FOR EXTENSION TYPE SPACES WITH RESPECT TO A SELECTION MAP*

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Dedicated with much admiration to Prof. Biagio Ricceri on the occasion of his 70th birthday

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Abstract

In this paper, we present new fixed point results for multivalued maps on extension spaces with respect to a map.

Keywords: fixed points, set-valued maps, extension spaces.
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1 Introduction

In this paper we consider two general classes of maps (motivated from the KLU [13], $HLPY$ [14] and Scalzo [20] maps) which have a selection property and we present new fixed point results. In particular we establish fixed point theorems in a variety of settings for extension type spaces with respect to a map. Our results improve and complement results in the literature; see [5, 9, 16–18] and the references therein. Note our theorems include as a special case results for ES(compact) and AES(compact) spaces.

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First we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map, that is, p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \stackrel{p}{\leftarrow} \Gamma \stackrel{q}{\to} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [10]. An upper semicontinuous map $\phi : X \to 2^Y$ (nonempty subsets of Y) with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. Another example is an acyclic map which we now describe. Let X and Z be subsets of Hausdorff topological spaces and let $F : X \to K(Z)$ i.e. F has nonempty compact values. Recall a nonempty topological space is said to be a acyclic if all its reduced Čech homology groups over the rationals are trivial. Now we consider maps $F : X \to Ac(Z)$ i.e. $F : X \to K(Z)$ with acyclic values (i.e. F has nonempty acyclic compact values). We say $F \in AC(X, Z)$ (i.e. F is an acyclic map) if $F : X \to Ac(Z)$ is upper semicontinuous.

Next we consider a general class of maps, namely the PK maps of Park (which include Kak and Ad maps). Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \; Fix\, F \neq \emptyset \;\text{ for all }\; F \in \mathcal{X}(Z,Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathcal{U}$ of maps is defined by the following properties:

(i). $\mathcal{U}$ contains the class C of single valued continuous functions;

(ii). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

(iii). $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \; \|x\| \leq 1\}$.

We say $F \in PK(X,Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Now we present some properties of PK maps which we will use in Section 2.

(i). PK is closed under compositions.

Let X, Y, Z be Hausdorff topological spaces. Let $F_1 \in PK(X,Y)$ and $F_2 \in PK(Y,Z)$. Suppose K is a compact subset of X. Then there exists a $G_1 \in \mathcal{U}_c(K,Y)$ with $G_1(x) \subseteq F_1(x)$ for each $x \in K$. Note $G_1(K)$ is compact so there exists a $G_2 \in \mathcal{U}_c(G_1(K),Z)$ with $G_2(y) \subseteq F_2(y)$ for each $y \in G_1(K)$. As a result $G_2\, G_1(x) \subseteq F_2\, G_1(x) \subseteq F_2\, F_1(x)$ for $x \in K$ and note $G_2\, G_i \in \mathcal{U}_c(K,Z)$.

(ii). Let $F \in PK(X,Y)$ and $Z \subseteq X$. Then $F \in PK(Z,Y)$.

This follows if we consider $F \circ i$ where $i : Z \to X$ is the inclusion. Alternatively, let K be compact in Z. Then K is compact in X so there exists a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

(iii). Let $F \in PK(X,Y)$ and $F(X) \subseteq W \subseteq Y$. Then $F \in PK(X,W)$.

Suppose K is a compact subset of X. Then there exists a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Now since $F(K) \subseteq W$ then $G(K) \subseteq W$. Let $\Psi : K \to 2^W$ be obtained by restricting the range of G and let Ω be open in W. Then $\Omega = W \cap U$ for some open set U of Y. Now since $G(K) \subseteq W$ then $\{x \in K : \Psi(x) \subseteq \Omega\} = \{x \in K : G(x) \subseteq U\}$ which is open in K. Thus $\Psi : K \to 2^W$ is upper semicontinuous so $\Psi\,(= G) \in \mathcal{U}_c(K,W)$.

Next we recall the following fixed point result for PK maps (see [19]). Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

Theorem 1. *Let X be an admissible convex set in a Hausdorff topological vector space and $F \in PK(X,X)$ be a closed compact map. Then F has a fixed point in X.*

Recall the Tychonoff cube T is the Cartesian product of copies of the unit interval and T lies in an appropriate locally convex topological vector space E (i.e. the linear span of the Tychonoff cube) [8, 9] (recall the product of real lines that contain T equipped with the product topology is a locally convex topological vector space). Note since any convex subset of a locally convex topological vector space is admissible then T is a convex admissible subset of E. Now Theorem 1 guarantees the following theorem (see also [16] for another proof).

Theorem 2. *Let $F \in PK(T,T))$ be a closed map. Then F has a fixed point in T.*

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$. Of course, given two single valued maps $f, g : X \to Y$ and $\alpha \in Cov(Y)$, then f and g are α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We now recall the following result from [3, 5].

Theorem 3. *Let X be a regular topological space, $F : X \to 2^X$ an upper semicontinuous map with closed values and suppose there exists a cofinal covering $\theta \subseteq Cov_X(\overline{F(X)})$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1. *From Theorem 3 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [4, page 298] to prove the existence of approximate fixed points* *(since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [12, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular [7]). Also note in Theorem 3 if F is compact valued, then the assumption that X is regular can be removed. We note here that when we apply Theorem 3 we will assume the space is uniform. Of course one could consider other appropriate spaces (like regular (Hausdorff) spaces) as well.*

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov\,(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0.

Next we describe the maps due to Wu [21]. Let X and Y be subsets lying in Hausdorff topological vector spaces and we say $\Phi \in W(X, Y)$ if $\Phi : X \to 2^Y$ and there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Next, we recall a selection theorem [1] (see the proof in Theorem 1.1 there) for Wu maps.

Theorem 4. *Let X be a paracompact subset of a Hausdorff topological vector space and Y a metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $\Phi \in W(X, Y)$ and let $\theta : X \to 2^Y$ be a lower semicontinuous map with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Then there exists a upper semicontinuous map $\Psi : X \to CK(Y)$ (collection of nonempty convex compact subsets of Y) with $\Psi(x) \subseteq \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Remark 2. *Let X be paracompact and Y a metrizable subset of a complete Hausdorff locally convex linear topological space E and $\Phi \in W(X, Y)$ with $\theta : X \to 2^Y$ a lower semicontinuous map and $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Note [15] that $\overline{co}\,\theta : X \to 2^Y$ (since $\overline{co}\,(\theta(x)) \subseteq \Phi(x) \subseteq Y$ for $x \in X$) is lower semicontinuous, so from Michael's selection theorem there exists a continuous (single valued) map $f : X \to Y$ with $f(x) \in \overline{co}\,(\theta(x))$ for $x \in X$, so consequently $f(x) \in \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z, W)$ [14] if W is convex and there exists a map $S : Z \to W$

(i.e., $S : Z \to P(W)$ (collection of subsets of W)) with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\, int\, S^{-1}(w) :\; w \in W\}$; here $S^{-1}(w) = \{z \in Z :\; w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\, S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. For the selection theorem below, see [14].

Theorem 5. *Let X be a paracompact subset of a Hausdorff topological space, Y a convex subset of a Hausdorff topological vector space and $F \in HLPY(X, Y)$ (let $S : X \to 2^Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup\{\, int\, S^{-1}(w) :\; w \in Y\}$). Then there exists a continuous (single–valued) map $f : X \to Y$ with $f(x) \in co\, S(x) \subseteq F(x)$ for all $x \in X$.*

Remark 3. *These maps are related to the DKT maps in the literature and $F \in DKT(Z, W)$ [6] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the Fan maps.*

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $T : X \to 2^Y$ has the strong continuous inclusion property (SCIP) [13] at $x \in X$ if there exists an open set $U(x)$ in X containing x and a $F^x : U(x) \to 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $co\, F^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. We write $T \in KLU(X, Y)$ if T has the SCIP at every $x \in X$.

In this paper our map T will be a compact map so T has the SCIP is equivalent to T has the CIP [11].

Remark 4. *These maps contain as a special case the Scalzo maps [20] in the literature (see [13 pg12]).*

Next we recall a selection theorem [13].

Theorem 6. *Let X be a paracompact subset of a Hausdorff topological space, Y a subset of a Hausdorff topological vector space and $T \in KLU(X, Y)$. Then there exists an upper semicontinuous map $G : X \to CK(Y)$ with $G(w) \subseteq co\, T(w)$ for all $w \in X$.*

2 Fixed point theory

We begin by describing the two general classes of maps. Let X be a subset of a Hausdorff topological space, and Y a subset of a Hausdorff topological vector space. We say $F \in HYPK(X,Y)$ if $F : X \to 2^Y$ and there exists an upper semicontinuous map $\Phi \in PK(X,Y)$ with compact values and with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$.

Let X be a subset of a Hausdorff topological space, and Y a subset of a Hausdorff topological space. We say $F \in HYPKC(X,Y)$ if $F : X \to 2^Y$ and there exists an upper semicontinuous map $\Phi \in PK(X,Y)$ with compact values and with $\Phi(x) \subseteq F(x)$ for $x \in X$.

Now we describe one of the spaces considered in this paper. Let X be a subset of a Hausdorff topological space, and $\Phi : X \to 2^X$ (to be described later).

Definition 1. *We say $X \in GES(compact)$ (w.r.t. Φ) if for any compact subset Z of a Hausdorff topological space and $A \subseteq Z$ closed in Z, and any homeomorphism $g : \overline{\Phi(X)} \to A$ there exists an upper semicontinuous map $\Psi \in PK(Z, \overline{\Phi(X)})$ with compact values and with $\Psi(x) \subseteq \Phi\, g^{-1}(x)$ for $x \in A$.*

Remark 5. *We note that we could replace $\overline{\Phi(X)}$ everywhere in this paper by a set K where $\overline{\Phi(X)} \subseteq K \subseteq X$. In particular, if X was compact in our theorems, one could replace $\overline{\Phi(X)}$ with X throughout.*

Example 1. *Let X be a subset of a Hausdorff topological space and $\Phi \in PK(X,X)$ be an upper semicontinuous map with compact values. Assume either (i). $\overline{\Phi(X)} \in ES(compact)$ or (ii). $X \in ES(compact)$. Then $X \in GES(compact)$ (w.r.t. Φ).*

To see this, let Z be a compact subset of a Hausdorff topological space and $A \subseteq Z$ closed in Z and let $g : \overline{\Phi(X)} \to A$ be a homeomorphism. Then $g^{-1} : A \to \overline{\Phi(X)}$ is continuous.

(i). Suppose $\overline{\Phi(X)} \in ES(compact)$.

Then g^{-1} extends to a continuous function $h : Z \to \overline{\Phi(X)}$ (note $h|_A = g^{-1}$) i.e. $h \in C(Z, \overline{\Phi(X)})$. Let $\Psi = \Phi\, h$. Note $\Phi \in PK(X,X)$ and (see Section 1) so $\Phi \in PK(\overline{\Phi(X)}, \overline{\Phi(X)})$ and as a result $\Psi \in PK(Z, \overline{\Phi(X)})$ is an upper semicontinuous map with compact values. Also, since $h|_A = g^{-1}$, for $x \in A$ we have $\Psi(x) = \Phi\, h(x) = \Phi\, g^{-1}(x)$.

(ii). Suppose $X \in ES(compact)$.

Note $g^{-1} : A \to X$ is continuous. Now, since $X \in ES(compact)$ then g^{-1} extends to a continuous function $h : Z \to X$ (note $h|_A = g^{-1}$), i.e. $h \in C(Z,X)$. Let $\Psi = \Phi\, h$. Note $\Phi \in PK(X,X)$ and (see Section 1) so $\Phi \in PK(X, \overline{\Phi(X)})$ and as a result $\Psi \in PK(Z, \overline{\Phi(X)})$ is an upper semicontinuous map with compact values. Also, since $h|_A = g^{-1}$, for $x \in A$ we have $\Psi(x) = \Phi\, h(x) = \Phi\, g^{-1}(x)$.

Theorem 7. *Let X be a subset of a Hausdorff topological vector space and $F \in HYPK(X,X)$ with $co\,F$ a compact map (so in particular there exists an upper semicontinuous compact map $\Phi \in PK(X,X)$ with compact values and with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Also assume $X \in GES(compact)$ (w.r.t. Φ). Then Φ (so consequently $co\,F$) has a fixed point.*

Proof. Let Φ be as in the statement of Theorem 7. Recall [9] every compact space is homeomorphic to a closed subset of the Tychonoff cube T, so as a result $K = \overline{\Phi(X)}$ can be embedded as a closed subset $K^\star$ of T; let $s : K \to K^\star$ be a homeomorphism. Since $X \in GES(\text{compact})$ (w.r.t. Φ) then there exists an upper semicontinuous map $\Psi \in PK(T,K)$ with compact values and with $\Psi(x) \subseteq \Phi\, s^{-1}(x)$ for $x \in K^\star$. Let $G = j\, s\, \Psi$, where $j : K^\star \hookrightarrow T$ is an inclusion. Note $G \in PK(T,T)$ is an upper semicontinuous compact map with compact values, so a closed map [2]. Now, Theorem 2 guarantees an $x \in T$ with $x \in G(x)$. Thus, there exists a $y \in \Psi(x)$ with $x = j\, s(y)$. Note $s(y) \in K^\star$ so $\Psi(x) \subseteq \Phi\, s^{-1}(x) = \Phi(y)$. As a result $y \in \Psi(x) \subseteq \Phi(y)$, i.e. $y \in \Phi(y) \subseteq co\,F(y)$.

The analogue of Theorem 7 for $HYPKC(X,X)$ maps is now immediate (note here we do not need to assume that X is a subset of a Hausdorff topological vector space but merely that X is a subset of a Hausdorff topological space).

Theorem 8. *Let X be a subset of a Hausdorff topological space and $F \in HYPKC(X,X)$ with F a compact map (so, in particular, there exists an upper semicontinuous compact map $\Phi \in PK(X,X)$ with compact values and with $\Phi(x) \subseteq F(x)$ for $x \in X$). Also assume $X \in GES(compact)$ (w.r.t. Φ). Then Φ (consequently, F) has a fixed point.*

Proof. Let Φ be as in the statement of Theorem 8. Now exactly the same argument as in Theorem 7 yields $y \in \Phi(y)$ so $y \in \Phi(y) \subseteq F(y)$.

A special case of Theorem 8 is the following (take $\Phi = F$).

Theorem 9. *Let X be a subset of a Hausdorff topological space and let $F \in PK(X,X)$ be an upper semicontinuous compact map with compact values. Also assume $X \in GES(compact)$ (w.r.t. F). Then F has a fixed point.*

Let X be a subset of a Hausdorff topological space and $\Phi : X \to 2^X$.

Definition 2. *We say $X \in GAES(compact)$ (w.r.t. Φ) if for any compact subset Z of a Hausdorff topological space and $A \subseteq Z$ closed in Z, any homeomorphism $g : \overline{\Phi(X)} \to A$, and any $\alpha \in Cov_X(\overline{\Phi(X)})$, there exists an upper semicontinuous map $\Psi_\alpha \in PK(Z, \overline{\Phi(X)})$ with compact values such that if $x \in A$ with $x \in g\,\Psi_\alpha(x)$ then Φ has an α–fixed point.*

Example 2. *Let X be a subset of a Hausdorff topological space and $\Phi \in PK(X, X)$ be an upper semicontinuous map with compact values. Assume either (i). $X \in AES(compact)$ or (ii). $\overline{\Phi(X)} \in AES(compact)$. Then $X \in GAES(compact)$ (w.r.t. Φ).*

To see this, let Z be a compact subset of a Hausdorff topological space and $A \subseteq Z$ closed in Z. Let $\alpha \in Cov_X(\overline{\Phi(X)})$ and let $g : \overline{\Phi(X)} \to A$ be a homeomorphism. Note $g^{-1} : A \to \overline{\Phi(X)}$ is continuous.

(i). Suppose $X \in AES(compact)$.

Now, $g^{-1} : A \to X$ is continuous. Let $\alpha' = \alpha \cup \{X \backslash \overline{\Phi(X)}\}$. Now, since $X \in AES(compact)$ there exists a continuous function $h_\alpha : Z \to X$ with $h_\alpha|_A$ α'–close to g^{-1}. Then $h_\alpha\, g : \overline{\Phi(X)} \to X$ and $i : \overline{\Phi(X)} \to X$ are α–close. To see this, let $x \in \overline{\Phi(X)} = K$. Now, let $y = g(x)$ and note $y \in A$. Then there exists a $V \in \alpha'$ with $g^{-1}(y) \in V$ and $h_\alpha(y) \in V$, i.e. $x \in V$ and $h_\alpha\, g(x) \in V$. Now, since $x \in K$ and $\alpha' = \alpha \cup \{X \backslash \overline{\Phi(X)}\}$ then $V \in \alpha$. Now let $\Psi_\alpha = \Phi\, h_\alpha$ and since $h_\alpha \in C(Z, X)$ and $\Phi \in PK(X, \overline{\Phi(X)})$ then $\Psi_\alpha \in PK(Z, \overline{\Phi(X)})$ is an upper semicontinuous map with compact values. Now suppose we had $x \in A$ and $x \in g\,\Psi_\alpha(x)$. Then $x \in g\,\Phi\, h_\alpha(x)$. Let $y = h_\alpha(x)$ so $y \in h_\alpha g\, \Phi(y)$, i.e. $y = h_\alpha g(w)$ for some $w \in \Phi(y)$. Now, since $y \in \overline{\Phi(X)}$ then there exists a $U \in \alpha$ with $h_\alpha g(w) \in U$ and $w \in U$, i.e. $y (= h_\alpha g(w)) \in U$ and $w \in U$. Thus $y \in U$ and $\Phi(y) \cap U \neq \emptyset$ (since $w \in U$ and $w \in \Phi(y)$). Thus, Φ has an α–fixed point.

(ii). Suppose $\overline{\Phi(X)} \in AES(compact)$.

Then there exists a continuous function $h_\alpha : Z \to \overline{\Phi(X)}$ with $h_\alpha|_A$ α–close to g^{-1}. Then, $h_\alpha\, g : \overline{\Phi(X)} \to \overline{\Phi(X)}$ and $i : \overline{\Phi(X)} \to \overline{\Phi(X)}$ are α–close. Now let $\Psi_\alpha = \Phi\, h_\alpha$ and note $\Psi_\alpha \in PK(Z, \overline{\Phi(X)})$ is an upper semicontinuous map with compact values. Now suppose we had $x \in A$ and $x \in g\,\Psi_\alpha(x)$. Let $y = h_\alpha(x)$ so $y = h_\alpha g(w)$ for some $w \in \Phi(y)$. Then there exists a $U \in \alpha$ with $h_\alpha g(w) \in U$ and $w \in U$, i.e. $y \in U$ and $w \in U$. Thus, $y \in U$ and $\Phi(y) \cap U \neq \emptyset$, i.e. Φ has an α-fixed point.

Remark 6. *In Definition 2 if we assume (a). for each $x \in A$ there exists a $U \in \alpha$ with $\Psi_\alpha(x) \subseteq U$ and $\Phi\, g^{-1}(x) \cap U \neq \emptyset$, then we immediately have (b). if $x \in A$ with $x \in g\,\Psi_\alpha(x)$ then Φ has an α–fixed point.*

To see this,let $x \in A$ with $x \in g\,\Psi_\alpha(x)$. Then there exists a $y \in \Psi_\alpha(x)$ with $x = g(y)$ and note $g(y) \in A$ so $x \in A$. Now, (a) implies that there exists *a* $U \in \alpha$ *with* $\Psi_\alpha(x) \subseteq U$ *and* $\Phi\, g^{-1}(x) \cap U \neq \emptyset$. *Thus, since* $y \in \Psi_\alpha(x)$ *we have* $y \in U$ *and* $\Phi(y) \cap U \neq \emptyset$ *(recall* $g^{-1}(x) = y$*), so* Φ *has an* α*–fixed point.*

In fact, in the above proof we just need a condition to guarantee "for each $x \in A$ *and each* $y \in X$ *with* $y \in \Psi_\alpha(x)$ *and* $x = g(y)$ *there exists a* $U \in \alpha$ *with* $y \in U$ *and* $\Phi\, g^{-1}(x) \cap U \neq \emptyset$*".*

(ii). In (i) if $\Phi = \phi$ *and* $\Psi_\alpha = \psi_\alpha$ *are single valued maps then (a) reads "*ψ_α *and* $\phi\, g^{-1}$ *are* α*–close".*

Theorem 10. *Let* X *be a subset of a Hausdorff topological vector space and* $F \in HYPK(X,X)$ *with* $co\, F$ *a compact map (so in particular there exists an upper semicontinuous compact map* $\Phi \in PK(X,X)$ *with compact values and with* $\Phi(x) \subseteq co\,(F(x))$ *for* $x \in X$*). Also assume* $X \in GAES(compact)$ *(w.r.t.* Φ*). Then* Φ *(so consequently* $co\, F$*) has a fixed point.*

Proof. Let Φ be as in the statement of Theorem 10. Let $\alpha \in Cov_X\,(K)$ where $K = \overline{\Phi(X)}$. Now K can be embedded as a closed subset $K^\star$ of T and let $s : K \to K^\star$ be a homeomorphism. Since $X \in GAES$(compact) (w.r.t. Φ), then there exists an upper semicontinuous map $\Psi_\alpha \in PK(T,K)$ with compact values such that if $x \in K^\star$ with $x \in s\,\Psi_\alpha(x)$ then Φ has an α–fixed point. Let $G_\alpha = j\, s\, \Psi_\alpha$, where $j : K^\star \hookrightarrow T$ is an inclusion. Note $G_\alpha \in PK(T,T)$ is an upper semicontinuous compact map with compact values, so a closed map. Now, Theorem 2 guarantees an $x \in T$ with $x \in G_\alpha(x)$, i.e. $x \in s\,\Psi_\alpha(x)$. From the above Φ has an α– fixed point (for each $\alpha \in Cov_X\,(K)$). Now, Theorem 3 and Remark 1 (note Hausdorff topological vector spaces are uniform spaces) guarantee that Φ (so consequently $co\, F$) has a fixed point.

The analogue of Theorem 10 for $HYPKC(X,X)$ maps is now immediate.

Theorem 11. *Let* X *be a subset of a Hausdorff topological space and* $F \in HYPKC(X,X)$ *with* F *a compact map (so in particular there exists an upper semicontinuous compact map* $\Phi \in PK(X,X)$ *with compact values and with* $\Phi(x) \subseteq F(x)$ *for* $x \in X$*). Also, assume* $X \in GAES(compact)$ *(w.r.t.* Φ*). Then* Φ *(consequently,* F*) has a fixed point.*

A special case of Theorem 11 is the following (take $\Phi = F$).

Theorem 12. *Let* X *be a subset of a Hausdorff topological space and let* $F \in PK(X,X)$ *be an upper semicontinuous compact map with compact* *values. Also assume* $X \in GAES(compact)$ *(w.r.t.* F*). Then* F *has a fixed point.*

Now we generalize the above results by considering admissible sets as in Section 1. Let W be a subset of a Hausdorff topological space and $\Phi : W \to 2^W$.

Definition 3. *We say* W *is GES admissible (w.r.t.* Φ*) if for all compact subsets* K *of* W*, all* $\alpha \in Cov_W(K)$*, there exists a continuous function* $\pi_\alpha : K \to W$ *such that*

(i). π_α *and* $i : K \hookrightarrow W$ *are* α*–close;*

(ii). $\pi_\alpha(K)$ *is contained in a subset* $C_\alpha \subseteq W$ *and* $C_\alpha \in GES(compact)$ *(w.r.t.* Φ*).*

Definition 4. *We say* W *is GAES admissible (w.r.t.* Φ*) if for all compact subsets* K *of* W*, all* $\alpha \in Cov_W(K)$*, there exists a continuous function* $\pi_\alpha : K \to W$ *such that*

(i). π_α *and* $i : K \hookrightarrow W$ *are* α*–close;*

(ii). $\pi_\alpha(K)$ *is contained in a subset* $C_\alpha \subseteq W$*,* $C_\alpha \in GAES(compact)$ *(w.r.t.* Φ*) and* C_α *is a uniform space.*

Remark 7. *In Definition 4 if* W *is a subset of a Hausdorff topological vector space then* W *is a uniform space and so automatically* C_α *is a uniform space (recall a subset of a uniform space is a uniform space). Thus* C_α *is a uniform space is redundent in Definition 4 if* W *is a subset of a Hausdorff topological vector space or more generally if* W *is a uniform space.*

Theorem 13. *Let* X *be a subset of a Hausdorff topological vector space and* $F \in HYPK(X, X)$ *with* $co\,F$ *a compact map (so in particular there exists an upper semicontinuous compact map* $\Phi \in PK(X, X)$ *with compact values and with* $\Phi(x) \subseteq co\,(F(x))$ *for* $x \in X$*). Also assume* X *is GES admissible (w.r.t.* Φ*). Then* Φ *(so consequently* $co\,F$*) has a fixed point.*

Proof. Let Φ be as in the statement of Theorem 13. Let $\alpha \in Cov_X\,(K)$ where $K = \overline{\Phi(X)}$. There exists a $\pi_\alpha \in C(K, C_\alpha)$ and $C_\alpha \in GES(\text{compact})$ (w.r.t. Φ) as described in Definition 3 (with $W = X$). Note (see Section 1) that $\Phi \in PK(C_\alpha, K)$ so $\Phi_\alpha = \pi_\alpha\,\Phi \in PK(C_\alpha, C_\alpha)$ is an upper semicontinuous compact map with compact values. Now (see the proof of Theorem 7 or alternatively see Theorem 8 or Theorem 9) guarantees that there exists an $x \in C_\alpha$ with $x \in \Phi_\alpha(x) = \pi_\alpha\,\Phi\,(x)$, i.e. $x = \pi_\alpha\,w$ for some $w \in \Phi(x)$.

Since π_α and i are α–close then there exists a $U \in \alpha$ with $\pi_\alpha(w) \in U$ and $i(w) \in U$, i.e. $w \in U$ and $x \in U$. As a result $x \in U$ and $\Phi(x) \cap U \neq \emptyset$ (since $w \in \Phi(x)$). Thus, Φ has an α– fixed point (for each $\alpha \in Cov_X(K)$) so Theorem 3 and Remark 1 guarantee that Φ (so consequently $co\,F$) has a fixed point.

The same analysis as above guarantees the following results.

Theorem 14. *Let X be a subset of a Hausdorff topological space, let X be a uniform space and let $F \in HYPKC(X,X)$ with F a compact map (so in particular there exists an upper semicontinuous compact map $\Phi \in PK(X,X)$ with compact values and with $\Phi(x) \subseteq F(x)$) for $x \in X$). Also assume X is GES admissible (w.r.t. Φ). Then Φ (so consequently F) has a fixed point.*

Theorem 15. *Let X be a subset of a Hausdorff topological space, let X be a uniform space and let $F \in PK(X,X)$ be an upper semicontinuous compact map with compact values. Also, assume X is GES admissible (w.r.t. F). Then F has a fixed point.*

Theorem 16. *Let X be a subset of a Hausdorff topological vector space and $F \in HYPK(X,X)$ with $co\,F$ a compact map (so in particular there exists an upper semicontinuous compact map $\Phi \in PK(X,X)$ with compact values and with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Also assume X is $GAES$ admissible (w.r.t. Φ). Then Φ (so consequently $co\,F$) has a fixed point.*

Proof. Let Φ be as in the statement of Theorem 16. Let $\alpha \in Cov_X(K)$ where $K = \overline{\Phi(X)}$. There exists a $\pi_\alpha \in C(K, C_\alpha)$ and $C_\alpha \in GAES$(compact) (w.r.t. Φ) as described in Definition 4). Let $\Phi_\alpha = \pi_\alpha \Phi$ and note $\Phi_\alpha \in PK(C_\alpha, C_\alpha)$ is an upper semicontinuous compact map with compact values. Now (see the proof of Theorem 10 or alternatively see Theorem 11 or Theorem 12) guarantees that there exists an $x \in C_\alpha$ with $x \in \Phi_\alpha(x) = \pi_\alpha \Phi(x)$. The same reasoning as in Theorem 13 guarantees that Φ has an α-fixed point (for each $\alpha \in Cov_X(K)$). Now, apply Theorem 3 and Remark 1.

The same analysis as above guarantees the following results.

Theorem 17. *Let X be a subset of a Hausdorff topological space, let X be a uniform space and let $F \in HYPKC(X,X)$ with F a compact map (so in particular there exists an upper semicontinuous compact map $\Phi \in PK(X,X)$ with compact values and with $\Phi(x) \subseteq F(x)$) for $x \in X$). Also assume X is $GAES$ admissible (w.r.t. Φ). Then Φ (so consequently F) has a fixed point.*

Theorem 18. *Let X be a subset of a Hausdorff topological space, let X be a uniform space and let $F \in PK(X,X)$ be an upper semicontinuous compact* *map with compact values. Also, assume X is GAES admissible (w.r.t. F). Then F has a fixed point.*

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