Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 18, No. 1/2026

A SPECTRAL CRITERION FOR EXACT DETECTABILITY OF A CLASS OF CONTINUOUS-TIME AND DISCRETE-TIME SYSTEMS WITH PERIODIC COEFFICIENTS*

Vasile Drăgan† Ioan-Lucian Popa‡ Samir Aberkane§
Eduardo Fontoura Costa¶

Dedicated to Biagio Ricceri on the occasion of his 70th anniversary

DOI 10.56082/annalsarscimath.2026.1.31

Abstract

This paper investigates exact detectability of systems with periodic coefficients in finite-dimensional real ordered Hilbert spaces, extending the classical framework of positive systems, which have applications in many fields. A spectral PBH-type criterion for detectability is established for both discrete and continuous-time systems in a unified treatment. We further propose a Barbasin–Krasovskii-type criterion for exponential stability by showing that the existence of a solution to

*Accepted for publication on September 22, 2025

†Vasile.Dragan@imar.ro, Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O.Box 1-764, RO-014700, Bucharest, Romania & Academy of the Romanian Scientists, 3 Ilfov, 050044, Bucharest, Romania

‡lucian.popa@uab.ro, Department of Computing, Mathematics and Electronics "1 Decembrie 1918" University of Alba Iulia, 510009 - Alba Iulia, Romania & Faculty of Mathematics and Computer Science, Transilvania University of Brasov, Iuliu Maniu Street 50, 500091, Brasov, Romania

§samir.aberkane@univ-lorraine.fr, Université de Lorraine, CRAN, UMR 7039, Campus Sciences, BP 70239, Vandoeuvre-les-Nancy Cedex, 54506, France CNRS, CRAN, UMR 7039, France

¶efcosta@icmc.usp.br, Instituto de Ciências Matemáticas e de Computação Universidade de São Paulo, São Carlos-SP, CEP 13566-590, Brazil

a dual system, combined with detectability, ensures stability. The obtained results provide a Lyapunov-like framework and open directions for studying stability in optimal control.

Keywords: detectability, exponential stability, Popov-Belevich-Hautus type criterion, Barbasin-Krasovski-type criterion.
MSC: 93A99, 93B28, 47A50, 47A60, 47A65.

1 Introduction

There are many analysis and control problems involving dynamical systems with outputs, where a natural question arises: can we infer convergence of the system's state from observing only the output? This type of property is commonly referred to as **detectability**, which can be traced back to the sisties [28]. The term reflects the idea that the state can be "perceived" through the output, or that the persistence of the state sufficiently far from the origin implies that, at some time, the output deviates proportionally from the origin. This is precisely the detectability notion defined in the seminal paper [1]. Beyond its obvious relevance in assessing stability of a systems' internal dynamics from output measurements, detectability is fundamental in optimal control problems involving minimization of an output norm. A classical example is linear quadratic control, where the quadratic stage cost can be seen as an output of the system. In such contexts, detectability ensures that finite cost implies stable internal dynamics (see for example [7] for Markov jump linear systems, or [16] for a rather general setting of stochastic systems).

Several variants of detectability have been proposed to accommodate different problem settings and system classes (e.g., [19, 9, 7]). Among these, **exact detectability** [9] has become prominent in linear stochastic systems, mainly due to its conceptual simplicity and applicability. Intuitively, a system is exact detectable if, for any initial condition, the state solution converges to zero whenever the output is identically zero. This simplicity facilitates adaptations to various scenarios, e.g. in [9] and [15], the state solution converges to zero in the mean square sense whenever the output is almost surely zero. Another key feature of exact detectability is that it is often amenable to verification via the so-called Popov-Bellevich-Hautus (PBH) test, as shown in [9] and [15] in different contexts.

In this paper, we study the notion of exact detectability in a general and abstract setting of systems with periodic coefficients in finite-dimensional real ordered Hilbert spaces, without relying on specific vector or matrix linear spaces as commonly done in the literature. The considered models can be seen as a broad generalization of positive systems, which arise in many applications, including queuing systems, chemical processes, biological systems, traffic modeling, population dynamics, *etc.* Some of the relevant general textbooks on deterministic positive systems are [18], [24], [22]. We first establish the PBH criterion and present it in Theorem 1 in a unified approach that encompasses both the discrete-time and continuous-time versions. The second part of the article is devoted to the application of the proposed exact detectability PBH-type conditions to the problem of exponential stability for the considered general class of dynamical systems defined on finite-dimensional real ordered Hilbert spaces. More specifically, we show that under exact detectability condition, we can relax existing stability results in the literature (see for example Corollary 2.4.4 from [14] and Theorem 2.4 from [13]) in order to develop a type of Barbasin-Krasovski criterion for stability. The stability results developed in [13, 14] are formulated via a backward linear differential/difference equation in a dual state z with a uniformly positive forcing term f, involving the self-adjoint operator of the system. It is shown in [13, 14] that the existence of a uniformly positive solution to such an equation is equivalent to the exponential stability of the original system. Under the detectability condition, we show that the exponential stability of the zero solution of the original system is also guaranteed when the backward linear differential/difference equation with a positive (not necessarily uniformly positive) forcing term admits a positive (not necessarily uniformly positive) solution. The obtained Barbasin-Krasovski type criterion can be interpreted as Lyapunov-like stability condition. The results are demonstrated in Theorems 4 and 5 in continuous-time and discrete-time contexts, respectively. Although we do not address control problems in this paper, the Barbasin-Krasovski criterion lays the foundation for future investigation of stability of optimal controllers.

The rest of the paper is organized as follows. Section 2 describes the models and notation used in the paper. The main results are given in Sections 3 and 4 – the PBH criterion and Barbasin-Krasovski criterion for exact detectability and exponential stability, respectively. Concluding remarks are given in Section 5.

2 Model description and basic definitions

2.1 Preliminary issues

In this work $(\mathcal{X}, \langle\cdot,\cdot\rangle_{\mathcal{X}}), (\mathcal{Y}, \langle\cdot,\cdot\rangle_{\mathcal{Y}})$ are real ordered finite-dimensional Hilbert spaces. The order relation on $\mathcal{X}$ is induced by a closed, solid, self dual, convex cone $\mathcal{X}^+$, while the order relation on $\mathcal{Y}$ is induced by a closed, solid, self dual, convex cone $\mathcal{Y}^+$. For definitions and useful properties of different kinds of convex cones we refer to [10, 11]. The following examples provide two of the most used finite dimensional ordered Hilbert spaces.

Example 1. $(\mathcal{X}, \mathcal{X}^+) \leftarrow (\mathbb{R}^n, \mathbb{R}^n_+)$ *where* $\mathbb{R}^n_+ = \{\mathbf{x} = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n | x_i \geq 0, 1 \leq i \leq n\}$. *The ordering relation induced by the convex cone* $\mathbb{R}^n_+$ *is known as the component wise ordering. The Hilbert space structure is induced by the usual inner product (euclidean inner product) on* $\mathbb{R}^n$. *One easily checks that* $\mathbb{R}^n_+$ *is a closed, solid, self dual, reproducing, convex cone.*

Example 2. *Let* $\mathcal{S}_n \subset \mathbb{R}^{n\times n}$ *be the linear space of symmetric matrices and* $\mathcal{S}_n^N \triangleq \underbrace{\mathcal{S}_n \times \mathcal{S}_n \times ... \times \mathcal{S}_n}_{N\ times}$. *Equipped with the inner product:*

$$\langle \mathbf{X}, \mathbf{Z} \rangle \triangleq \sum_{i=1}^{N} Tr[X_i Z_i], \tag{1}$$

for all $\mathbf{X} = (X_1, X_2, ..., X_N)$ *and* $\mathbf{Z} = (Z_1, Z_2, ..., Z_N)$ *from* $\mathcal{S}_n^N$ *it becomes a Hilbert real space. The order relation on this Hilbert space is induced by the convex cone* $\mathcal{S}_n^{N+} = \{\mathbf{X} \in \mathcal{S}_n^N | \mathbf{X} = (X_1, X_2, ..., X_N), X_i \geq 0, 1 \leq i \leq N\}$. *One checks that* $\mathcal{S}_n^N$ *is a closed, solid, self dual, reproducing, convex cone (see for example Chapter 2 in [14]). In (1),* $Tr[\cdot]$ *stands for the trace of a matrix.*

To ease the exposition, we denote $\mathbf{L}[\mathcal{X}]$ the set of the linear operators $T : \mathcal{X} \to \mathcal{X}$ and $\mathbf{L}[\mathcal{X}, \mathcal{Y}]$ the set of linear operators $C : \mathcal{X} \to \mathcal{Y}$. An operator $T \in \mathbf{L}[\mathcal{X}] (C \in \mathbf{L}[\mathcal{X}, \mathcal{Y}])$ is named positive operator on the ordered Hilbert space $(\mathcal{X}, \mathcal{X}^+)$ if $T\mathcal{X}^+ \subset \mathcal{X}^+$ $(C\mathcal{X}^+ \subset \mathcal{Y}^+)$.

2.2 A continuous-time linear system

On the ordered Hilbert space $(\mathcal{X}, \mathcal{X}^+)$ we consider the continuous-time linear system:

$$\dot{x}_c(t) = \mathcal{L}_c(t) x_c(t) \tag{2a}$$

$$y_c(t) = \mathcal{C}_c(t) x_c(t) \tag{2b}$$

$t \in \mathbb{R}$, where $x_c(t) \in \mathcal{X}$ is the state of the system and $y_c(t) \in \mathcal{Y}$ is an output. In (2a), $\mathcal{L}_c(\cdot) : \mathbb{R} \to \mathbf{L}[\mathcal{X}]$ and in (2b), $\mathcal{C}_c(\cdot) : \mathbb{R} \to \mathbf{L}[\mathcal{X}, \mathcal{Y}]$ are continuous operator-valued functions.

If $x_c(t; t_0, x_0)$ is the solution of the differential equation (2a) associated to the initial pair $(t_0, x_0) \in \mathbb{R} \times \mathcal{X}$, the corresponding output is

$$y_c(t; t_0, x_0) = \mathcal{C}_c(t) x_c(t; t_0, x_0), \quad t \in \mathbb{R}. \tag{3}$$

Definition 1. *We say that the linear differential equation (2a), or equivalently, that the operator-valued function $\mathcal{L}_c(\cdot)$ defines a positive evolution on the ordered Hilbert space $\left(\mathcal{X}, \mathcal{X}^+\right)$ if, for any initial time $t_0 \in \mathbb{R}$, $x_c(t; t_0, x_0) \in \mathcal{X}^+$ for any $t \geq t_0$, whenever $x_0 \in \mathcal{X}^+$.*

Regarding the system (2) we make the assumption:

$\mathbf{H}_c$)

a) the linear differential equation (2a) defines a positive evolution on the linear ordered space $\left(\mathcal{X}, \mathcal{X}^+\right)$;

b) for each $t \in \mathbb{R}, \mathcal{C}_c(t)$ is a positive operator, that is $\mathcal{C}_c(t)\mathcal{X}^+ \subset \mathcal{Y}^+$;

c) $\mathcal{L}_c(\cdot)$, $\mathcal{C}_c(\cdot)$ are periodic operator-valued functions with common period $\theta_c > 0$.

Let $\mathbf{T}_c(t, t_0) : \mathcal{X} \to \mathcal{X}$ be the linear evolution operator defined by the linear differential equation (2a) as

$$\mathbf{T}_c(t, t_0) x_0 \triangleq x(t; t_0, x_0),$$

for all $t, t_0 \in \mathbb{R}, x_0 \in \mathcal{X}$.

Useful properties of a linear evolution operator defined by a linear differential equation on a Banach space may be found, for example, in Chapter 3 from [8], [17, Chapter 6], [23]. Here, we recall two of these properties which are true under the assumption $\mathbf{H}_c$). First, let us remark that if the assumption $\mathbf{H}_c$) *a*) holds, then

$$\mathbf{T}_c(t, t_0)\mathcal{X}^+ \subset \mathcal{X}^+ \tag{4}$$

for all $t \geq t_0$, $t, t_0 \in \mathbb{R}$. Furthermore, if the assumption $\mathbf{H}_c$) c) is fulfilled, then

$$\mathbf{T}_c(t + j\theta_c, t_0 + j\theta_c) = \mathbf{T}_c(t, t_0) \tag{5}$$

for all $t, t_0 \in \mathbb{R}$, $j \in \mathbb{Z}$.

Remark 1. *Reasoning as in Section 2 from [4] one may show that the spectrum of the linear operator* $\mathbf{T}_c(t, t_0)$ *does not depend on* $(t, t_0) \in \mathbb{R} \times \mathbb{R}$.

We define

$$\mathbb{T}_{c,\theta_c}(t_0) \stackrel{\Delta}{=} \mathbf{T}_c(t_0 + \theta_c, t_0) \tag{6}$$

for all $t_0 \in \mathbb{R}$ named the *monodromy operator* associated to the linear differential equation with periodic coefficients (2a). According to the Remark 1, the spectrum $\sigma[\mathbb{T}_{c,\theta_c}(t_0)]$ of the monodromy operator does not depend upon t_0. The elements of the set $\sigma[\mathbb{T}_{c,\theta_c}(t_0)]$ are called *characteristic multipliers*. The relation between the characteristic multipliers and different kinds of stability of the zero solution of a differential equation of the form (2a) can be found in Chapter 3 from [23] or [29].

Definition 2. *We say that the system* (2.2) *is:*

a) ***exact detectable at*** t_0, *if* $\lim_{t\to\infty} x(t; t_0, x_0) = 0$ *for any initial state* $x_0 \in \mathcal{X}^+$ *for which* $y(t; t_0, x_0) = 0$, *for any* $t \geq t_0$*;*

b) ***exact detectable****, if it is exact detectable for any initial time* $t_0 \in \mathbb{R}$.

One of our aims is to provide a Popov-Bellevich-Hautus (PBH) type criterion which allows to decide if a linear system with periodic coefficients as (2) is or not exact detectable. This criterion will be stated using the characteristic multipliers of the differential equation (2a).

2.3 A discrete-time linear system

We consider the discrete-time linear system

$$x_d(t+1) = \mathcal{L}_d(t)x_d(t) \tag{7a}$$

$$y_d(t+1) = \mathcal{C}_d(t)x_d(t) \tag{7b}$$

$t \in \mathbb{Z}$, where $x_d(t) \in \mathcal{X}$ denotes the states of the system, while $y_d(t)$, $t \in \mathbb{Z}$, stands for an output of the system and $\{\mathcal{L}_d(t)\}_{t\in\mathbb{Z}} \subset \mathbf{L}[\mathcal{X}]$, $\{\mathcal{C}_d(t)\}_{t\in\mathbb{Z}} \subset \mathbf{L}[\mathcal{X}, \mathcal{Y}]$.

Regarding the system (7) we make the assumption:

$\mathbf{H}_d$)

a) $\{\mathcal{L}_d(t)\}_{t\in\mathbb{Z}}$ and $\{\mathcal{C}_d(t)\}_{t\in\mathbb{Z}}$ are periodic sequences with common period $\theta_d \geq 1$;

b) for each $t \in \mathbb{Z}$, $\mathcal{L}_d(t)$ and $\mathcal{C}_d(t)$ are positive operators on the ordered Hilbert space $\big(\mathcal{X}, \mathcal{X}^+\big)$.

Remark 2. *Definition 1 may be adapted mutatis mutandis to the discrete-time linear equation (7a). One sees that the equation (7a) defines a positive evolution on the ordered Hilbert space $\big(\mathcal{X}, \mathcal{X}^+\big)$ if and only if for each $t \in \mathbb{Z}$, $\mathcal{L}_d(t)$ is a positive operator.*

The linear evolution operator on the space $\mathcal{X}$ defined by the discrete time linear equation (DTLE) (7a) is defined by:

$$\mathbf{T}_d(t, t_0) = \begin{cases} \mathcal{L}_d(t-1)\mathcal{L}_d(t-2)\dots\mathcal{L}_d(t_0), & \text{if } t > t_0 \\ I_{\mathcal{X}}, & \text{if } t = t_0 \end{cases}$$

where $I_{\mathcal{X}}$ is the identity operator on the space $\mathcal{X}$.

Based on the properties of the operators $\mathcal{L}_d(t)$ from the assumption $\mathbf{H}_d$) we emphasize the following properties of the linear evolution operator $\mathbf{T}_d(t, t_0)$:

$$\mathbf{T}_d(t, t_0)\mathcal{X}^+ \subset \mathcal{X}^+ \tag{8}$$

for all $t \geq t_0$, $t, t_0 \in \mathbb{Z}$,

$$\mathbf{T}_d(t + j\theta_d, t_0 + j\theta_d) = \mathbf{T}_d(t, t_0) \tag{9}$$

for any $t, t_0, j \in \mathbb{Z}$.

The solution $x_d(t; t_0, x_0)$ of the equation (7a) corresponding to the initial pair $(t_0, x_0) \in \mathbb{Z} \times \mathcal{X}$ has the representation

$$x_d(t; t_0, x_0) = \mathbf{T}_d(t, t_0)x_0,$$

for all $t \geq t_0$. The output corresponding to this trajectory is:

$$y_d(t; t_0, x_0) = \mathcal{C}_d(t)x_d(t; t_0, x_0),$$

for all $t \geq t_0$.

Reasoning as in the proof of Proposition 3.1 from [4], it can be shown that the eigenvalues of the linear operator $\mathbf{T}_d(t; t_0)$ do not depend upon the pair $(t, t_0) \in \mathbb{Z} \times \mathbb{Z}$.

We set

$$\mathbb{T}_{d,\theta_d}(t_0) \triangleq \mathbf{T}_d(t_0 + \theta_d, t_0), \tag{10}$$

for all $t_0 \in \mathbb{Z}$. By analogy to the case of a linear differential equation with periodic coefficients, in the case of a discrete-time linear equation with periodic coefficients, the linear operator introduced in (10) will be named monodromy operator of the DTLE (7a). As in the continuous-time case, the eigenvalues of the monodromy operator will be named characteristic multipliers. The set $\sigma[\mathbb{T}_{d,\theta_d}(t_0)]$ of the characteristic multipliers of the monodromy operator does not depend upon $t_0 \in \mathbb{Z}$.

Definition 3. *We say that the discrete-time linear system (7) is:*

a) ***exact detectable at initial time*** t_0 *if* $\lim_{t\to+\infty} x_d(t;t_0,x_0) = 0$ *for any initial states* $x_0 \in \mathcal{X}^+$ *for which* $y_d(t;t_0,x_0) = 0$*, for all* $t \geq t_0$*;*

b) ***exact detectable*** *if it is exact detectable at any initial time* $t_0 \in \mathbb{Z}$*.*

Another aim of this work is to provide a PBH type criterion that allows us to decide if a discrete-time linear system of the form of (7) is or not exact detectable, in the sense of Definition 3.

Starting from the similarity between the definitions of the concept of exact detectability in the continuous- time case system and the discrete-time case system, we consider a more general framework which allows us an unified approach for the derivation of a PBH criterion for the linear systems of the form (2) and those of the form (7).

2.4 A unified approach of exact detectability for the case of continuous-time and discrete-time systems with periodic coefficients

We consider a linear system on the Hilbert space $\mathcal{X}$ which exploits the similarity between (2) and (7):

$$\big(\delta^+ x\big)(t) = \mathcal{L}(t)x(t) \tag{11a}$$

$$y(t) = \mathcal{C}(t)x(t) \tag{11b}$$

$t \in \mathbb{K}$ where $\big(\delta^+ x\big)(t) = \dot{x}(t)$ when $\mathbb{K} \leftarrow \mathbb{R}$ and $\big(\delta^+ x\big)(t) = x(t+1)$ when $\mathbb{K} \leftarrow \mathbb{Z}$. In (11), $x(t) \in \mathcal{X}$ are the states of the system and $y(t) \in \mathcal{Y}$ is the output of the system, $\mathcal{X}, \mathcal{Y}$ being the same finite dimensional Hilbert spaces, where were defined the continuous-time linear system (2) and the discrete-time linear system (7). When $\mathbb{K} \leftarrow \mathbb{R}$, in 11, $\mathcal{L}(t)$ coincides with $\mathcal{L}_c(t)$ and $\mathcal{C}(t)$ is $\mathcal{C}_c(t), t \in \mathbb{R}$ while $\mathbb{K} \leftarrow \mathbb{Z}$, in (11) $\mathcal{L}(t)$ coincides with $\mathcal{L}_d(t)$ and $\mathcal{C}(t)$ becomes $\mathcal{C}_d(t)$, $t \in \mathbb{Z}$.

Let $x(t;t_0,x_0)$, $t \geq t_0 \in \mathbb{K}$ be the solution of the equation (11a) corresponding to the initial pair $(t_0, x_0) \in \mathbb{K} \times \mathcal{X}$. The corresponding output is

$y(t; t_0, x_0) = \mathcal{C}(t)x(t; t_0, x_0)$. We have the representation formula

$$x(t; t_0, x_0) = \mathbf{T}(t, t_0)x_0 \tag{12}$$

for all $t \geq t_0 \in \mathbb{K}$, $x_0 \in \mathcal{X}$, where

$$\mathbf{T}(t, t_0) = \begin{cases} \mathbf{T}_c(t, t_o), & \text{if } \mathbb{K} \leftarrow \mathbb{R} \\ \mathbf{T}_d(t, t_0), & \text{if } \mathbb{K} \leftarrow \mathbb{Z}. \end{cases} \tag{13}$$

Definition 4. *We say that the linear system (11) is:*

a) ***exact detectable at initial time*** $t_0 \in \mathbb{K}$ *if for any* $x_0 \in \mathcal{X}^+$ *for which* $y(t; t_0, x_0) = 0$, *for all* $t \geq t_0 \in \mathbb{K}$, *it follows that* $\lim_{t \to +\infty} x(t; t_0, x_0) = 0$;

b) ***exact detectable*** *if it is exact detectable at any initial times* $t_0 \in \mathbb{K}$.

One sees that in the case when $\mathbb{K} \leftarrow \mathbb{R}$, the Definition 4 is just Definition 2, while if $\mathbb{K} \leftarrow \mathbb{Z}$, the Definition 4 recovers Definition 3.

In the next section, we shall derive the PBH criterion for exact detectability of the system (11). The PBH type criterion derived for the system (11) will contain as special cases the PBH criteria for the continuous-time linear system (2) and the discrete-time linear system (7).

3 A PBH criterion for exact detectability

3.1 Some preliminary issues

We assume the assumptions $\mathbf{H}_c$) and $\mathbf{H}_d$) hold. In this case, the operator-valued functions $\mathcal{L}(\cdot)$ and $\mathcal{C}(\cdot)$, involved in (11) are periodic functions of period θ, where

$$\theta = \begin{cases} \theta_c & \text{if } \mathbb{K} \leftarrow \mathbb{R} \\ \theta_d & \text{if } \mathbb{K} \leftarrow \mathbb{Z}. \end{cases} \tag{14}$$

We set

$$\mathbb{T}_\theta(t_0) \stackrel{\Delta}{=} \mathbf{T}(t_0 + \theta, t_0), \tag{15}$$

where $\mathbf{T}(\cdot, \cdot)$ is defined by (13) and θ is that from (14).

From (13) and (15) we may infer that under the assumptions $\mathbf{H}_c$), $\mathbf{H}_d$), $\mathbb{T}_\theta(t_0)$ is a positive operator on the ordered Hilbert space $\left(\mathcal{X}, \mathcal{X}^+\right)$, that is

$$\mathbb{T}_\theta(t_0)\mathcal{X}^+ \subset \mathcal{X}^+,$$

for all $t_0 \in \mathbb{K}$. The operator $\mathbb{T}_\theta(t_0)$ is named the monodromy operator associated with the equation with periodic coefficients (11), and its eigenvalues will be named characteristic multipliers.

The set $\sigma[\mathbb{T}_\theta(t_0)]$ of the characteristic multipliers does not depend upon $t_0 \in \mathbb{K}$.

In the developments from this section, an important role is played by the subset of the characteristic multipliers

$$\sigma^+[\mathbb{T}_\theta(t_0)] = \{\mu \in \sigma[\mathbb{T}_\theta(t_0)] \mid \exists 0 \neq y \in \mathcal{X}^+ \text{such that } \mathbb{T}_\theta(t_0)y = \mu y\}.$$

One sees that $\sigma^+[\mathbb{T}_\theta(t_0)] \subset \mathbb{R}_+$.

Definition 5. *We say that $\mu \in \sigma^+[\mathbb{T}_\theta(t_0)]$ is a characteristic multiplier unobservable at $t_0 \in \mathbb{K}$ if there exist $z \in \mathcal{X}^+, z \neq 0$ which satisfies the equalities*

$$\mathbb{T}_\theta(t_0)z = \mu z \tag{16a}$$

$$\mathcal{C}(t)\mathbf{T}(t,t_0)z = 0, \quad t_0 \leq t < t_0 + \theta, k \in \mathbb{K}. \tag{16b}$$

Let ρ_θ be the spectral radius of the linear operator $\mathbb{T}_\theta(t_0)$. Applying Theorem 19.2 from [11], we may conclude that $\rho_\theta \in \sigma^+[\mathbb{T}_\theta(t_0)]$.

At the end of this subsection, let us recall the version for the linear evolution operator $\mathbf{T}(\cdot,\cdot)$ introduced in (13) of the equalities (5) and (9):

$$\mathbf{T}(t + j\theta, t_0 + j\theta) = \mathbf{T}(t,t_0), \tag{17}$$

for any $j \in \mathbb{Z}, t \geq t_0,\ t, t_0 \in \mathbb{K}$.

3.2 A PBH-type criterion for exact detectability

In the case where $\mathbb{K} \leftarrow \mathbb{Z}$, $[t_0, t_0 + \theta)$ stands for the subset of integers $\{t_0, t_0 + 1, \ldots, t_0 + \theta - 1\}$. The main result of this section is stated in the next theorem:

Theorem 1. *Assume that assumptions $\mathbf{H}_c$) (for the continuous-time case) and $\mathbf{H}_d$) (for the discrete-time case) are fulfilled. Let θ (defined as in (14)) be the period of the operator-valued functions $\mathcal{L}(\cdot)$, $\mathcal{C}(\cdot)$ that describe the system (11). Then, for each $t_0 \in \mathbb{K}$ the following are equivalent:*

(i) the linear system (11) is exact detectable at the initial time $t_0 \in \mathbb{K}$;

(ii) there are no characteristic multipliers $\mu \in \sigma^+[\mathbb{T}_\theta(t_0)], \mu \geq 1$, which are unobservable at t_0.

Proof. (see Appendix A). □

In the last part of this section, we rewrite Definition 5 and the statement of Theorem 1 for the case of a continuous-time linear system of the form (2) and for the case of a discrete-time linear system of the form (7).

The particular case of a continuous-time linear system with periodic coefficients

Using the notation introduced in Section 2.2 we obtain the following form of Definition 5 and of the adapted PBH criterion for the system (2).

Definition 6. *We say that* $\mu \in \sigma^+[\mathbb{T}_{c,\theta_c}(t_0)]$ *is a characteristic multiplier which is unobservable at initial time* t_0 *for the linear system (2) if there exists* $z \in \mathcal{X}^+$, $z \neq 0$, *satisfying the equalities:*

$$\mathbb{T}_{c,\theta_c}(t_0)z = \mu z \tag{18a}$$

$$\mathcal{C}_c(t)\mathbf{T}_c(t,t_0)z = 0 \tag{18b}$$

for any $t \in [t_0, t_0 + \theta_c)$.

Theorem 2. *Assume that assumption* $\mathbf{H}_c$) *is fulfilled. Under these conditions, for each* $t_0 \in \mathbb{R}$, *the following are equivalent:*

(i) the continuous-time linear system (2) is exact detectable at t_0;

(ii) there are no characteristic multipliers $\mu \in \sigma^+[\mathbb{T}_{c,\theta_c}(t_0)], \mu \geq 1$, *which are unobservable at initial time* t_0, *for the system (2).*

The particular case of a discrete-time linear system with periodic coefficients

Using the notations introduced in Section 2.3 we obtain the following form of Definition 5 and of the PBH criterion for the system(7).

Definition 7. *We say that* $\mu \in \sigma^+[\mathbb{T}_{d,\theta_d}(t_0)]$ *is a characteristic multiplier unobservable at initial time* $t_0 \in \mathbb{Z}$ *for the linear system (7) if then exists* $z \in \mathcal{X}^+$, $z \neq 0$ *satisfying the equalities:*

$$\mathbb{T}_{d,\theta_d}(t_0)z = \mu z \tag{19a}$$

$$\mathcal{C}_d(t)\mathbf{T}_d(t,t_0)z = 0 \tag{19b}$$

for all $t \in \{t_0, t_0 + 1, \ldots, t_0 + \theta_d - 1\}$.

Theorem 3. *Assume that assumption $\boldsymbol{H}_d$) is fulfilled. Under these conditions, for each $t_0 \in \mathbb{Z}$ the following are equivalent:*

(i) the discrete-time linear system (7) is exact detectable at the initial time t_0;

(ii) there are no characteristic multipliers $\mu \in \sigma^+[\mathbb{T}_{d,\theta_d}(t_0)]$, $\mu \geq 1$, which are unobservable at t_0 for the linear system (7).

In the next section, we shall use the PBH type criterion provided by Theorem 2 and Theorem 3, to derive Barbashin-Krasovski type criteria for the exponential stability of a linear differential equation of type (2a) and of a discrete-time linear equation of type (7a).

4 A Barbasin-Krasovski type criterion for exponential stability of zero solution of a linear equation

In this section, we apply the results stated in Theorem 2 and Theorem 3 to obtain a version of the result of Barbasin - Krasovski [3] for the case of linear differential equations and difference equations defined on an ordered Hilbert space.

4.1 The case of a continuous-time linear differential equation

On the finite dimensional ordered Hilbert space $\left(\mathcal{X}, \mathcal{X}^+\right)$ we consider the linear differential equation:

$$\dot{x}_c(t) = \mathcal{L}_c(t)x_c(t). \tag{20}$$

The ordering relation on $\mathcal{X}$ is induced by the closed, solid, self dual, convex cone $\mathcal{X}^+$. Regarding the differential equation (20) we make the assumption:

$\mathbf{H}_c^1$)

a) the operator-valued function $\mathcal{L}_c(\cdot) : \mathbb{R} \to \mathbf{L}[\mathcal{X}]$ is continuous and periodic with period $\theta_c > 0$;

b) the linear differential equation (20) generates a positive evolution on the ordered Hilbert space $\left(\mathcal{X}, \mathcal{X}^+\right)$.

Example 3. *On the ordered space* $(\mathbb{R}^n, \mathbb{R}^n_+)$ *the equation (20) takes the form*

$$\dot{x}_c(t) = A(t)x_c(t) \tag{21}$$

where $A(t)$ *is the matrix associated to the operator* $\mathcal{L}_c(t)$ *with respect to the canonical basis of* $\mathbb{R}^n$, $A(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$ *is a matrix valued function continuous and periodic, with period* θ_c. *According to [12], the linear differential equation (21) defines a positive evolution on the ordered Hilbert space* $(\mathbb{R}^n, \mathbb{R}^n_+)$ *if and only if for each* $t \in \mathbb{R}, A(t)$ *is a Metzlez matrix. This means that the off diagonal elements of the matrix* $A(t)$ *are non-negative. This kind of differential equation models phenomena for which the state parameters remain non-negative, with applications in many areas of science [18], e.g., in queuing systems where the state components represent probabilities [18, Chap. 16], in chemical processes involving mass and heat balance equations [26], in telecommunication engineering where the problem of Power allocation in a mobile telecommunication network with n users which transmit signals to a receiving station can be modeled as an optimal control for a positive system [6], or compartmental models which are systems composed of interconnected reservoirs (named compartments), exchanging flows of a common resource. Such models are for example used in biology to describe storage and flows of certain substances through different components of a biological organism. The amount of resource in each compartment being intrinsically a positive variable, positive models are appropriate to describe such phenomena (see [22, 6]). We can also cite some consensus problems appearing in networked control systems (network of agents) can be also efficiently tackled under the positive system paradigm [27] and applications in epidemiological modeling [2, 5].*

Example 4. *On the ordered Hilbert space* $(\mathcal{S}, \mathcal{S}_n^+)$ *we consider the linear differential equation*

$$\dot{X}(t) = A_0(t)X(t) + X(t)A_0^T(t) + \sum_{k=1}^{r} A_k(t)X(t)A_k^T(t). \tag{22}$$

The linear differential equation (22) is defined by the operator-valued function $\mathcal{L}(\cdot) : \mathbb{R} \to \mathcal{S}_n$ *described by*

$$\mathcal{L}(t)X = A_0(t)X + XA_0^T(t) + \sum_{k=1}^{r} A_k(t)XA_k^T(t). \tag{23}$$

This kind of differential equation occurs in connection with the derivation of criteria for exponential stability in mean square of a stochastic linear *differential equation of the form:*

$$dx(t) = A_0(t)x(t)dt + \sum_{k=1}^{r} A_k(t)x(t)dw_k(t), t \geq 0,$$

where $\{w(t)\}_{t\geq 0}$ $(w(t) = (w_1(t)\ w_2(t)\ \dots\ w_r(t))^T)$ *is a r-dimensional standard Wiener process defined on a given probability space* $(\Omega, \mathcal{F}, \mathcal{P})$ *(see [20]).*

Let us remark that if $A_k(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}, 0 \leq k \leq r$ are continuous matrix valued functions, periodic with common period $\theta_c > 0$, then the operator-valued function $\mathcal{L}(\cdot)$ described by (23) is continuous and periodic of period θ_c. Moreover, Theorem 2.6 (i) from [14] allows to deduce that the linear differential equation (22) defines a positive evolution on the ordered space $(\mathcal{S}_n, \mathcal{S}_n^+)$.

For the reader's convenience, we recall the definition of the exponential stability of the zero solution of a continuous-time linear differential equation (CTLDE) (20).

Definition 8. *We say that the zero solution of the CTLDE (20) is exponentially stable if its solutions* $x(\cdot; t_0, x_0)$ *satisfy:*

$$||x(t; t_0, x_0)|| \leq \beta e^{-\alpha(t-t_0)}||x_0||, \tag{24}$$

for all $t \geq t_0$, $t, t_0 \in \mathbb{R}$, $x_0 \in \mathcal{X}$, *where* $\alpha > 0, \beta \geq 1$ *are constants not depending upon* t, t_0, x_0.

Definition 9. *A vector valued function* $h(\cdot) : \mathcal{I} \subset \mathbb{R} \to \mathcal{X}$ *is uniform positive on* $\mathcal{I}$ *if there exists a constant* $\delta > 0$ *and a vector* $\xi \in Int\mathcal{X}^+$ *such that*

$$h(t) \succeq \delta\xi \tag{25}$$

for any $t \in \mathcal{I}$.

Remark 3. *If (25) holds for a vector* $\xi \in Int\mathcal{X}^+$, *then for any* $\zeta \in Int\mathcal{X}^+$, *there exists a constant* $\nu > 0$ *which may depends upon* ζ *such that* $h(t) \succeq \nu\zeta$ *for all* $t \in \mathcal{I}$. *Indeed if* $\zeta \in Int\mathcal{X}^+$ *there exists* $\epsilon > 0$ *which may depends upon* ζ *such that*

$$0 \prec \epsilon\zeta \prec \xi.$$

This allows to write (via (25)) that

$$h(t) \succeq \epsilon\delta\zeta,$$

for any $t \in \mathcal{I}$.

Further, we denote $h(t) \succ\succ 0$, $t \in \mathcal{I}$ to show that $h(\cdot)$ is a uniform positive vector valued function on the interval $\mathcal{I}$. In the case when $h(t) \succeq 0, t \in \mathcal{I}$, the function $h(\cdot)$ will be named positive on $\mathcal{I}$.

A set of criteria that allows us to decide if the zero solution of a CTLDE of the form (20) is exponentially stable is provided, for example, by Corollary 2.4.4 from [14]. So, employing the equivalence $(i) \Leftrightarrow (iv)$ from Corollary 2.4.4 from the aforementioned reference we may infer that the zero solution of the differential equation (20) is exponentially stable, if and only if there exists a continuous vector-valued function $f(\cdot) : \mathbb{R} \to \mathcal{X}$ periodic with period θ_c uniformly positive on $\mathbb{R}$ with the property that the backward linear differential equation:

$$-\dot{z}(t) = \mathcal{L}_c^*(t)z(t) + f(t) \tag{26}$$

has a solution $\tilde{z}(\cdot) : \mathbb{R} \to \mathcal{X}$ that is a uniform positive and periodic function of period θ_c.

Our aim is to show that under the suitable exact detectability property, the exponential stability of the zero solution of the equation (20) is also guaranteed in the case where the backward linear differential equation (26) with $f(t) \succeq 0$, $t \in \mathbb{R}$, instead of $f(t) \succ\succ 0$ has a solution $\tilde{z}(\cdot) : \mathbb{R} \to \mathcal{X}$, which is a θ_c periodic function such that $\tilde{z}(t) \succeq 0$.

Attaching a suitable output to the differential equation (20) we obtain the following linear system

$$\dot{x}_c(t) = \mathcal{L}_c(t)x_c(t) \tag{27a}$$

$$y_c(t) = \langle f(t), x_c(t)\rangle_{\mathcal{X}} \tag{27b}$$

where $f(t)$, $t \in \mathbb{R}$ is the free term of equation (26). Bearing in mind that (20) has the properties of the assumption $\mathbf{H}_c^1$) and that the cone $\mathcal{X}^+$ is a self-dual convex cone we may infer that assumption $\mathbf{H}_c$) holds in the case of the system (27) too, because $f(\cdot)$ is a continuous θ_c periodic function and is satisfying $f(t) \succeq 0$ for all $t \in \mathbb{R}$. This means that we may apply Theorem 2 to obtain a PBH criterion for the exact detectability of this system. This allows us to prove a Barbasin-Krasovski type criterion for exponential stability of the zero solution of the linear differential equation (20).

Theorem 4. *Assume:*

a) the assumption $\mathbf{H}_c^1$) *holds;*

b) the continuous vector valued function $f : \mathbb{R} \to \mathcal{X}^+$*, not necessarily uniform positive, periodic with period* θ_c*, is such that the corresponding system of the form (27) is exact detectable.*

Under these conditions, the following are equivalent:

(i) the zero solution of the linear differential equation (20) is exponentially stable;

(ii) the backward linear differential equation of the form (26) whose free term $f(\cdot)$ is satisfying the conditions from b), has a solution $\tilde{z}(\cdot) : \mathbb{R} \to \mathcal{X}^+$ periodic with period θ_c.

Proof. The implication $(i) \implies (ii)$ follows from Theorem 2.3.7 (ii) from [14] in the case of the equation (26). It remains to prove the implication $(ii) \implies (i)$. Let $\mathbf{T}_c(t, t_0)$, $t, t_0 \in \mathbb{R}$, be the linear evolution operator on $\mathcal{X}$ defined by the differential equation (20). Let $\mathbb{T}_{c,\theta_c}(t_0) \triangleq \mathbf{T}_c(t_0 + \theta_c, t_0), t_0 \in \mathbb{R}$, be the monodromy operator defined by the differential equation (20). Let ρ_{c,θ_c} be the spectral radius of the monodromy operator $\mathbb{T}_{c,\theta_c}(t_0)$. We have to show that $\rho_{c,\theta_c} < 1$. Based on the assumption $\mathbf{H}_c^1$) b) we may infer that $\mathbb{T}_{c,\theta_c}(t_0)$ is a positive operator on the ordered space $\left(\mathcal{X}, \mathcal{X}^+\right)$. According to Theorem 19.2 from [11] applied in the case of the operator $\mathbb{T}_{c,\theta_c}(t_0)$ we obtain that there exists a vector $\tilde{z}_0 \neq 0$ in $\mathcal{X}^+$ which is satisfying

$$\mathbb{T}_{c,\theta_c}(t_0)\tilde{z}_0 = \rho_{c,\theta_c}\tilde{z}_0. \tag{28}$$

Further writing (26) for its solution $\tilde{z}(\cdot)$ we obtain the following representation formula:

$$\tilde{z}(t) = \mathbf{T}_c^*(t_0 + \theta_c, t)\tilde{z}(t_0 + \theta_c) + \int_t^{t_0+\theta_c} \mathbf{T}_c^*(s, t)f(s)ds \tag{29}$$

for any $t < t_0 + \theta_c, t_0 \in \mathbb{R}$ is arbitrary but fixed. Taking $t_0 = 0$ and $t = 0$ in (29) we get:

$$\tilde{z}(0) = \mathbb{T}_{c,\theta_c}^*(0)\tilde{z}(\theta_c) + \Psi(\theta_c) \tag{30}$$

where

$$\Psi(\theta_c) \triangleq \int_0^{\theta_c} \mathbf{T}_c^*(s, 0)f(s)ds. \tag{31}$$

Invoking again the assumption $\mathbf{H}_c^1$) b) we deduce that for any $s \geq t$, $\mathbf{T}_c(s, t)$ is a positive operator on the ordered space $\left(\mathcal{X}, \mathcal{X}^+\right)$. Bearing in mind that $\mathcal{X}^+$ is a self dual convex cone we may infer via Proposition 2.1.9 from [14] that $\mathbf{T}_c^*(s, t)$ is also a positive operator. Hence, $\mathbf{T}_c^*(s, t)f(s) \succeq 0$ for all $s \geq t$ because $f(s) \succeq 0$ for all $s \in \mathbb{R}$. This allows us to conclude via (31) that

$\Psi(\theta_c) \succeq 0$ because $\mathcal{X}^+$ is a closed, convex cone. Bearing in mind again that $\mathcal{X}^+$ is a self dual convex cone we deduce via (30) that

$$\begin{aligned} 0 \preceq \ &\langle \Psi(\theta_c), \tilde{z}_0\rangle_{\mathcal{X}} = \langle \tilde{z}(0), \tilde{z}_0\rangle_{\mathcal{X}} - \langle \mathbb{T}_{c,\theta_c}(0)\tilde{z}(\theta_c), \tilde{z}_0\rangle_{\mathcal{X}} \\ &= \langle \tilde{z}(0), \tilde{z}_0\rangle_{\mathcal{X}} - \langle \tilde{z}(\theta_c), \mathbb{T}_{c,\theta_c}(0)\tilde{z}_0\rangle_{\mathcal{X}} = \langle \tilde{z}(0), (1-\rho_{c,\theta_c})\tilde{z}_0\rangle_{\mathcal{X}}. \end{aligned} \tag{32}$$

For the last equality we took into account (28) and that $\tilde{z}(0) = \tilde{z}(\theta_c)$. If $\rho_{c,\theta_c} \geq 1$ we obtain from (32) that

$$\langle \Psi(\theta_c), \tilde{z}_0\rangle_{\mathcal{X}} = 0.$$

Further, employing (31) we get

$$0 = \int_0^{\theta_c} \langle \mathbf{T}_c^*(s,0) f(s), \tilde{z}_0\rangle_{\mathcal{X}} ds = \int_0^{\theta_c} \langle f(s), \mathbf{T}_c(s,t)\tilde{z}_0\rangle_{\mathcal{X}} ds. \tag{33}$$

Invoking again the fact that $\mathcal{X}^+$ is a self dual, convex cone and that $\mathbf{T}_c(s,t)$ is a positive operator, we deduce that

$$\langle f(s), \mathbf{T}_c(s,0)\tilde{z}_0\rangle_{\mathcal{X}} \geq 0,$$

for all $s \in [0, \theta_c]$. Hence, (33) yields:

$$\langle f(s), \mathbf{T}_c(s,0)\tilde{z}_0\rangle_{\mathcal{X}} = 0 \tag{34}$$

for any $0 \leq s \leq \theta_c$. From (28) and (34) we conclude that ρ_{c,θ_c} is a characteristic multiplier unobservable at $t = 0$ for the system (27). Since the system (27) is exact detectable we may infer via $(i) \implies (ii)$ from Theorem 2, that $\rho_{c,\theta_c} < 1$. Thus the proof ends. □

Remark 4. *In the case of the differential equation (22) the corresponding backward linear differential equation in the form (26) is:*

$$\dot{Y}(t) + A_0^T(t)Y(t) + Y(t)A_0(t) + \sum_{k=1}^{r} A_k^T(t)Y(t)A_k(t) + F(t) = 0 \tag{35}$$

where $F(\cdot) : \mathbb{R} \to \mathcal{S}_n^+$ is a matrix valued function continuous and θ_c periodic. According to (1) written for $N = 1$, we obtain the following system of the form (27) associated to the equation (22):

$$\dot{X}(t) = \mathcal{L}(t)X(t) \tag{36a}$$

$$Y(t) = Tr[C(t)X(t)C^T(t)], \quad t \in \mathbb{R} \tag{36b}$$

where $C(t)$ is provided by the factorization $F(t) = C^T(t)C(t)$, $t \in \mathbb{R}$ and $\mathcal{L}(\cdot)$ is the linear operator (23). Let $X(\cdot; t_0, X_0)$ be the solution of the differential equation (36a) corresponding to the initial pair $(t_0, X_0) \in \mathbb{R} \times \mathcal{S}_n^+$. The output (36b) determined by the trajectory is

$$y(t; t_0, X_0) = Tr[C(t)X(t; t_0, X_0)C^T(t)],$$

$t \in \mathbb{R}$. Hence $y(t; t_0, X_0) = 0$ for any $t \geq t_0$ if and only if $\tilde{Y}(t; t_0, X_0) \triangleq C(t)X(t; t_0, X_0)C^T(t) = 0$ for all $t \geq t_0$ because $X(t; t_0, X_0) \in \mathcal{S}_n^+$, for all $t \geq t_0$. Applying Theorem 4.1 from [15] we may conclude that the system (36) is exact detectable in the sense of Definition 2 if and only if the stochastic linear system

$$dx(t) = A_0(t)x(t)dt + \sum_{k=1}^{r} A_k(t)x(t)dw_k(t)$$
$$y(t) = C(t)x(t),$$

is exact detectable in the sense of Definition 2.1 from [15]. This allows us to conclude that the result proved in Theorem 4 from above, recover as a special case the result proved in Theorem 3.1 from [16].

4.2 The case of discrete-time linear equation

On the finite dimensional ordered Hilbert space $(\mathcal{X}, \mathcal{X}^+)$ we consider the discrete-time linear equation

$$x_d(t+1) = \mathcal{L}_d(t)x_d(t),\ t \in \mathbb{Z}. \tag{37}$$

The order relation " $\succeq$ " on $\mathcal{X}$ is induced by the convex cone $\mathcal{X}^+$, having the same properties as in the case of the differential equation (20). Regarding the operators $\mathcal{L}_d(t)$ involved in (37) we make the following assumption:

$\mathbf{H}_d^1$)

a) $\{\mathcal{L}_d(t)\}_{t\in\mathbb{Z}} \subset \mathbf{L}[\mathcal{X}]$ is a periodic sequence of period $\theta_d \geq 1$;

b) for each $t \in \mathbb{Z}, \mathcal{L}_d(t) : \mathcal{X} \to \mathcal{X}$ is a positive linear operator on the ordered space $(\mathcal{X}, \mathcal{X}^+)$.

Example 5. *On the ordered space $(\mathbb{R}^n, \mathbb{R}_+^n)$ we consider the discrete time linear equation:*

$$x(t+1) = M(t)x(t) \tag{38}$$

where, for each $t \in \mathbb{Z}$, $M(t)$ is the matrix with respect to the canonical basis in $\mathbb{R}^n$ of the operator $\mathcal{L}_d(t)$. If $\{M(t)\}_{t\in\mathbb{Z}} \subset \mathbb{R}^{n\times n}$ is a periodic sequence of period $\theta_d \geq 1$ and if for each $t \in \mathbb{Z}$, the elements of the matrix $M(t)$ are non-negative, then the conditions from the assumption $\boldsymbol{H}_d^1$) are satisfied in the case of the equation (38). The difference equation (38) *appears in applications whose states are naturally non-negative, for example, price and population modeling in [18, Chaps. 12, 13].*

Example 6. *On the ordered Hilbert space $\left(\mathcal{S}_n, \mathcal{S}_n^+\right)$ we consider the discrete-time linear equation*

$$X(t+1) = \sum_{k=0}^{r} M_k(t)X(t)M_k^T(t),\ t \in \mathbb{Z}. \tag{39}$$

This discrete-time linear equation is defined by the linear operator

$$\mathcal{L}_d(t)X = \sum_{k=0}^{r} M_k(t)XM_k^T(t). \tag{40}$$

This kind of discrete-time linear equation occurs in connection with the derivation of criteria for exponential stability in mean square of a discrete-time stochastic linear equation:

$$x(t+1) = \left(M_0(t) + \sum_{k=1}^{r} w_k(t)M_k(t)\right)x(t),\ t \in \mathbb{Z}_+ \tag{41}$$

where, $\{w(t)\}_{t\in\mathbb{Z}_+}$ $\left(w(t) = (w_1(t)\ w_2(t)\ \ldots\ w_r(t))^T\right)$ is an r-dimensional independent random vector with zero mean and covariance $\mathbb{E}[w(t)w^T(t)] = I_r$. For the reader convenience, we refer to [25].

From (40) it is clear that if for each $0 \leq k \leq r$, $\{M_k(t)\}_{t\in\mathbb{Z}_+}$ is a periodic sequence of period $\theta_d \geq 1$, then the operators $\mathcal{L}_d(t)$ have the properties from the assumption $\boldsymbol{H}_d^1$).

The discrete-time counterpart of Definition 8 is:

Definition 10. *We say that the zero state equilibrium of the equation (37) is exponentially stable if its solutions $x(t; t_0, x_0)$ have a decay of the form*

$$||x(t; t_0, x_0)|| \leq \beta\delta^{t-t_0}||x_0||,$$

for any $t \geq t_0$, $t, t_0 \in \mathbb{Z}$, where $\beta \geq 1$, $\delta \in (0, 1)$ are constants not depending upon t, t_0, x_0.

A set of criteria which allows us to deduce if the zero state equilibrium of a discrete-time linear equation of type (37) under the assumption $\mathbf{H}_d^1$) is provided, for example, by Theorem 2.4 from [13]. So, according to the equivalence $(i) \Leftrightarrow (vi)$ from Theorem 2.4 together with Theorem 2.5 (ii) from the aforementioned reference, it follows that the zero state equilibrium of the equation (37) is exponentially stable if and only if there exists a sequence $\{f_d(t)\}_{t\in\mathbb{Z}} \subset Int\mathcal{X}^+$, periodic with period θ_d with the property that the discrete-time backward affine equation

$$z(t) = \mathcal{L}_d^*(t)z(t+1) + f_d(t) \tag{42}$$

has a periodic solution $\{\tilde{z}(t)\}_{t\in\mathbb{Z}} \subset Int\ \mathcal{X}^+$ of period θ_d. Our aim is to show that under a suitable exact detectability property, the exponential stability of the zero state equilibrium of (37) is guaranteed also in the case when a discrete time backward affine equation of the form (42) with the free term $\{f_d(t)\}_{t\in\mathbb{Z}} \subset \mathcal{X}^+$ (instead of $\{f_d(t)\}_{t\in\mathbb{Z}} \subset Int\ \mathcal{X}^+$) has a periodic solution $\{\zeta(t)\}_{t\in\mathbb{Z}} \subset \mathcal{X}^+$ (instead of $\{\zeta(t)\}_{t\in\mathbb{Z}} \subset Int\ \mathcal{X}^+$).

Attaching a suitable output to the system (37) we obtain the following discrete-time linear system:

$$x(t) = \mathcal{L}_d(t)x(t) \tag{43a}$$

$$y_d(t) = \langle f_d(t), x(t)\rangle_{\mathcal{X}}. \tag{43b}$$

The following result is the discrete-time counterpart of the result proved in Theorem 4.

Theorem 5. *Assume:*

a) the assumption $\boldsymbol{H}_d^1$) is fulfilled;

b) the vector valued sequence $\{f_d(t)\}_{t\in\mathbb{Z}} \subset \mathcal{X}^+$, periodic of period θ_d, is such that the corresponding system of type (43) is exact detectable.

Under these conditions, the following are equivalent:

(i) the zero state equilibrium of the discrete-time linear equation (37) is exponentially stable;

(ii) the discrete-time backward affine equation of the form (42) whose free term $f_d(t)$, $t \in \mathbb{Z}$ is satisfying the conditions stated in b) has a solution $\{\tilde{\zeta}(t)\}_{t\in\mathbb{Z}} \subset \mathcal{X}^+$ which is periodic with period θ_d.

Proof. (Hint) The implication $(i) \implies (ii)$ follows applying Theorem 2.5 (ii) from [13].

To prove the implication $(ii) \implies (i)$ we consider the linear evolution operator $\mathbf{T}_d(t, t_0)$, $t \geq t_0$, $t, t_0 \in \mathbb{Z}$ defined by the discrete-time linear equation (37). Let $\mathbb{T}_{t,\theta_d}(t_0) \stackrel{\Delta}{=} \mathbb{T}_d(t_0 + \theta_d, t_0)$ be the monodromy operator associated to the equation (37). We have to prove that its spectral radius is satisfying $\rho_{d,\theta_d} < 1$. Applying Theorem 19.2 from [11] in the case of the linear and positive operator $\mathbb{T}_{d,\theta_d}(0)$ we deduce that there exists $\tilde{\zeta}_0 \neq 0$ in $\mathcal{X}^+$ satisfying

$$\mathbb{T}_{d,\theta_d}(0)\tilde{\zeta}_0 = \rho_{d,\theta_d}\tilde{\zeta}_0. \tag{44}$$

Further, writing (42) for its solution $\{\tilde{\zeta}(t)\}_{t \in \mathbb{Z}}$ we obtain the representation formula

$$\tilde{\zeta}(t) = \mathbf{T}_d^*(t_0 + \theta_d, t)\tilde{\zeta}(t_0 + \theta_d) + \sum_{s=t}^{t_0+\theta_d-1} \mathbf{T}_d^*(s, t) f_d(s), \tag{45}$$

for all $t_0 \in \mathbb{R}, t < t_0 + \theta_d$. Taking $t_0 = t = 0$ in (45) we get:

$$\tilde{\zeta}(0) = \mathbb{T}_{d,\theta_d}^*(0)\tilde{\zeta}(\theta_d) + \Psi(\theta_d) \tag{46}$$

where

$$\Psi(\theta_d) = \sum_{s=0}^{\theta_d-1} \mathbf{T}_d^*(s, 0) f_d(s). \tag{47}$$

Reasoning as in the proof of the Theorem 4 we obtain via (46), (47) that

$$\langle f_d(s), \mathbf{T}_d(s, 0)\tilde{\zeta}_0 \rangle_{\mathcal{X}} = 0, \tag{48}$$

$0 \leq s \leq \theta_d - 1$. From (44) and (48) we may conclude that ρ_{d,θ_d} is a characteristic multiplier unobservable for the system (43). Since the system (43) is exact detectable, we may infer via the implication $(i) \implies (ii)$ from Theorem 3 that $\rho_{d,\theta_d} < 1$. Thus the proof ends. □

5 Conclusion

In this paper, we established a spectral, PBH-type criterion for detectability in systems with periodic coefficients in finite-dimensional real ordered Hilbert spaces. We further developed a Barbasin–Krasovski-type criterion for exponential stability, showing that the existence of solutions to a backward differential/difference equation with a non-necessarily uniform positive forcing term, combined with detectability, guarantees stability. While the PHB test provides a more amenable way to numerically check if a system is exact detectable, the Barbasin–Krasovski criterion offers a foundation for future investigations on stability in optimal control problems.

Appendix A. Proof of Theorem 1

We prove the implication $(i) \Rightarrow (ii)$ by contradiction. Let us assume that there exists a characteristic multiplier $\mu \in \sigma^+[\mathbb{T}_\theta(t_0)], \mu \geq 1$, which is unobservable at t_0. This means that there exist $z_0 \in \mathcal{X}^+$, $z_0 \neq 0$, satisfying the equalities:

$$\mathbb{T}_\theta(t_0)z_0 = \mu z_0 \tag{49a}$$
$$C(s)\mathbf{T}(s,t_0)z_0 = 0, \quad \text{for all} \quad s \in [t_0, t_0+\theta). \tag{49b}$$

Let $x(t;t_0,z_0)$, $t \geq t_0, t, t_0 \in \mathbb{K}$, be the solution of the equation (11a) corresponding to the initial pair $(t_0, z_0) \in \mathbb{K} \times \mathcal{X}^+$.

Let $\left[\dfrac{t-t_0}{\theta}\right]$ be the integer part of the real number $\dfrac{t-t_0}{\theta}$. Implying (12), (15), (17), (49a) we may write successively

$$\begin{aligned} x(t;t_0,z_0) =& \mathbf{T}\Big(t, t_0 + \Big[\frac{t-t_0}{\theta}\Big]\theta\Big)\mathbf{T}\Big(t_0 + \Big[\frac{t-t_0}{\theta}\Big]\theta, t_0 + \Big[\frac{t-t_0}{\theta}\Big]\theta - \theta\Big)\dots \\ & \dots \mathbf{T}(t_0+\theta,t_0)z_0 = \mu^{\left[\frac{t-t_0}{\theta}\right]}\mathbf{T}\left(t, t_0 + \Big[\frac{t-t_0}{\theta}\Big]\theta\right) z_0 \end{aligned} \tag{50}$$

for all $t \geq t_0, t, t_0 \in \mathbb{K}$.

Employing again (17) together with the periodicity property of the operator-valued function $C(\cdot)$ we obtain

$$\begin{aligned} y(t;t_0,z_0) =& C(t)x(t;t_0,z_0) = \mu^{\left[\frac{t-t_0}{\theta}\right]} C(t)\mathbf{T}\Big(t, t_0 + \big[\frac{t-t_0}{\theta}\big]\theta\Big)z_0 \\ =& \mu^{\left[\frac{t-t_0}{\theta}\right]} C\big(t - \Big[\frac{t-t_0}{\theta}\Big]\theta\big)\mathbf{T}(t - \Big[\frac{t-t_0}{\theta}\Big]\theta, t_0)z_0. \end{aligned} \tag{51}$$

From the definition of the integer part of a real number, we know that

$$\Big[\frac{t-t_0}{\theta}\Big] \leq \frac{t-t_0}{\theta} < \Big[\frac{t-t_0}{\theta}\Big] + 1.$$

This leads to

$$t_0 \leq s := t - \left[\frac{t - t_0}{\theta}\right]\theta < t_0 + \theta.$$

So, (49b) and (51) allow us to conclude that $y(t; t_0, z_0) = 0$, for any $t \geq t_0$, $t, t_0 \in \mathbb{K}$. Since (i) holds, we deduce via Definition 4 that

$$\lim_{t\to\infty} x(t; t_0, z_0) = 0. \tag{52}$$

On the other hand, from (50) one sees that (52) is true if and only if $0 \leq \mu < 1$ or $z_0 = 0$, which is in contradiction to the supposition that $\mu \geq 1$ and $y_0 \neq 0$. Hence, the implication $(i) \implies (ii)$ is true.

Let us assume now that (ii) holds and show that (i) holds too. Let $x_0 \in \mathcal{X}^+$, $x_0 \neq 0$, with the property that

$$y(t; t_0, x_0) = C(t)x(t; t_0, x_0) = 0, \text{ for all } t \geq t_0, \ t, t_0 \in \mathbb{K}. \tag{53}$$

We have to show that $\lim_{t\to\infty} ||x(t)|| = 0$ where $x(t) = \mathbf{T}(t, t_0)x_0$ is the solution of the equation (11a) corresponding to the initial pair (t_0, x_0) and $|| \cdot ||$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{X}}$. Based on the assumption $\mathbf{H}_c$) a) for the continuous-time case, and the assumption $\mathbf{H}_d$) b) in the discrete time case, we may infer that

$$x(t) \in \mathcal{X}^+, \text{ for all } t \geq t_0, \ t, t_0 \in \mathbb{K}. \tag{54}$$

Let $\mathfrak{Z}$ be the set of all finite sums with nonnegative coefficients formed with the terms of the sequence $\{x(t_0 + l\theta)\}_{l\in\mathbb{Z}_+}$. So, $z \in \mathfrak{Z}$, if and only if

$$z = \sum_{j=1}^{M} \alpha_j x(t_0 + l_j\theta), \tag{55}$$

where $\alpha_j \geq 0$ are arbitrary. In (55) the number M is not prefixed. It depends upon z. From (54) and (55), we see that $\mathfrak{Z} \subset \mathcal{X}^+$ and it is a pointed convex cone. We show that:

$$\mathbb{T}_\theta(t_0)\mathfrak{Z} \subset \mathfrak{Z} \tag{56a}$$

$$C(t_0)\mathfrak{Z} = \{0\}. \tag{56b}$$

Indeed from (12), (15), (17), (55) we deduce that if $z \in \mathfrak{Z}$ is arbitrary, we have successively

$$\begin{aligned}\mathbb{T}_\theta(t_0)z &= \sum_{j=1}^{M} \alpha_j \mathbf{T}(t_0+\theta, t_0)x(t_0+l_j\theta) \\ &= \sum_{j=1}^{M} \alpha_j \mathbf{T}(t_0+\theta, t_0)\mathbf{T}(t_0+l_j\theta, t_0)x_0 \\ &= \sum_{j=1}^{M} \alpha_j \mathbf{T}(t_0+(l_j+1)\theta, t_0)x_0 \\ &= \sum_{j=1}^{M} \alpha_j x(t_0+(l_j+1)\theta) \in \mathfrak{Z}.\end{aligned}$$

This means that (56a) is true. Further, periodicity property of the operator-valued function $C(\cdot)$ together with (53) yield

$$\begin{aligned}C(t_0)z &= \sum_{j=1}^{M} \alpha_j C(t_0)x(t_0+l_j\theta) \\ &= \sum_{j=1}^{M} \alpha_j C(t_0+l_j\theta)x(t_0+l_j\theta; t_0, x_0) = 0.\end{aligned}$$

Thus we have shown that (56b) holds too.

Let

$$\mathcal{H}^+ \triangleq Cl\,\mathfrak{Z}$$

be the closure of the set $\mathfrak{Z}$ with respect to the topology on $\mathcal{X}$, induced by the norm defined by the inner product $\langle\cdot,\cdot\rangle_{\mathcal{X}}$. Hence $\mathcal{H}^+ \subset \mathcal{X}^+$ because $\mathcal{X}^+$ is a closed set. Moreover, $\mathcal{H}^+$ is a closed, pointed, convex cone. Further, (56) is extended in a natural way to

$$\mathbb{T}_\theta(t_0)\mathcal{H}^+ \subset \mathcal{H}^+ \quad (57a)$$

$$C(t_0)\mathcal{H}^+ = \{0\}. \quad (57b)$$

Let

$$\mathcal{H} \triangleq \mathcal{H}^+ - \mathcal{H}^+. \quad (58)$$

This means that $z \in \mathcal{H}$ if and only if there exist $z_k \in \mathcal{H}^+$, $k = 1, 2$ such that $z = z_1 - z_2$. One sees that $\mathcal{H} \subset \mathcal{X}$ is a closed linear subspace. Hence, the restriction to this subspace of the inner product $\langle\cdot,\cdot\rangle_{\mathcal{X}}$, will induce a Hilbert space structure on $\mathcal{H}$. From (57) and (58), we deduce that

$$\mathbb{T}_\theta(t_0)\mathcal{H} \subset \mathcal{H}.$$

Let $\tilde{\mathbb{T}}_\theta(t_0) = \mathbb{T}_\theta(t_0)|_{\mathcal{H}}$ be the restriction of the operator $\mathbb{T}_\theta(t_0)$ to the invariant subspace $\mathcal{H}$. Bearing in mind (57a), we may infer that $\tilde{\mathbb{T}}_\theta(t_0)$ is a linear and positive operator on the ordered Hilbert space $(\mathcal{H}, \mathcal{H}^+)$. From (58) it follows that $\mathcal{H}^+$ is a closed, pointed, reproducing, convex cone on $\mathcal{H}$. Employing Lemma 3.1 from [10] we may conclude that $\mathcal{H}^+$ is a closed, solid, normal, reproducing, convex cone.

Let $\tilde{\rho}_\theta$ be the spectral radius of the linear operator $\tilde{\mathbb{T}}_\theta(t_0)$. Applying Theorem 19.2 from [11] in the case of the positive linear and bounded operator $\tilde{\mathbb{T}}_\theta(t_0)$ defined on the ordered Hilbert space $\mathcal{H}^+$, we obtain that there exists $\tilde{x}_0 \in \mathcal{H}^+, \tilde{x}_0 \neq 0$, satisfying

$$\tilde{\mathbb{T}}_\theta(t_0)\tilde{x}_0 = \tilde{\rho}_\theta\tilde{x}_0. \tag{59}$$

Further, we show that

$$C(t)\mathbf{T}(s,t_0)\tilde{x}_0 = 0, \ \text{ for all } \ s \in [t_0, t_0+\theta). \tag{60}$$

First, let us assume that $\tilde{x}_0 \in \mathfrak{Z}$. When $s = t_0$, (60) reduces to $C(t_0)\tilde{x}_0 = 0$ which is true because of (56b). To prove (60) for $s \in (t_0, t_0+\theta)$, we use (17) (55) together with the periodicity property of the function $C(\cdot)$ and obtain:

$$\begin{aligned} C(s)\mathbf{T}(s,t_0)\tilde{x}_0 &= \sum_{j=1}^{M} \alpha_j C(s)\mathbf{T}(s,t_0)x(t_0+l_j\theta) \\ &= \sum_{j=1}^{M} \alpha_j C(s+l_j\theta)\mathbf{T}(s+l_j\theta, t_0+l_j\theta)x(t_0+l_j\theta) \\ &= \sum_{j=1}^{M} \alpha_j C(s+l_j\theta)x(s+l_j\theta). \end{aligned}$$

Further, (53) allows us to conclude that $C(s)\mathbf{T}(s,t_0)\tilde{x}_0 = 0$ if $s \in (t_0, t_0+\theta)$. Thus, we have shown that (60) holds when $\tilde{x}_0 \in \mathfrak{Z}$.

Finally, we may infer that (60) still holds in the case when $\tilde{x}_0 \in \mathcal{H}^+$, because $\mathfrak{Z}$ is a dense subset included in $\mathcal{H}^+$. From (59) and (60), we deduce that $(\mu, z_0) \leftarrow (\tilde{\rho}_\theta, \tilde{x}_0)$ is satisfying (49). If the statement (ii) is true, it follows that $\tilde{\rho}_\theta < 1$, because $\tilde{x}_0 \in \mathcal{X}^+$ is a nonzero eigenvector associated with the eigenvalue $\mu \leftarrow \tilde{\rho}_\theta$. Hence, there exist $\beta \geq 1$, $\delta \in (0,1)$ such that

$$||x(t_0 + l_j\theta)|| = ||\tilde{\mathbb{T}}_\theta^l(t_0)x_0|| \leq \beta\delta^l||x_0||$$

for any $l \in \mathbb{Z}_+$. Let $t \geq t_0$ be arbitrary l in $\mathbb{K}$. We chose $l \in \mathbb{Z}_+$ with the property that $l\theta \leq t - t_0 < (l+1)\theta$. We have

$$\begin{aligned} ||x(t)|| &= ||\mathbf{T}(t, t_0 + l\theta)x(t_0 + l\theta)|| \\ &\leq ||\mathbf{T}(t, t_0 + l\theta)||\ \beta\delta^l\ ||x_0|| \\ &\leq \beta\delta^{-1}\delta_1^{t-t_0}||\mathbf{T}(t, t_0 + l\theta)|| \cdot ||x_0||, \end{aligned} \tag{61}$$

where $\delta_1 \triangleq \delta^{1/\theta} \in (0,1)$. Further, we have:

$$||\mathbf{T}(t, t_0 + l\theta)|| \leq e^{\gamma\theta}, \ \text{ for all } \ t \geq t_0 \in \mathbb{K},\ l \in \mathbb{Z}_+,$$

with $\gamma > 0$ is a constant depending upon $\sup_{t\in\mathbb{K}} ||\mathcal{L}(t)||$. Thus, (61) becomes

$$||x(t)|| \leq \beta_1\delta_1^{t-t_0}||x_0||, \ \text{ for all } \ t \geq t_0 \in \mathbb{K}, \tag{62}$$

$\beta_1 = \beta\delta^{-1}e^{\gamma\theta}$. So, (62) gives $\lim_{t\to\infty} ||x(t)|| = 0$. Thus, the proof ends.

References

[1] B.D.O. Anderson and J.B. Moore, Detectability and stabilizability of time-varying discrete-time linear systems, *SIAM J. Control Optim.* 1 (1981), 20-32.

[2] M. Ait Rami, V.S. Bokharaie, O. Mason and F.R. Wirth, Stability criteria for SIS epidemiological models under switching policies, *Discrete Contin. Dyn. Syst. Ser. B* 19 (2014), 2865-2887.

[3] E.M. Barbashin and N.N. Krasovskii, Stability of motion in the large, *Dokl. Akad. Nauk* 86 (1952), 453-456.

[4] S. Bittanti and P. Colaneri, *Periodic Systems: Filtering and Control*, Springer-Verlag London, 2009.

[5] F. Blanchini, P. Colaneri and R.H. Middleton, A convexity result for the optimal control of a class of positive nonlinear systems, *IFAC Proc.* 47 (2014), 1507-1512.

[6] P. Bolzern and P. Colaneri, *Positive Markov Jump Linear Systems*, New Foundations and Trends, 2015.

[7] E.F. Costa and J.B. do Val, On the observability and detectability of continuous-time Markov jump linear systems, *SIAM J. Control Optim.* 41 (2002), 1295-1314.

[8] Ju.L. Daleckii and M.G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, Translations of Mathematical Monographs, vol. 43, American Mathematical Society, 1974.

[9] T. Damm, On detectability of stochastic systems, *Automatica J. IFAC* 43 (2007), 928–933.

[10] T. Damm, *Rational Matrix Equations in Stochastic Control Lecture Notes in Control and Information Sciences*, 297, Berlin, Germany, Springer, 2004.

[11] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[12] V. Drăgan, T. Damm, G. Freiling and T. Morozan, *Differential equations with positive evolutions and some applications*, *Res. Math.* 48 (2005), 206-236.

[13] V. Drăgan, T. Morozan and A.-M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, Springer, 2010.

[14] V. Drăgan, T. Morozan and A.-M. Stoica, *Mathematical Methods in Robust Control of Linear Stochastic Systems*, 2nd ed. New York, NY, USA, Springer, 2013.

[15] V. Drăgan, E.F. Costa, I.-L. Popa and S. Aberkane, Exact detectability of discrete-time and continuous-time linear stochastic systems: a unified approach, *IEEE Trans. Automat. Control* 67 (2022), 5730-5745.

[16] V. Drăgan, E.F. Costa, I.-L. Popa and S. Aberkane, Exact detectability: application to generalized Lyapunov and Riccati equations, *Syst. Control Lett.* 157 (2021), 105032.

[17] B.K. Driver, *Analysis Tools with Applications*, Springer, 2003.

[18] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, John Wiley & Sons, 2011.

[19] G. Freiling and A. Hochhaus, On a class of rational matrix differential equations arising in stochastic control, *Linear Algebra Appl.* 379 (2004), 43-68.

[20] A. Friedman, *Stochastic Differential Equations and Applications*, vol. I, New York, 1975.

[21] E. Gershon, U. Shaked and I. Yaesh, *H-infinity Control and Estimation of State-Multiplicative Linear Systems*, Lecture Notes in Control and Information Sciences, London, Springer, 2005.

[22] W. Haddad, V. Chellaboina and Q. Hui, *Nonengative and Compartmental Dynamical Systems*, Princeton University Press, Princeton, NJ, 2010.

[23] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*, Academic Press, 1966.

[24] T. Kaczorek, *Positive 1D and 2D Systems*, Springer Verlag, London, UK, 2002.

[25] R.Z. Khasminskii, *Stochastic Stability of Differential Equations*, Sythoff and Noordhoff, Alpen aan den Ryn, 1980.

[26] H. Leyva, R. Femat, F.A. Carrillo and G. Quiroz-Compean, On positive systems with multivariable positive control: process stabilization, *J. Process Control* 132 (2023), 103-128.

[27] R. Olfati-Saber, J.A. Fax and R.M. Murray, Consensus and cooperation in networked multi-agent systems, *Proc. IEEE.* 95 (2007), 215-233.

[28] W.M. Wohnam, On a matrix Riccati equation of stochastic control, *SIAM J. Control Optim.* 6 (1968), 681-685.

[29] V.A. Yakubovich and V.M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients*, Translated from Russian by D. Louvish. John Wiley & Sons, 1975.