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FIXED POINT THEOREMS FOR A CLASS OF MAPPINGS OF CONTRACTIVE TYPE*

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Dedicated to Biagio Ricceri on the occasion of his 70th anniversary

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Abstract

We introduce the notion of α-ψ-$\mathcal{R}$ contractive mappings that act on a metric space. We establish the existence and uniqueness of fixed points for this class of mappings and provide a sequence of iterates which approximate their fixed points. Some examples are presented and the relationships with some previous results are described.

Keywords: binary relation, complete metric space, fixed point theorem, metric space.

MSC: 47H10, 54H25.

1 Introduction

For over a century and counting, fixed point theory has continued to draw the attention of many researchers due to its many applications. Fixed point methods have thus been applied in diverse areas of studies such as variational inequality problems and complementarity problems. Many studies in this direction go back to the classical contraction principle [4] which remains

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an indispensable tool in this theory. Since then, there have been many extensions and generalizations of this principle. These extensions have taken place, for instance, in the form of generalizing the metric spaces or weakening the contraction assumption.

In 2012, Samet et al. [10] introduced the notion of an α-ψ contraction. They established existence and uniqueness results regarding the fixed points of such mappings in complete metric spaces. Alam and Imdad [2] extended the classical Banach contraction principle to a complete metric space equipped with a binary relation. As reported by [2], this is a weaker contractive condition; unlike the usual condition, it is only required to hold on those elements that are related by the underlying relation and not on the whole space.

In this paper, we introduce α-ψ-$\mathcal{R}$ contraction mappings in the wake of the works of Samet et al. [10] and Alam and Imdad [2]. We establish some existence and uniqueness results regarding fixed points of the said mappings in a complete metric space. This class of mappings is of particular interest because these mappings offer a robust framework for examining the convergence and stability of mathematical systems. These mappings operate on a complete metric space equipped with arbitrary binary relations. These binary relations are extensions of various previously studied binary relations such as partial order, preorder and transitive relations. We also present the relationship of these new mappings with previously studied mappings.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of positive integers, nonnegative integers, rational numbers and real numbers, respectively.

Definition 1. *[6]. Let X be a nonempty set. A subset $\mathcal{R}$ of $X \times X$ is called a binary relation on X. For each pair $x, y \in X$, exactly one of the following conditions holds:*

(i) $(x, y) \in \mathcal{R}$, *which amounts to saying that x is $\mathcal{R}$-related to y or that x relates to y under $\mathcal{R}$.*

(ii) $(x, y) \notin \mathcal{R}$, *which means that x is not $\mathcal{R}$-related to y or that x does not relate to y under $\mathcal{R}$.*

It is trivial to see that $X \times X$ and $\emptyset$, which are subsets of $X \times X$, are binary relations on X which are respectively referred to as the universal relation (or the full relation) and the empty relation. The identity relation (or the diagonal relation) is given by

$$\triangle_X = \{(x, x) : x \in X\}.$$

In what follows, $\mathcal{R}$ stands for a nonempty binary relation. To keep things simple, we only write "binary relation" instead of "nonempty binary relation".

Definition 2. *[2]. Let $\mathcal{R}$ be a binary relation defined on a nonempty set X. The points x and y are called $\mathcal{R}$-comparable if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. Whenever this is the case, we denote it by $[x, y] \in \mathcal{R}$.*

Definition 3. *[6, 9]. A binary relation $\mathcal{R}$ defined on a nonempty set X is said to be*

(i) reflexive, if $(x, x) \in \mathcal{R} \ \forall \ x \in X$;

(ii) irreflexive, if $(x, x) \notin \mathcal{R} \ \forall \ x \in X$;

(iii) symmetric, if $(x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$;

(iv) antisymmetric, if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R}$ imply that $x = y$;

(v) transitive, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ imply that $(x, z) \in \mathcal{R}$;

(vi) complete, connected or dichotomous, if $[x, y] \in \mathcal{R} \ \forall \ x, y \in X$;

(vii) weakly complete, weakly connected or trichotomous if $[x, y] \in \mathcal{R}$ or $x = y$ for all $x, y \in X$.

Definition 4. *[5, 6, 9, 12]. A binary relation $\mathcal{R}$ defined on a nonempty set X is called*

(i) a strict order (or a sharp order), if $\mathcal{R}$ is irreflexive and transitive;

(ii) a near-order, if $\mathcal{R}$ antisymmetric and transitive;

(iii) a pseudo-order, if $\mathcal{R}$ is reflexive and antisymmetric;

(iv) a quasi-order or a preorder, if $\mathcal{R}$ is reflexive and transitive;

(v) a partial order, if $\mathcal{R}$ is reflexive, antismmetric and transitive;

(vi) a simple order, if $\mathcal{R}$ is a weakly complete strict order;

(vii) a weak order, if $\mathcal{R}$ is a complete preorder;

(viii) a total order, a linear order or a chain order, if $\mathcal{R}$ is a complete partial order;

(ix) a tolerance, if $\mathcal{R}$ is reflexive and symmetric;

(x) an equivalence, if $\mathcal{R}$ is reflexive, symmetric and transitive.

It is obvious that the universal relation $X \times X$ defined on a nonempty set X is a complete equivalence relation.

Definition 5. *[6]. Let X be a nonempty set and let $\mathcal{R}$ be a binary relation on X.*

(a) The inverse, transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by

$$\mathcal{R}^{-1} = \{(x, y) \in X \times X : (y, x) \in \mathcal{R}\}.$$

(b) The reflexive closure of $\mathcal{R}$, denoted by $\mathcal{R}^{\circ}$, is defined to be the set $\mathcal{R} \cup \triangle_X$ (that is, $\mathcal{R}^{\circ} := \mathcal{R} \cup \triangle_X$). Indeed, $\mathcal{R}^{\circ}$ is the smallest reflexive relation on X containing $\mathcal{R}$.

(c) The symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (that is, $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$). Indeed, $\mathcal{R}^s$ is the smallest symmetric relation on X containing $\mathcal{R}$.

Proposition 1. *[2]. For a binary relation $\mathcal{R}$ defined on a nonempty set X,*

$$(x, y) \in \mathcal{R}^s \Leftrightarrow [x, y] \in \mathcal{R}.$$

Definition 6. *[2]. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if we have $(x_n, x_{n+1}) \in \mathcal{R}$, for all $n \in \mathbb{N}_0$.*

Definition 7. *[1, 2]. Let S be a self-mapping on a nonempty set X. A binary relation $\mathcal{R}$ defined on X is called S-closed if, for any $x, y \in X$, we have*

$$(x, y) \in \mathcal{R} \Rightarrow (Sx, Sy) \in \mathcal{R}.$$

Proposition 2. *[1]. If $\mathcal{R}$ is S-closed, then, for all $n \in \mathbb{N}_0$, $\mathcal{R}$ is also S^n-closed, where S^n denotes the nth iterate of S.*

Definition 8. *[11]. Let $\mathcal{R}$ be a binary relation on a nonempty set X. A subset K is called $\mathcal{R}$-directed if for each $x, y \in K$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.*

Definition 9. *[7]. Let $\mathcal{R}$ be a binary relation on a nonempty set X. For $x, y \in X$, a path of length k (k is a natural number) in $\mathcal{R}$ from x to y is a finite sequence $\{w_i\}_{i=0}^{k} \subset X$ satisfying the following conditions:*

(i) $x = w_0$ *and* $y = w_k$;

(ii) $(w_i, w_{i+1}) \in \mathcal{R}$ *for each* $i \in [0, k-1]$.

Observe that a path of length k consists of $k+1$ elements, which are not necessarily distinct.

Definition 10. *[3]. The metric space (X, d) is said to be $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.*

Every complete metric space is $\mathcal{R}$-complete for any binary relation $\mathcal{R}$. In particular, the notion of $\mathcal{R}$-completeness coincides with usual completeness under the universal relation.

Definition 11. *[3]. The mapping S is $\mathcal{R}$-continuous at $x \in X$ if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $Sx_n \xrightarrow{d} Sx$. The mapping S is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at every point $x \in X$.*

Every continuous mapping is $\mathcal{R}$-continuous for any binary relation. In particular, the notion of $\mathcal{R}$-continuity coincides with usual continuity under the universal relation.

We also use the following notations.

(i) $F(S)$ denotes the set of fixed points of S. That is $F(S) = \{x \in S : x = Sx\}$.

(ii) $X(S, \mathcal{R}) := \{x \in X : (x, Sx) \in \mathcal{R}\}$.

(iii) $\Upsilon(x, y, \mathcal{R})$ is the class of all paths in $\mathcal{R}$ from x to y.

3 Main results

We now present our main results. First, we introduce the notion of an α-ψ-$\mathcal{R}$-preserving contractive mapping.

Let Ψ be the family of increasing functions $\psi : [0, +\infty) \to [0, +\infty)$ that satisfy $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ.

Lemma 1. *For every function $\psi : [0, +\infty) \to [0, +\infty)$ the following statement holds: If ψ is increasing, then for each $t \geq 0$, we have*

$$\lim_{n\to\infty} \psi^n(t) = 0 \Rightarrow \psi(t) < t,$$

Definition 12. *A mapping $S : X \to X$ is said to be an α-ψ-$\mathcal{R}$-preserving contractive mapping if there exist two functions, $\alpha : X \times X \to [0, +\infty)$ and $\psi \in \Psi$, such that*

$$\alpha(x, y)d(Sx, Sy) \leq \psi(d(x, y)) \quad \forall \quad (x, y) \in \mathcal{R}.$$

Remark 1. *If $S : X \to X$ is an α-ψ-$\mathcal{R}$-preserving contractive mapping, where $\mathcal{R}$ is the universal relation, $\alpha(x, y) = 1$ for all $(x, y) \in \mathcal{R}$ and $\psi(t) = \beta t$ for some $\beta \in [0, 1)$ and all $t \geq 0$, then S reduces to a Banach strict contraction.*

Definition 13. *Let $S : X \to X$ and $\alpha : X \times X \to [0, +\infty)$. Then S is said to be $\mathcal{R}$ α-admissible if*

$$(x, y) \in \mathcal{R}, \quad \alpha(x, y) \geq 1 \Rightarrow \alpha(Sx, Sy) \geq 1.$$

Example 1. *Let $X = (0, +\infty)$. Define a binary relation $\mathcal{R} = \{(x, y) \in X^2 : x \geq y, \quad x \in \mathbb{Q}\}$ on X. Define $S : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Sx = \ln x$ for all $x \in X$ and $\alpha(x, y) = 2$ for $(x, y) \in \mathcal{R}$. Then S is $\mathcal{R}$ α-admissible.*

We are now in position to state and prove our first main result.

Theorem 1. *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation on X and S be an α-ψ-$\mathcal{R}$-preserving contractive mapping such that*

(1) (X, d) *is* $\mathcal{R}$*-complete.*

(2) $X(S, \mathcal{R}) \neq \emptyset$.

(3) S *is* α*-admissible.*

(4) $\mathcal{R}$ *is* S*-closed.*

(5) there exists $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq 1$.

(6) S *is* $\mathcal{R}$ *continuous.*

Then S has a fixed point.

Proof. Since the set $X(S,\mathcal{R})$ is nonempty, we can choose an arbitrary point $x_0 \in X(S,\mathcal{R})$. Now, define a sequence of successive points with x_0 as its initial point as follows:

$$x_{n+1} = S^{n+1}x_0 = Sx_n, \quad n = 0, 1, \cdots . \tag{1}$$

Since $(x_0, Sx_0) \in \mathcal{R}$, using the S-closedness of $\mathcal{R}$ and Proposition 2, we have $(S^n x_0, S^{n+1}x_0) \in \mathcal{R}$, which by (1) implies that $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \geq 0$. Consequently, the sequence $\{x_n\}$ is $\mathcal{R}$-preserving. Since S is α-admissible, we have

$$\alpha(x_0, x_1) = \alpha(x_0, Sx_0) \geq 1 \Rightarrow \alpha(Sx_0, Sx_1) = \alpha(x_1, x_2) \geq 1.$$

Using mathematical induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \quad n = 0, 1, \cdots . \tag{2}$$

Now, from the definition of an α-ψ-$\mathcal{R}$-preserving contractive mapping, it follows that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Sx_n, Sx_{n+1}) \\ &\leq \alpha(x_n, x_{n+1}) d(Sx_n, Sx_{n+1}) \\ &\leq \psi(d(x_n, x_{n+1})) \\ &\vdots \\ &\leq \psi^{n+1} d(x_0, Sx_0), \quad n = 0, 1, \cdots . \end{aligned} \tag{3}$$

Assume that $\varepsilon > 0$ and $n(\varepsilon)$ are such that $\sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, Sx_0)) < \varepsilon$.

Let $n, m \in \mathbb{N}$ with $m > n \geq n(\varepsilon)$. Then it follows from (3) and the triangle inequality that

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) \\ &\leq \sum_{j=n}^{m-1} \psi^j(d(x_0, Sx_0)) \\ &\leq \sum_{n \geq n(\varepsilon)} \psi^n(d(x_0, Sx_0)) < \varepsilon, \end{aligned}$$

which implies that the sequence $\{x_n\}$ is Cauchy in X. Hence $\{x_n\}$ is an $\mathcal{R}$-preserving Cauchy sequence. Since X is $\mathcal{R}$ complete, there exists $x^* \in X$

such that $x_n \xrightarrow{d} x^*$. Next, we show, using (6), that x^* is a fixed point of S. Since $\{x_n\}$ is $\mathcal{R}$-preserving, the $\mathcal{R}$ continuity of S, when combined with $x_n \xrightarrow{d} x^*$, implies that $x_{n+1} = Sx_n \xrightarrow{d} Sx^*$. Using the uniqueness of limits, we get $Sx^* = x^*$. Thus, x^* is a fixed point of S, as asserted. □

In the following theorem we remove the assumption of the $\mathcal{R}$ continuity of S. First, we prove the following important proposition.

Proposition 3. *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation on X and let S be a self-mapping of X. Let $\psi \in \Psi$ and $\alpha : X \times X \to [0, +\infty)$ be such that $\alpha(x, y) = \alpha(y, x)$. Then the following conditions are equivalent:*

(I) $\alpha(x, y)d(Sx, Sy) \leq \psi(d(x, y))\ \forall x, y \in X$ *with* $(x, y) \in \mathcal{R}$;

(II) $\alpha(x, y)d(Sx, Sy) \leq \psi(d(x, y))\ \forall x, y \in X$ *with* $[x, y] \in \mathcal{R}$.

Proof. The implication $(II) \Rightarrow (I)$ is trivial. Conversely, suppose that (I) holds. Take $x, y \in X$ with $[x, y] \in \mathcal{R}$. If $(x, y) \in \mathcal{R}$, then (II) directly follows from (I). Otherwise, if $(y, x) \in \mathcal{R}$, then using the symmetry of d and I, we obtain

$$\alpha(x, y)d(Sx, Sy) = \alpha(y, x)d(Sy, Sx) \leq \psi(d(y, x)) = \psi(d(x, y)).$$

This shows that $(I) \Rightarrow (II)$. □

Theorem 2. *Let (X, d) be a metric space, $\mathcal{R}$ be a binary relation on X and let S be an α-ψ-$\mathcal{R}$-preserving contractive mapping such that*

(1) (X, d) *is* $\mathcal{R}$*-complete.*

(2) $X(S, \mathcal{R}) \neq \emptyset$.

(3) S *is* α*-admissible.*

(4) $\mathcal{R}$ *is* S*-closed.*

(5) there exists $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$.

(6) $\mathcal{R}$ *is d-self closed.*

If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \xrightarrow{d} x^*$ as $n \to \infty$, then $\alpha(x_n, x^*) \geq 1$. Then S has a fixed point.

Proof. We have already shown that $\{x_n\}$ is a $\mathcal{R}$-Cauchy sequence. Now suppose $\mathcal{R}$ is d-self closed. Since the sequence $\{x_n\}$ is $\mathcal{R}$-preserving and $x_n \xrightarrow{d} x^*$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x^*] \in \mathcal{R}$ for all $k \in \mathbb{N}$. Using the definition of S, Proposition 3, the facts that $[x_{n_k}, x^*] \in \mathcal{R}$ and $x_{n_k} \xrightarrow{d} x^*$, we find that

$$\begin{aligned} d(x_{n_k+1}, Sx^*) &= d(Sx_{n_k}, Sx^*) \\ &\leq \alpha(x_{n_k}, x) d(Sx_{n_k}, Sx^*) \\ &\leq \psi(d(x_{n_k}, x^*)) \to 0 \quad \text{as} \quad k \to \infty, \end{aligned}$$

which implies that $x_{n_k+1} \xrightarrow{d} Sx^*$. It follows from the uniqueness of the limit that $Sx^* = x^*$. Thus, x^* is a fixed point of S. □

We now show that the point x^* in the previous results is the unique fixed point of S.

Theorem 3. *In addition to the hypotheses of Theorem 1 and Theorem 2, assume that $S(X)$ is $\mathcal{R}^s$-connected and for each $i \in [0, k-1]$, we have $\alpha(S^n w_i, S^n w_{i+1}) \geq 1$. Then the fixed point x^* is unique.*

Proof. Assume that $x^*, y^* \in F(S)$. Then

$$S^n x^* = x^* \quad \text{and} \quad S^n y^* = y^*. \tag{4}$$

Since $x^*, y^* \in S(X)$ and $S(X)$ is $\mathcal{R}^s$-connected, there exists a path connecting x^* to y^* such that

$$x^* = w_0, \quad y^* = w_k \quad \text{and} \quad [w_i, w_{i+1}] \in \mathcal{R} \quad \text{for each} \quad i \in [0, k-1]. \tag{5}$$

Using Proposition 2 and the fact that $\mathcal{R}$ is S-closed, we find that

$$[S^n w_i, S^n w_{i+1}] \in \mathcal{R} \quad \text{for each} \quad i \in [0, k-1].$$

For each $n \in \mathbb{N}_0$, we have

$$\begin{aligned} d(S^n w_i, S^n w_{i+1}) &= d(S(S^{n-1} w_i), S(S^{n-1} w_{i+1})) \\ &\leq \alpha(S^{n-1} w_i, S^{n-1} w_{i+1}) d(S^n w_i, S^n w_{i+1}) \\ &\leq \psi(d(S^{n-1} w_i, S^{n-1} w_{i+1})). \end{aligned}$$

Also,

$$\begin{aligned} d(S^{n-1} w_i, S^{n-1} w_{i+1}) &= d(S(S^{n-2} w_i), S(S^{n-2} w_{i+1})) \\ &\leq \alpha(S^{n-2} w_i, S^{n-2} w_{i+1}) d(S^{n-1} w_i, S^{n-1} w_{i+1}) \\ &\leq \psi(d(S^{n-2} w_i, S^{n-2} w_{i+1})). \end{aligned}$$

Proceeding in the same manner, we arrive at

$$d(S^n w_i, S^n w_{i+1}) \leq \psi^n(d(w_i, w_{i+1})). \tag{6}$$

Now, using the triangle inequality, we obtain

$$\begin{aligned} d(x^*, y^*) &= d(S^n w_0, S^n w_k) \\ &\leq d(S^n w_0, S^n w_1) + d(S^n w_1, S^n w_2) + \cdots + d(S^n w_{k-1}, S^n w_k) \\ &= \sum_{i=0}^{n} d(S^n w_i, S^n w_{i+1}) \\ &\leq \sum_{i=0}^{n} \psi^n(d(w_i, w_{i+1})) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{7}$$

Therefore, $x^* = y^*$, which implies that the fixed point of S is indeed unique, as asserted. □

We now present some special cases of our main result.

(1) Under the universal relation (that is, $\mathcal{R} = X^2$), Theorem 1 and Theorem 2 coincide with the results of [11, Theorem 2.1] and [11, Theorem 2.2], respectively.

(2) Suppose that $\alpha(x, y) = 1$ for all $x, y \in X$ with $(x, y) \in \mathcal{R}$ and that $\psi(t) = \beta t$ for some $\beta \in [0, 1)$. Then Theorems 1 and 2 are the results established in [2].

(3) Assume in (2) above that the relation $\mathcal{R}$ is the universal relation. Then the result reduces to the celebrated Banach contraction principle.

4 Illustrative example

In this section, we illustrate our results by presenting the following example:

Example 2. *Let $X = \mathbb{R}$ be endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Define the binary relation $\mathcal{R} = \{(x, y) \in \mathbb{R} : x - y \geq 0,\ x \in \mathbb{Q}\}$ on X. Consider the mapping $S : X \to X$ defined by $Sx := \frac{7}{16} + \frac{x}{8}$. Next, define the mapping $\alpha : X \times X \to [0, +\infty)$ by $\alpha(x, y) := 1$ for all $x, y \in X$ with $(x, y) \in \mathcal{R}$. We claim that S is an α-ψ-$\mathcal{R}$ contractive mapping with $\psi(t) = \frac{t}{4}$ for all $t \geq 0$. Indeed, for all $x, y \in X$ with $(x, y) \in \mathcal{R}$, we have*

$$\alpha(x, y)d(Tx, Ty) = 1\left|\frac{7}{16} + \frac{x}{8} - \left(\frac{7}{16} + \frac{y}{8}\right)\right| = \frac{1}{8}|x - y| \leq \frac{1}{4}d(x, y).$$

Also, $\mathcal{R}$ is S-closed and S is continuous. In addition, there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. In fact, for $x_0 = 1$,

$$\alpha(1, S1) = \alpha\left(1, \frac{9}{16}\right) = 1.$$

Now, let $x, y \in X$ with $(x, y) \in \mathcal{R}$ be such that $\alpha(x, y) \geq 1$. By the definitions of S and α, we have

$$Sx = \frac{7}{16} + \frac{x}{8},\ Sy = \frac{7}{16} + \frac{y}{8},\ Sx \geq Sy \text{ and } \alpha(Sx, Sy) = 1.$$

Therefore, all the hypotheses of Theorem 1 are satisfied and there exists a fixed point of S, namely $\frac{1}{2}$. Since the range of $T(X)$ is the entire set of real numbers, $S(X)$ is $\mathcal{R}^s$-connected and thus the fixed point is unique.

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