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A FIXED POINT RESULT IN GENERALIZED METRIC SPACES WITH GRAPHS*

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Dedicated to Biagio Ricceri on the occasion of his 70th anniversary

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Abstract

We introduce a new class of generalized metric spaces with graphs and prove a fixed point result for Rakotch type contractive mappings.

Keywords: fixed point, generalized metric, generalized nonexpansive mapping, graph, Rakotch contraction.
MSC: 47H09, 47H10, 54E35.

1 Introduction

For more than sixty years, there has been a lot of research activity concerning the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 4, 5, 8–10, 13, 15–20, 24, 25] and references cited therein. This activity stems from Banach's classical theorem [1] regarding the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility, common fixed point

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problems and variational inequalities, which find important applications in engineering, medical and the natural sciences [3, 6, 7, 22–25].

One of the important directions in fixed point theory is the study of operators on some spaces with distances which are not metrics. In particular, an example of such a space is the modular space studied in [11, 12, 14]. In our recent work [21] we have introduced certain generalized metric spaces by extending the concept of a modular space studied in [11, 12, 14] and have established a fixed point theorem for certain Rakotch type contractive operators. In the present paper we introduce a new class of generalized metric spaces with graphs which contains the class of spaces considered in [21] and prove a fixed point result for Rakotch type contractive mappings.

To this end, we first recall the notion of a modular space.

Let X be a vector space. A functional $\rho : X \to [0, \infty]$ is called a *modular* [11, 12, 14] if the following properties hold:

(1) $\rho(x) = 0$ if and only $x = 0$;

(2) $\rho(-x) = \rho(x)$ for all $x \in X$;

(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for each $x, y \in X$ and each $\alpha, \beta \geq 0$ satisfying $\alpha + \beta = 1$.

The vector space

$$X_\rho := \{x \in X : \ \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$$

is called a *modular space*.

Assume that ρ is a modular defined on a vector space X. We say that the modular ρ satisfies a Δ_2-type condition if there exists a number $M > 0$ such that

$$\rho(2x) \leq M\rho(x), \ x \in X_\rho. \tag{1}$$

The authors of [12] considered a modular function space L_ρ (which is a particular case of a modular space) with a modular ρ satisfying a Δ_2-type condition. They showed that if T is a self-mapping of a closed subset K of L_ρ such that for some $c \in [0, 1)$,

$$\rho(T(x) - T(y)) \leq c\rho(x, y) \text{ for all } x, y \in K$$

and such that there exists a point $x_0 \in K$ satisfying

$$\sup\{\rho(2T^p(x_0)) : \ p = 1, 2, \dots\} < \infty,$$

then T has a fixed point.

Assume now that ρ is a modular defined on the vector space X. For each $x, y \in X$, define

$$d(x, y) := \rho(x - y).$$

It is easy to see that for each $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$ and that $d(x, y) = d(y, x)$.

Assume that ρ satisfies the Δ_2-type condition (1) with a number $M > 0$. Then for each $x, y, z \in X_\rho$, we have

$$d(x, z) = \rho(x - z) = \rho((x - y) + (y - z))$$
$$= \rho(2(2^{-1}(x - y) + 2^{-1}(y - z))) \leq M\rho(2^{-1}(x - y) + 2^{-1}(y - z))$$
$$\leq M(\rho(x - y) + \rho(y - z)) \leq Md(x, y) + Md(y, z).$$

It is clear that d, the distance in X_ρ associated with the modular ρ, is not a metric in general. This leads us to the following definition, which was introduced in [21].

Assume that X is a nonempty set, $d : X \times X \to [0, \infty]$, $M > 0$, and that for each $x, y, z \in X$,

$$d(x, y) = 0 \text{ if and only if } x = y,$$

$$d(x, y) = d(y, x)$$

and

$$d(x, z) \leq Md(x, y) + Md(y, z).$$

We call the pair (X, d) a generalized metric space. For each point $x \in X$ and each number $r > 0$, set

$$B_d(x, r) := \{y \in X : \ d(x, y) \leq r\}.$$

Clearly, the concept of a generalized metric space is a generalization of both a modular space and a metric space. It allows us to unify the study of these two important classes of spaces. For specific examples of modular spaces see, for instance, [11, 14].

We equip the space X with the uniformity determined by the base

$$\mathcal{U}(\epsilon) := \{(x, y) \in X \times X : \ d(x, y) \leq \epsilon\}, \ \epsilon > 0.$$

This uniform space is metrizable (by a metric $\tilde{d}$). We also equip the space X with the topology induced by this uniformity and assume that the uniform space X is complete.

Let $\{x_n\}_{n=1}^\infty \subset X$ and $x \in X$. Clearly, $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence if and only if for each $\epsilon > 0$, there exists a natural number $n(\epsilon)$ such that $d(x_n, x_m) \leq \epsilon$ for every pair of integers $n, m \geq n(\epsilon)$.

A set $E \subset X$ is said to be bounded if

$$\sup\{d(x,y): x,y \in E\} < \infty.$$

Assume that $\phi : [0,\infty) \to [0,1]$ is a decreasing function such that

$$\phi(t) < 1 \text{ for all } t > 0.$$

In [21] we proved the following fixed point result for Rakotch type contractive operators.

Theorem 1. *Let K be a nonempty closed subset of X and let $T : K \to X$ satisfy*

$$d(T(x), T(y)) \le \phi(d(x,y))d(x,y)$$

for each $x, y \in K$ satisfying $d(x,y) < \infty$. Assume that for each integer $n \ge 1$, there exists a point $x_n \in K$ such that

$$T^n(x_n) \text{ exists and belongs to } K$$

and that the set

$$E := \{T^i(x_n): n = 1, 2, \dots \text{ and } i \in \{0, \dots, n\}\}$$

is bounded. Then there exists a point $x_* \in K$ such that $T(x_*) = x_*$. Moreover, this fixed point is unique if $d(x,y) < \infty$ for each pair $x, y \in K$.

2 Generalized metric spaces with graphs

Let X be a nonempty set and let a mapping $d : X \times X \to [0,\infty)$ be given. Assume that for each $x, y \in X$, $d(x,y) = 0$ if and only if $x = y$ and that the following properties hold.

(P1) For each $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in X$ satisfy $d(x,y) \le \delta$, then $d(y,x) \le \epsilon$.

(P2) For each $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, y, z \in X$ satisfying $d(x,y)$, $d(y,z) \le \delta$, we have $d(x,z) \le \epsilon$.

For each $\epsilon > 0$, define

$$\mathcal{U}(\epsilon) = \{(x,y) \in X \times X : d(x,y), d(y,x) \le \epsilon\}. \tag{2}$$

It is not difficult to see that $\mathcal{U}(\epsilon)$, $\epsilon > 0$, is a base of a uniformity which is metrizable by a metric d_1. We assume that the metric space (X, d_1) is complete and that it is endowed with a graph G. We denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of its edges.

Let a nonempty set $K \subset X$ be closed, assume that

$$\sup\{d(x,y) :\ x, y \in K\} < \infty \tag{3}$$

and that there exists a natural number n_0 such that the following property holds.

(P3) for each $x, y \in K$, there exists a natural number $n \le n_0$ and points $x_i \in K$, $i = 0, \dots, n$, such that

$$x_0 = x,\ x_n = y$$

and for each $i \in \{0, \dots, n-1\}$, at least one of the following inclusions holds:

$$(x_i, x_{i+1}) \in E(G),\ (x_{i+1}, x_i) \in E(G).$$

Assume that $T : K \to K$, $\phi : [0,\infty) \to [0,1)$ is a decreasing function,

$$\phi(t) < 1,\ t > 0, \tag{4}$$

and for each $(x, y) \in E(G)$,

$$(T(x), T(y)) \in E(G), \tag{5}$$

$$d(T(x), T(y)) \le \phi(d(x,y))d(x,y). \tag{6}$$

Set $T^0(x) = x$, $x \in K$.

Theorem 2. *There exists a point $x_* \in K$ such that $T^n(x) \to x_*$ in K as $n \to \infty$.*

3 Proof of Theorem 2

We begin with the following auxiliary result.

Lemma 1. *Assume that $x, y \in K$, $(x,y) \in E(G)$, $\epsilon > 0$, and*

$$d(x,y) > \epsilon. \tag{7}$$

Then

$$d(T(x), T(y)) \le d(x,y) - \epsilon(1 - \phi(\epsilon)).$$

Proof. By (4), (6) and (7),

$$d(x,y) - d(T(x),T(y)) \geq d(x,y)(1-\phi(d(x,y))) \geq d(x,y)(1-\phi(\epsilon)) \geq \epsilon(1-\phi(\epsilon)).$$

This completes the proof of Lemma 1. □

Set

$$D_0 = \sup\{d(x,y) : \ x, y \in K\}. \tag{8}$$

Proof. (*of Theorem 2*) Let $\epsilon > 0$. Property (P2) implies that there exists a number $\epsilon_0 \in (0, \epsilon)$ such that the following property holds:

(P4) for each natural number $n \leq n_0$ and each $\xi_i \in K$, $i = 0, \dots, n_0$, satisfying $d(\xi_i, \xi_{i+1}) \leq \epsilon_0$, $i = 0, \dots, n_0 - 1$, we have

$$d(\xi_0, \xi_n) \leq \epsilon.$$

Fix an integer

$$k_0 > D_0 \epsilon_0^{-1} (1 - \phi(\epsilon_0))^{-1}. \tag{9}$$

Assume that

$$\xi, \eta \in K, \ (\xi, \eta) \in E(G). \tag{10}$$

We claim that there exists an integer $n \in [0, k_0]$ such that

$$d(T^n(\xi), T^n(\eta)) \leq \epsilon_0.$$

Suppose to the contrary that this is not true. Then for each integer $n \in \{0, \dots, k_0\}$,

$$d(T^n(\xi), T^n(\eta)) > \epsilon_0$$

and in view of (10) and Lemma 1,

$$d(T^{n+1}(\xi), T^{n+1}(\eta)) \leq d(T^n(\xi), T^n(\eta)) - \epsilon_0(1 - \phi(\epsilon_0)). \tag{11}$$

By (8) and (11), we have

$$D_0 \geq d(\xi, \eta) \geq d(\xi, \eta) - d(T^{k_0}(\xi), T^{k_0}(\eta))$$

$$= \sum_{i=0}^{k_0 - 1} (d(T^i(\xi), T^i(\eta)) - d(T^{i+1}(\xi), T^{i+1}(\eta)) \geq k_0 \epsilon_0 (1 - \phi(\epsilon_0)),$$

$$k_0 \leq D_0 \epsilon_0^{-1} (1 - \phi(\epsilon_0))^{-1}.$$

This, however, contradicts (9). The contradiction we have reached proves that there does exist an integer $n \in [0, k_0]$ such that

$$d(T^n(\xi), T^n(\eta)) \leq \epsilon_0,$$

as claimed.

Thus the following property holds:

(P5) For each $\xi, \eta \in K$ satisfying $(\xi, \eta) \in E(G)$, the inequality

$$d(T^n(\xi), T^n(\eta)) \leq \epsilon_0$$

holds true for each integer $n \geq k_0$.

Assume now that $x, y \in K$. Property (P3) implies that there exist a natural number $q \leq n_0$ and points $\xi_i \in K$, $i = 0, \ldots, q$, such that

$$\xi_0 = x, \ \xi_q = y,$$

$$(\xi_i, \xi_{i+1}) \in E(G), \ i = 0, \ldots, q-1.$$

Property (P5) and the above relations imply that for each integer $n \geq k_0$,

$$d(T^n(\xi_i), T^n(\xi_{i+1})) \leq \epsilon_0, \ i = 0, \ldots, q-1.$$

When combined with property (P5), this implies that

$$d(T^{k_0}(x), T^{k_0}(y)) \leq \epsilon \tag{12}$$

for each $x, y \in K$.

Fix $\xi_0 \in K$. By (12),

$$d(T^{k_0}(\xi), T^{k_0}(y)) \leq \epsilon, \text{ for each } y \in K.$$

This implies that

$$d(T^{k_0}(\xi), T^n(y)) \leq \epsilon \text{ for each integer } n \geq k_0 \text{ and for each point } y \in K.$$

Since ϵ is an arbitrary positive number, this implies that for each $x \in K$, $\{T^n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence and there exists

$$\lim_{n\to\infty} T^n(x).$$

Let $\delta > 0$. Since ϵ is an arbitrary positive number, in view of (12), there exists a natural number n_δ such that

$$d_1(T^{n_\delta}(x), T^{n_\delta}(y)) \leq \delta, \ x, y \in K.$$

This implies that for each $x, y \in K$ and each integer $n \geq n_\delta$, we have

$$d_1(T^{n_\delta}(x), T^n(y)) \leq \delta$$

and

$$d_1(T^{n_\delta}(x), \lim_{n\to\infty} T^n(y)) \leq \delta.$$

Since δ is an arbitrary positive number, we conclude that

$$\lim_{n\to\infty} T^n(y_1) = \lim_{n\to\infty} T^n(y_2)$$

for each $y_1, y_2 \in K$, and that for each $x, y \in K$ and each integer $n \geq n_\delta$, we have

$$d_1(\lim_{n\to\infty} T^n(x), T^n(y)) \leq d_1(\lim_{n\to\infty} T^n(x), T^{n_\delta}(x)) + d_1(T^{n_\delta}(x), T^n(y)) \leq 2\delta.$$

Since δ is an arbitrary positive number, this completes the proof of Theorem 2. □

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