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FINITE ENERGY WEAK SOLUTIONS TO SOME DIRICHLET PROBLEMS WITH VERY SINGULAR DRIFTS AND NONLINEAR ADVECTION TERMS*

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Dedicated to Biagio Ricceri on the occasion of his 70th anniversary

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Abstract

In this paper, we study existence and main properties of weak solutions for a class of boundary value problems.

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MSC: 35D30, 35J25, 35J62.

1 Introduction

In this paper, we study the the boundary value problem (1) (and (6)) below; the main results are existence of solutions (Theorem 1) and positivity of solutions (Theorem 2). This last result does not imply uniqueness, due to the presence of the nonlinear term $b(u)$; now, we have no feeling about uniqueness or multiplicity. In order to investigate multiplicity, a useful tool can be the result by *B. Ricceri* in [12].

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Consider

$$\begin{cases} -\mathrm{div}(M(x)Du) + E(x)\cdot DH(u) + \, u = f(x), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where the set Ω is a bounded, open subset of $\mathbb{R}^N$ $(N \geq 3)$.

$M : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$ is a bounded and measurable matrix such that (for $0 < \alpha \leq \beta$)

$$\alpha|\xi|^2 \leq M(x)\xi\cdot\xi, \quad |M(x)| \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N; \tag{2}$$

$$f \in L^\infty(\Omega). \tag{3}$$

The vector field $E(x)$ is very singular:

$$E \in (L^2(\Omega))^N \tag{4}$$

and H is a function such that:

$$\begin{cases} h(s) \text{ is a continuous increasing real function with } h(0) = 0,\, |h(s)| = h(|s|), \\ H(t) = \displaystyle\int_0^t h(s)\,ds. \end{cases} \tag{5}$$

We prove the existence of weak (bounded) solutions of the boundary value problem (1), that is

$$\left\{ \begin{array}{l} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \\ \displaystyle\int_\Omega M(x)DuD\varphi + \int_\Omega \varphi[E(x)\cdot Du]h(u) + \int_\Omega u\,\varphi = \int_\Omega f(x)\,\varphi(x), \\ \forall\, \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega). \end{array} \right. \tag{6}$$

We point out that the transport part $E(x)\cdot DH(u) = [E(x)\cdot Du]h(u)$ is very singular. In particular, the assumption (4) on the drift term $E(x)$ is weaker than the boundedness assumption used in some papers and also weaker than the assumption $E \in (L^N(\Omega))^N$ used in [3]; on the other hand, we assume that the right hand side $f(x)$ is a bounded function.

We follow the nonlinear duality approach of [4], [3] and, despite the presence of $E(x)$ (which only belongs to L^2) and the nonlinear advection term $DH(u)$, we will prove the existence of a solution $u \in W_0^{1,2}(\Omega)$, thanks to the impact of the zero order term. Another possible way could be the use of the method of [10].

We observe that in several papers the vector field is less singular than in (4); moreover, it often has constant components. Of course, the case where $E(x)$ has constant components (or, more generally, is divergence-free) simplifies the proofs of a priori estimates, thanks to the Divergence Theorem applied to our boundary value problem with zero boundary data (see also [6]). We recall that there are papers with advection terms, but right hand side of (1) function of the solution u (e.g. [8], [9]), instead of external force $f(x)$.

Moreover, we emphasize that the boundary value problem (1) has a surprising aspect: there exists a finite energy solution ($u \in W_0^{1,2}(\Omega)$), despite the fact that the term $E(x) \cdot DH(u) \in L^1$.

Our approach hinges on a duality approach, even if the differential operator is nonlinear in the first order term.

2 Existence

The main result of this paper is the following theorem.

Theorem 1. *Under the assumptions (2), (3), (4), (5), there exists a weak (bounded) solution u of the boundary value problem (1), that is of (6). Moreover u satisfies the following a priori estimates*

$$\begin{cases} \|u\|_\infty \le \|f\|_\infty, \\ \displaystyle\int_\Omega |Du|^2 \le 2\frac{\|f\|^2_{L^\infty(\Omega)}\ \mu(\Omega)}{\alpha} + \frac{h(\|f\|_{L^\infty(\Omega)})^2}{\alpha^2}\int_\Omega |E|^2. \end{cases} \tag{7}$$

In order to prove the above theorem, we define a sequence of approximating problems and we prove two a priori estimates on the sequence of the solutions of these problems.

2.1 Approximating problems

Our starting point is the following nonlinear Dirichlet problem, for a given n,

$$\begin{cases} u_n \in W_0^{1,2}(\Omega): \\ -\mathrm{div}(M(x)Du_n) + \dfrac{E(x)\cdot Du_n}{(1+\frac{1}{n}|E|)(1+\frac{1}{n}|u_n|)}\dfrac{h(u_n)}{1+\frac{1}{n}|h(u_n)|} + u_n = f(x). \end{cases} \tag{8}$$

For our study, we need to start with bounded weak solutions u_n, whose existence will be established below.

To this end, we consider the following equivalent Dirichlet problem, where the equivalence holds for fixed n and is understood in the sense of a simple rewriting of the equation upon setting $j = \frac{1}{n}, B = -\frac{E}{1+\frac{1}{n}|E|}$ and $\widetilde{h} = \frac{h}{1+\frac{1}{n}|h|}$

$$u_j \in W_0^{1,2}(\Omega): \ -\mathrm{div}(M(x)Du_j) + u_j = \frac{[B(x) \cdot Du_j]}{(1 + j|u_j|)}\widetilde{h}(u_j) + b(x), \tag{9}$$

where $j \in \mathbb{R}^+$, $\widetilde{h}(s)$ is a bounded continuous function, $B \in (L^\infty(\Omega))^N$, $b \in L^\infty(\Omega)$ and the matrix M still satisfies (2).

Here we follow the Appendix of [3]. Due to the properties of the differential operator $L(v) = -\mathrm{div}(M(x)Dv)$, (9) can be rewritten as a fixed point problem

$$u_j \in W_0^{1,2}(\Omega): \ u_j = L^{-1}\left(\frac{B(x) \cdot Du_j}{(1 + j|u_j|)}\widetilde{h}(u_j) + b(x)\right). \tag{10}$$

Define the operator

$$T(v) = L^{-1}\left(\frac{B(x) \cdot Dv}{(1 + j|v|)}\widetilde{h}(v) + b(x)\right).$$

Since $T : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is well defined and compact, in order to apply the Schaefer fixed point theorem ([13]; see also Theorem 4 in 9.2.2 of [11]), we only need to prove that the set

$$X = \{v \in W_0^{1,2}(\Omega) : v = t\,T(v), \text{ for some } t \in [0, 1]\}$$

is bounded in order to prove that T has a fixed point.

If $v \in X$, then

$$L(v) = t\left(\frac{B(x) \cdot Dv}{(1 + j|v|)}\widetilde{h}(v) + b(x)\right) \tag{11}$$

Now we use v as test function in the weak formulation of (11) and we have (using $t \in [0, 1]$, $\frac{|v|}{(1+j|v|)} \leq \frac{1}{j}$ and the Hölder inequality)

$$\left| \begin{array}{l} \alpha \displaystyle\int_\Omega |Dv|^2 \leq \|B\|_{L^\infty} \int_\Omega |Dv| \frac{|v|}{(1 + j|v|)}\widetilde{h}(v) + \|b\|_{L^\infty} \int_\Omega |v| \\ \leq \|B\|_{L^\infty} \displaystyle\sup_{s\in\mathbb{R}} |\widetilde{h}(s)| \int_\Omega |Dv| \frac{|v|}{(1 + j|v|)} \frac{1}{(1 + j|v|)} + \|b\|_{L^\infty} \int_\Omega |v| \\ \leq \|B\|_{L^\infty} \displaystyle\sup_{s\in\mathbb{R}} |\widetilde{h}(s)| \frac{[\mu(\Omega)]^{\frac{1}{2}}}{j} \left[\int_\Omega |Dv|^2\right]^{\frac{1}{2}} + \|b\|_{L^\infty} \int_\Omega |v|, \end{array} \right.$$

which proves that $\|v\|_{W_0^{1,2}(\Omega)} \leq C(j, \alpha, \beta, b, B, \widetilde{h})$; this implies that X is bounded when j, B, b and $\widetilde{h}(s)$ are fixed). Hence problem (10) has a fixed point and (9) has a solution.

Now we prove that the function $u_j \in L^\infty(\Omega)$. We take $G_k(u_j)$ as test function in (9) and we have

$$\alpha \int_\Omega |DG_k(u_j)|^2 \leq \|B\|_{L^\infty} \sup_{s\in\mathbb{R}} |\widetilde{h}(s)| \int_\Omega |Du_j||G_k(u_j)| + \|b\|_{L^\infty} \int_\Omega |G_k(u_j)|.$$

$$\leq \|B\|_{L^\infty} \sup_{s\in\mathbb{R}} |\widetilde{h}(s)| \|u_j\|_{W_0^{1,2}(\Omega)} \left[\int_\Omega |G_k(u_j)|^2 \right]^{\frac{1}{2}} + \|b\|_{L^\infty} \int_\Omega |G_k(u_j)|.$$

Here the Stampacchia method (see [14]) gives the boundedness of the function u_j (of course for j, B, b, $\widetilde{h}$ fixed). Thus it follows that (8) has a bounded weak solution.

2.2 Estimates

The second tool is the following nonlinear auxiliary Dirichlet problem

$$\begin{cases} \psi_n \in W_0^{1,2}(\Omega): \\ -\mathrm{div}(M^*(x)D\psi_n) - \mathrm{div}\left(\psi_n \dfrac{E(x)}{(1+\frac{1}{n}|E|)(1+\frac{1}{n}|u_n|)} \dfrac{h(u_n)}{1+\frac{1}{n}|h(u_n)|}\right) + \psi_n \\ = u_n|u_n|^{q-2}, \end{cases} \tag{12}$$

where $q > 2$. Note that a weak solution $\psi_n \in W_0^{1,2}(\Omega)$ of (8) exists thanks to Schauder fixed point theorem and that ψ_n is a bounded function (or see [1], [2]).

Since every solution u_n and ψ_n is a bounded function, we will use, in the following proofs, nonlinear compositions of u_n and ψ_n as test functions.

We point out that, even if the Dirichlet problems (8) and (12) are nonlinear, we can adapt a duality method in order to prove our a priori estimates.

Now we prove an L^1 estimate on ψ_n (see also [2]).

Lemma 1. *Under the assumptions (2), (3), (4), (5), we prove the estimate*

$$\int_\Omega |\psi_n(x)| \leq \int_\Omega |u_n|^{q-1}. \tag{13}$$

Proof. Recall that, $\forall\, k \geq 0$,

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\ \frac{s}{|s|}, & \text{if } |s| > k. \end{cases}$$

We use $T_k(\psi_n)$ as test function in the weak formulation of (12)

$$\alpha \int_\Omega |DT_k(\psi_n)|^2 + \int_\Omega \psi_n \, T_k(\psi_n) \le \int_\Omega k|E(x)| \, n \, |DT_k(\psi_n)| + k \int_\Omega |u_n|^{q-1}$$

and we have (we use the Young inequality with $B \in (0, \alpha)$)

$$(\alpha - B) \int_\Omega |DT_k(\psi_n)|^2 + \int_\Omega \psi_n T_k(\psi_n) \le \frac{k^2}{4B} n^2 \int_\Omega |E(x)|^2 + k \int_\Omega |u_n|^{q-1}.$$

Now we drop a positive term and we have

$$\int_\Omega \frac{\psi_n \, T_k(\psi_n)}{k} \le \frac{1}{4B} \, k \, n^2 \int_\Omega |E(x)|^2 + \int_\Omega |u_n|^{q-1}. \tag{14}$$

In the above inequality, the limit $k \to 0$ gives, thanks to the Fatou lemma, estimate (13). □

Lemma 2. *Under the assumptions (2), (3), (4), (5), we prove the estimate*

$$\|u_n\|_\infty \le \|f\|_\infty. \tag{15}$$

Proof. Now we use a duality approach: by (8) (with test function ψ_n) and (12) (with test function u_n), we deduce that

$$\left| \int_\Omega |u_n|^q \right| = \left| \int_\Omega f \psi_n \right| \le \|f\|_{L^\infty(\Omega)} \| |u_n|^{q-1} \|_{L^1(\Omega)}, \tag{16}$$

which implies (use the Hölder inequality with exponents q and q')

$$\int_\Omega |u_n|^q \le \|f\|_{L^\infty(\Omega)} \left[\int_\Omega |u_n|^q \right]^{\frac{1}{q'}} \mu(\Omega)^{\frac{1}{q}},$$

that is

$$\left[\int_\Omega |u_n|^q \right]^{\frac{1}{q}} \le \mu(\Omega)^{\frac{1}{q}} \|f\|_{L^\infty(\Omega)}.$$

Then we can pass to the limit as $q \to \infty$ (recall that every u_n is a bounded function) and we conclude that (15) holds: the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$. □

Remark 1. *We point out that the estimate (15) does not depend neither on $E(x)$, nor on $H(s)$.*

Lemma 3. *Under the assumptions (2), (3), (4), (5), we prove the energy estimate*

$$(\alpha - B)\int_\Omega |Du_n|^2 \leq \|f\|^2_{L^\infty(\Omega)}\mu(\Omega) + \frac{h(\|f\|_{L^\infty(\Omega)})^2}{4B}\int_\Omega |E|^2 \tag{17}$$

where $B \in (0,\alpha)$.

Proof. Now we use u_n as test function in the weak formulation of (8) and we have, thanks to (15) and dropping a positive term,

$$\begin{aligned} &\alpha \int_\Omega |Du_n|^2 \leq h(\|u_n\|_\infty)\int_\Omega |E||Du| + \|f\|_\infty \|u_n\|_\infty \mu(\Omega) \\ &\leq h(\|f\|_\infty)\|E\|_2 \|Du\|_2 + \|f\|^2_\infty \mu(\Omega); \end{aligned}$$

then the Young inequality, with $B \in (0,\alpha)$, gives

$$\alpha \int_\Omega |Du_n|^2 \leq \|f\|^2_{L^\infty(\Omega)}\ \mu(\Omega) + B\int_\Omega |Du_n|^2 + \frac{h(\|f\|_{L^\infty(\Omega)})^2}{4B}\int_\Omega |E|^2,$$

which implies (17). □

Thus there exists a function $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and a subsequence $\{u_{n_k}\}$ such that

$$u_{n_k} \text{ converges weakly in } W_0^{1,2}(\Omega) \text{ and a.e. to } u. \tag{18}$$

2.3 Existence

Proof of Theorem 1. Thanks to (18), it is possible to pass to the limit in the weak formulation of (8), and we prove the existence of a bounded weak solution u of (6). Moreover with the limits in (15), (17) we see that u satisfies the a priori estimates

$$\begin{cases} \|u\|_\infty \leq \|f\|_\infty, \\ \displaystyle\int_\Omega |Du|^2 \leq \frac{\|f\|^2_{L^\infty(\Omega)}\ \mu(\Omega)}{(\alpha - B)} + \frac{h(\|f\|_{L^\infty(\Omega)})^2}{4B(\alpha - B)}\int_\Omega |E|^2. \end{cases}$$

Then, by minimization on B, we obtain (7).

Remark 2. *If we go back to (8), thanks to (17) and (4), we can say that*

$$-\mathrm{div}(M(x)Du_n) = y_n, \ \ \textit{sequence bounded in } L^1.$$

Thus we can use a result by [7] to say that the sequence $Du_{n_k}(x)$ *converges a.e. to* $Du(x)$. *This a.e. convergence and (18) imply*

$$u_{n_k} \textit{ converges to } u \textit{ strongly in } W_0^{1,q}(\Omega),\ \forall\, q < 2. \tag{19}$$

2.4 Positivity of solutions

In this subsection, we prove a basic result in the study of Dirichlet problems: the positivity of the solutions: if $f(x) \geq 0$, then $u(x) \geq 0$.

The standard proof (the use of $u^-(x)$ as test function) does not work, since the differential operator is not coercive, because of the presence of the term of order one (the advection term).

We write again (8) (with weak bounded solution u_n) and a dual problem, similar to (12) (with weak bounded solution ψ_n)

$$\begin{cases} -\mathrm{div}(M(x)Du_n) + \dfrac{E(x) \cdot Du_n}{(1+\frac{1}{n}|E|)(1+\frac{1}{n}|u_n|)} \dfrac{h(u_n)}{1+\frac{1}{n}|h(u_n)|} + u_n = f \\ -\mathrm{div}(M^*(x)D\psi_n) - \mathrm{div}\Big(\psi_n \dfrac{E(x)}{(1+\frac{1}{n}|E|)(1+\frac{1}{n}|u_n|)} \dfrac{h(u_n)}{1+\frac{1}{n}|h(u_n)|}\Big) + \psi_n \\ = u^-. \end{cases}$$

Again note that a weak solution $\psi_n \in W_0^{1,2}(\Omega)$ exists thanks to Schauder fixed point theorem and that ψ_n is a bounded function (or see [1], [2]); $u(x)$ is the bounded weak solution of Theorem 1.

We use again a duality approach: by (8) (with test function ψ_n) and by the last Dirichlet problem (with test function u_n), we deduce that

$$\int_\Omega u_n u^- = \int_\Omega f\psi_n,$$

where $\int_\Omega f\psi_n \geq 0$, since $\psi_n \geq 0$, proved in [1], [2] since $u_- \geq 0$. We pass to the limit and we deduce that

$$-\int_\Omega u^- u^- = \int_\Omega u u^- \geq 0,$$

that is $u^- = 0$: $u(x) \geq 0$. Thus we proved the following positivity theorem.

Theorem 2. *Assume (2), (3), (4), (5),*

$$f(x) \geq 0.$$

Then u, bounded weak solution of (6), is such that

$$u(x) \geq 0.$$

□

2.5 Open problems

- In the previous Theorem: if $f(x) \geq 0$ and not equal to zero a.e., "how much positive is" the weak solution $u(x)$?
  Is it possible to prove, as in [5], that the the set $\{u(x) = 0\}$ has zero measure?

- $h(s) = s|s|^{p-2}$ and

$$\begin{cases} u_p \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \\ \displaystyle\int_\Omega M(x) Du_p D\varphi + \int_\Omega \varphi[E(x) \cdot Du_p]\, u_p |u_p|^{p-2} + \int_\Omega u_p\, \varphi \\ = \int_\Omega f(x)\, \varphi(x), \\ \forall\, \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \\ \|u_p\|_\infty \leq \|f\|_\infty \end{cases} \tag{20}$$

  Asymptotic beaviour, as $p \to \infty$, of the sequence $\{u_p\}$ (bounded in $L^\infty(\Omega)$).

- $H(s) = \sqrt{s^+}$.

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