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BOUNDED SOLUTIONS FOR AN INCOMPLETE CAUCHY PROBLEM INVOLVING A NON-CONVEX FUNCTION*

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Dedicated with friendship and appreciation to Professor Biagio Ricceri on the occasion of his 70th anniversary

Abstract

Consider in a real Hilbert space $\big(H,\ (\cdot,\cdot),\ |\cdot|\big)$ the following incomplete Cauchy problem,

$$(ICP) \qquad \begin{cases} u''(t) = \nabla\phi(u(t)),\ t \geq 0, & (E) \\ u(0) = u_0, & (IC) \end{cases}$$

where $u_0 \in H$ is a given initial state, and $\phi : H \to \mathbb{R}$ is a C^1, non-convex function (preferably quasiconvex, as explained below). We call (ICP) an *incomplete Cauchy problem* because the usual additional Cauchy condition $u'(0) = v_0$ is missing. In this paper, we establish sufficient conditions on the non-convex function ϕ guaranteeing the existence of bounded solutions on $[0, \infty)$ of (ICP) for any $u_0 \in H$.

Keywords: second order differential equation, gradient of a C^1 function, bounded solutions.

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1 Introduction

First of all, we recall that the existence of bounded solutions for the problem (ICP) formulated in our abstract above has been proved long ago in the case when $\phi : H \to (-\infty, \infty]$ is a proper (i.e., not identically $+\infty$), lower semicontinuous, convex function with a nonempty set of minimum points (see [1, Chap. V, p. 315]). More precisely, in this case, for any $u_0 \in \overline{D(\phi)}$, there exists a unique bounded solution of problem (ICP) with the subdifferential $\partial\phi$ instead of $\nabla\phi$. That is why here we concentrate our attention on the case when ϕ is non-convex. By showing the existence of bounded solutions on $[0, \infty)$ for (ICP), we legitimize the existing results on the asymptotic behavior of the solutions to equation (E) as $t \to \infty$, where ϕ is a quasiconvex function (i.e., its level sets $\{x \in H;\, \phi(x) \leq \alpha\}$, $\alpha \in \mathbb{R}$, are convex) (see [5], [6]).

Specifically, in this paper we provide an answer to the long standing open problem concerning the existence of bounded solutions for the problem (ICP) with a C^1 function ϕ in two cases:

1. the gradient $\nabla\phi$ is a Lipschitz operator;
2. the function ϕ satisfies

$$a|v|^2 \leq \phi(v) \leq b|v|^2, \ \forall v \in H, \tag{1}$$

where $0 < a < b < \infty$ are given numbers.

2 The case when ϕ is a C^1 function with $\nabla\phi$ Lipschitzian

In this case we have the following result:

Theorem 1. *Assume that $\phi : H \to \mathbb{R}$ is differentiable and $\nabla\phi$ is a Lipschitz operator on H. Then for all u_0, $v_0 \in H$ there exists a unique function $u \in C^2([0, \infty); H)$ satisfying equation (E) on $[0, \infty)$ and the Cauchy conditions $u(0) = u_0$, $u'(0) = v_0$.*

Proof. Using the substitution $v(t) = u'(t)$ we are led to the following Cauchy problem in the product space $X := H \times H$ equipped with the usual scalar product and norm:

$$\begin{cases} \frac{d}{dt}(u, v) = (v, \nabla\phi(u)), \ t \geq 0, \\ (u(0), v(0)) = (u_0, v_0). \end{cases}$$

Let T be an arbitrary but fixed positive number. By applying Banach's Contraction Principle, we easily derive the existence of a unique solution $(u, v) \in C^1([0, T]; X)$ of the above problem considered on the interval $[0, T]$. Of course, this solution can uniquely be extended to the whole half axis $[0, \infty)$. So $u = u(t)$ belongs to $C^2([0, \infty); H)$ and it is the unique solution of equation (E) satisfying the Cauchy conditions $u(0) = u_0$, $u'(0) = v_0$. Hence, the conclusion of Theorem 1 holds true. □

2.1 Existence of bounded solutions on $[0, \infty)$ for (ICP)

In the context of Theorem 1, we need to identify additional conditions on ϕ that guarantee the boundedness of $u = u(t)$ on $[0, \infty)$. Such situations are possible. In what follows, we identify a class of C^1 functions ϕ with Lipschitzian gradients such that for every $u_0 \in H$, $u = u(t; u_0, v_0)$ in Theorem 1 be bounded on $[0, \infty)$ for some $v_0 \in H$.

For the moment, let us consider for example the function $\phi : H \to \mathbb{R}$ defined by

$$\phi(v) = \frac{|v|^2}{1 + |v|^2},\ v \in H.$$

This function is not convex (but is quasiconvex), and its Fréchet derivative is given by

$$\nabla\phi(v) = \frac{2}{(1 + |v|^2)^2}\, v,\ v \in H.$$

Furthermore, by an elementary computation it follows that $\nabla\phi : H \to H$ is a Lipschitz operator. Therefore, according to Theorem 1 above, for every u_0, $v_0 \in H$, there exists a unique function $u = u(t; u_0, v_0) \in C^2([0, \infty); H)$ satisfying equation (E) and the Cauchy conditions $u(0) = u_0$ and $u'(0) = v_0$. According to [3, 1177–1178], for every $u_0 \in H$, $u(t; u_0, v_0)$ is bounded on $[0, \infty)$ for some $v_0 \in H$.

Indeed, if we multiply equation (E) by $u'(t)$ we get

$$\frac{1}{2}\frac{d}{dt}|u'(t)|^2 = \frac{d}{dt}\phi(u(t)), \quad t \geq 0,$$

or, equivalently,

$$\frac{1}{2}|u'(t)|^2 = \phi(u(t)) + C, \quad t \geq 0,$$

where C is a real constant. We choose $v_0 = u'(0)$ such that $C = 0$, and consider the following related Cauchy problem in H, denoted (CP),

$$\begin{cases} u'(t) = -\frac{\sqrt{2}}{\sqrt{1+|u(t)|^2}}u(t), & t \geq 0, \\ u(0) = u_0. \end{cases} \tag{2}$$

It is easily seen that the operator $P : H \to H$ defined by

$$Pv = -\frac{\sqrt{2}}{\sqrt{1+|v|^2}}v, \quad v \in H$$

is Lipschitzian. Therefore the above problem (CP) has a unique solution $u = u(t, u_0, v_0) \in C^\infty([0, \infty); H)$, with v_0 chosen above (by applying Banach's Contraction Principle in the space $C([0, T]; H)$, $T > 0$, equipped with a Bielecki norm, with extension to $[0, \infty)$).

Now, multiplying equation (2) by $u(t)$, we find

$$\frac{d}{dt}|u(t)|^2 \leq 0, \quad t \geq 0,$$

so $|u(t)| \leq |u_0|, \quad t \geq 0$. Hence $u = u(t, u_0, v_0)$ is bounded on $[0, \infty)$. In fact this u satisfies (ICP). Indeed, if we differentiate the equation

$$\sqrt{1+|u|^2}u' + \sqrt{2}u = 0, \quad t \geq 0,$$

derived from (2), we obtain

$$\frac{(u, u')}{\sqrt{1+|u|^2}}u' + \sqrt{1+|u|^2}u'' = -\sqrt{2}u', \quad t \geq 0. \tag{3}$$

Using again (2) we obtain

$$(u, u') = -\frac{\sqrt{2}|u|^2}{\sqrt{1+|u|^2}}, \tag{4}$$

and so by (3) and (4) we derive

$$u'' = \frac{2}{(1+|u|^2)^2}u, \quad t \geq 0,$$

hence u satisfies equation (E). Therefore, the function $\phi : H \to \mathbb{R}$,

$$\phi(v) = \frac{|v|^2}{1+|v|^2}, \quad v \in H,$$

is a good example for the existence of bounded solutions on $[0,\infty)$ for problem (ICP).

Note that the same example was considered in [4] in the particular case $H=\mathbb{R}$ which allows using elementary student level arguments.

Now, for $\lambda,\mu>0$, define $\phi_{\lambda\mu}: H\to\mathbb{R}$ by

$$\phi_{\lambda\mu}(v)=\frac{\lambda|v|^2}{1+\mu|v|^2},\quad v\in H.$$

It is easily seen (by using arguments similar to those corresponding to the case $\lambda=\mu=1$) that problem (ICP) with $\phi=\phi_{\lambda\mu}$ has bounded solutions on $[0,\infty)$ for all $\lambda>0,\ \mu>0$. Therefore, we have a class of functions $\{\phi_{\lambda\mu},\ \lambda,\mu>0\}$ generating bounded solutions for (ICP). Many other classes of such functions could also be considered for applications.

3 The case when ϕ is a C^1 function satisfying condition (1)

In this case we have the following result:

Theorem 2. *If $\phi: H\to\mathbb{R}$ is a C^1 function satisfying condition (1) above, then problem (ICP) has a solution bounded on $[0,\infty)$.*

Proof. First of all, it follows by condition (1) that $\phi(0)=0$ and $\phi=\phi(v)$ attains its global minimum at $v=0$, so $\nabla\phi(0)=0$. Therefore, in this case problem (ICP) admits the null solution. In what follows we will assume that $u_0\neq 0$.

Consider the Sobolev space $X=W^{1,2}\big((0,\infty);H\big)$, i.e., the space of all $w\in L^2\big((0,\infty);H\big)$ with derivatives $w'\in L^2\big((0,\infty);H\big)$, equipped with the inner product

$$((w_1,w_2))=\int_0^\infty (w_1(t),w_2(t))\,dt+\int_0^\infty (w_1'(t),w_2'(t))\,dt,\quad \forall w_1,\,w_2\in X,$$

and the corresponding norm

$$\|w\|=\Big(\int_0^\infty |w(t)|^2\,dt+\int_0^\infty |w'(t)|^2\,dt\Big)^{1/2},\ \forall w\in X,$$

so $\big(X,((\cdot,\cdot)),\|\cdot\|\big)$ is a real Hilbert space.

Now, consider the subspace $X_0=W_0^{1,2}\big((0,\infty);H\big)$, which is defined as the closure of $C_0^\infty\big((0,\infty);H\big)$ in X. In other words, $X_0=W_0^{1,2}\big((0,\infty);H\big)$

consists of all $w \in W^{1,2}\big((0,\infty);H\big)$ with $w(0)=0$, being a Hilbert subspace of $X = W^{1,2}\big((0,\infty);H\big)$ with the same scalar product and norm.

Now, let us define the function $F : X_0 \to \mathbb{R}$ by

$$F(w) = \frac{1}{2}\int_0^\infty |w'(t) - e^{-t}u_0|^2\,dt + \int_0^\infty \phi(w(t) + e^{-t}u_0)\,dt,\ w \in X_0.$$

According to our condition (1), F is well defined on X_0 and coercive (i.e., $F(w)$ converges to ∞ as $\|w\| \to \infty$).

By the coercivity of F it follows that $\forall M > 0$ the set $\{w \in X_0;\ F(w) < M\}$ is bounded in X_0. Let (w_n) be a minimizing sequence in X_0 satisfying

$$Inf_{X_0}F \le F(w_n) < Inf_{X_0}F + \frac{1}{n},\ \forall n \in \mathbb{N}. \tag{5}$$

Then (w_n) is bounded in X_0 so, as X_0 is a Hilbert space (hence reflexive), one can extract a subsequence, again denoted (w_n), which converges weakly in X_0 to some $\tilde{u} \in X_0$.

On the other hand, as w_n converges weakly to $\tilde{u}$ in X_0, it follows that w_n converges weakly in $L^2\big((0,\infty);H\big)$ (to $\tilde{u}$), and w_n' also converges weakly in $L^2\big((0,\infty);H\big)$ to the derivative $\tilde{u}'$. Let us explain this in detail for completeness. Denote by z the weak limit of w_n' in $L^2\big((0,\infty);H\big)$. Notice that for all $n \in \mathbb{N}$, $w = \psi(t)\zeta$ with $\psi \in C_0^\infty(0,\infty)$ and $\zeta \in H$, we have

$$\int_0^\infty (w_n'(t),\zeta)\,\psi(t)\,dt = -\int_0^\infty \big(w_n(t),\zeta\big)\,\psi'(t)\,dt,$$

which implies by passing to limit as $n \to \infty$,

$$\int_0^\infty (z(t),\zeta)\,\psi(t)\,dt = -\int_0^\infty (\tilde{u}(t),\zeta)\,\psi'(t)\,dt,$$

and so

$$\int_0^\infty (z(t),\zeta)\,\psi(t)\,dt = \int_0^\infty (\tilde{u}'(t),\zeta)\,\psi(t)\,dt,$$

for all $\zeta \in H$ and $\psi \in C_0^\infty(0,\infty)$. Hence $z = \tilde{u}'$ and consequently $w_n' \to \tilde{u}'$ weakly in $L^2\big((0,\infty);H\big)$, as asserted.

On the other hand, for $T > 0$ arbitrary but fixed, the sequence (w_n) is bounded in $C([0,T];H)$ and equi-continuous on $[0,T]$, as one can easily deduce from

$$w_n(t) = \int_0^t w_n'(s)\,ds,\ n \in \mathbb{N}.$$

So by the Arzelà-Ascoli Criterion w_n converges to $\tilde{u}$ in $C([0,T];H)$. Since $T>0$ was arbitrary and $\phi \in C^1$, we have

$$\lim_{n\to\infty} \phi(w_n(t)+e^{-t}u_0) = \phi(\tilde{u}(t)+e^{-t}u_0),$$

uniformly on every interval $[0,T]$, where $t \to \phi(\tilde{u}(t)+e^{-t}u_0)$ belongs to $C([0,\infty);\mathbb{R}) \cap L^1\big((0,\infty);\mathbb{R}\big)$.

Moreover, for every $T \in (0,\infty)$, we have

$$\liminf_{n\to\infty} \int_0^\infty \phi(w_n(t)+e^{-t}u_0)\,dt \geq \int_0^T \phi(\tilde{u}(t)) + e^{-t}u_0)\,dt,$$

which implies

$$\liminf_{n\to\infty} \int_0^\infty \phi(w_n(t)+e^{-t}u_0)\,dt \geq \int_0^\infty \phi(\tilde{u}(t)) + e^{-t}u_0)\,dt.$$

On the other hand, as $w_n' \to \tilde{u}'$ weakly in $L^2\big((0,\infty);H\big)$, we have

$$\liminf_{n\to\infty} \int_0^\infty |w_n'(t)-e^{-t}u_0|^2\,dt \geq \int_0^\infty |\tilde{u}'(t)-e^{-t}u_0|^2\,dt.$$

Therefore, $F(\tilde{u}) \leq \liminf_{n\to\infty} F(w_n)$. In fact, taking into account (5), we can conclude that $\inf_{X_0} F = \lim_{n\to\infty} F(w_n) = F(\tilde{u})$. As $\tilde{u}$ is a minimum point of F on X_0, we have

$$\nabla F(\tilde{u}) = 0. \tag{6}$$

Now, for $\lambda > 0$ and $v \in X_0$, we have

$$\frac{F(\tilde{u}+\lambda v)-F(\tilde{u})}{\lambda} = \frac{1}{2\lambda}\int_0^\infty \big(|\tilde{u}'(t)-e^{-t}u_0+\lambda v'(t)|^2 - |\tilde{u}'(t)-e^{-t}u_0|^2\big)\,dt +$$
$$\frac{1}{\lambda}\int_0^\infty \big(\phi(\tilde{u}(t)+e^{-t}u_0+\lambda v(t)) - \phi(\tilde{u}(t)+e^{-t}u_0)\big)\,dt.$$

Hence, for $\lambda \to 0^+$ we get (see also equation (6) above)

$$0 = \int_0^\infty (\tilde{u}'(t)-e^{-t}u_0, v'(t))\,dt + \int_0^\infty (\nabla\phi(\tilde{u}(t)+e^{-t}u_0), v(t))\,dt,$$

for all $v \in C_0^\infty\big((0,\infty);H\big)$. Therefore, choosing in the last equation $v(t) = \alpha(t)\xi$, with $\alpha \in C_0^\infty(0,\infty)$ and $\xi \in H$, we get

$$\begin{cases} -\tilde{u}''(t) - e^{-t}u_0 + \nabla\phi\big(\tilde{u}(t)+e^{-t}u_0\big) = 0, \ t \geq 0, \\ \tilde{u} \in X_0, \end{cases}$$

hence $u(t) = \tilde{u}(t) + e^{-t}u_0$ satisfies problem (ICP). Note that both the functions $f(t) = |u(t)|^2$, $f'(t) = 2((u(t), u'(t))$ belong to $L^1\big((0,\infty);\mathbb{R}\big)$. It follows from

$$f(t) = |u_0|^2 + \int_0^t f'(s)\,ds$$

that $\sup_{t\geq 0}|u(t)| < \infty$, so problem (ICP) has a solution bounded on $[0,\infty)$. □

Remark 1. *Notice that in Theorem 2 we did not assume that ϕ is a quasiconvex function, but this situation is not excluded. One can say that in Theorem 2 the function ϕ is almost convex, but not necessarily convex.*

3.1 Graphical representations in the case $H = \mathbb{R}$

We notice that in the case $H = \mathbb{R}$ there are infinitely many quasiconvex C^1 functions $y = \phi(v)$ satisfying condition (1) above, that may be alternatively convex and concave on subintervals, i.e., their graphs are smooth wavy curves situated in the region between the graphs of the functions $y = a|v|^2 = av^2$ and $y = b|v|^2 = bv^2$, as illustrated in Figures 1–3 below, wherein the plottings were obtained using Matlab.

Example 1. *Consider the functions $g,\ h : \mathbb{R} \to \mathbb{R}$, $g(v) = 0.1v^2$, $h(v) = 0.02v^2$, and the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\phi(v) = \begin{cases} h(v)\,,\ \ 0 \leq v \leq 2, \\ (1-\theta(v))h(v) + \theta(v)\,F(v),\ \ v \geq 2, \\ \phi(-v),\ \ v \leq 0, \end{cases}$$

where

$$\begin{aligned} \theta(v) &= s\left(\frac{v}{2} - 1\right),\ \ v \in \mathbb{R}, \\ s(t) &= \begin{cases} 0,\ \ t \leq 0, \\ 3t^2 - 2t^3,\ \ 0 < t < 1, \\ 1,\ \ t \geq 1, \end{cases} \end{aligned}$$

and

$$F(v) = 0.056v^2 + 0.036\sin(4v),\ \ v \in \mathbb{R}.$$

Then ϕ is quasiconvex, of class $C^1(\mathbb{R})$, ϕ' is Lipschitzian, and its graph is plotted in Figure 1 below on a time interval.

Example 2. *We take now the functions* $g,\ h:\mathbb{R}\to\mathbb{R}$*,* $g(v)=0.1v^2$*,* $h(v)=0.01v^2$*, and* $\phi:\mathbb{R}\to\mathbb{R}$*,*

$$\phi(v)=\begin{cases} g(v)-(g(v)-h(v))S_1(v),\ v\in(-\infty,-5)\,,\\ g(v),\ v\in[-5,0],\\ h(v),\ v\in[0,6],\\ h(v)+(g(v)-h(v))S_2(v),\ v\in(6,\infty)\,,\end{cases}$$

where

$$S_1(v)=0.5(1-\exp(-2(-5-v)))^2(1+0.4\sin(1.5(-5-v))),\ v\in\mathbb{R},$$

$$S_2(v)=0.25(1-\exp(-2(v-6)))^2(1+0.3\sin(1.5(v-6))),\ v\in\mathbb{R}.$$

We easily infer that ϕ *is of class* $C^1(\mathbb{R})$ *and* ϕ' *is Lipschitzian. The plotting of the graph of* ϕ *on a time interval is given in Figure 2 below.*

Example 3. *Another example is represented by the functions* $g,\ h:\mathbb{R}\to\mathbb{R}$*,* $g(v)=0.1v^2$*,* $h(v)=0.01v^2$*, and* $\phi:\mathbb{R}\to\mathbb{R}$*,*

$$\phi(v)=\begin{cases} h(v)+(g(v)-h(v))S_1(v),\ v\in(-\infty,-6)\,,\\ h(v),\ v\in[-6,0],\\ g(v),\ v\in[0,5],\\ g(v)-(g(v)-h(v))S_2(v),\ v\in(5,\infty)\,,\end{cases}$$

where

$$S_1(v)=0.25(1-\exp(-2(-6-v)))^2(1+0.4\sin(1.5(-6-v))),\ v\in\mathbb{R},$$

$$S_2(v)=0.7(1-\exp(-2(v-5)))^2(1+0.3\sin(1.5(v-5))),\ v\in\mathbb{R}.$$

Then ϕ *is of class* $C^1(\mathbb{R})$ *and* ϕ' *is Lipschitzian, the plotting of the graph of* ϕ *on a time interval being provided in Figure 3 below.*

4 Conclusion

In this paper, we have identified classes of non-convex C^1 functions ϕ such that the problem given in (ICP) has bounded solutions on $[0,\infty)$. Thus we have legitimized the efforts in [2], [5], [6] towards establishing results on the asymptotic behavior of the *bounded solutions* to problem (ICP) as $t\to\infty$.

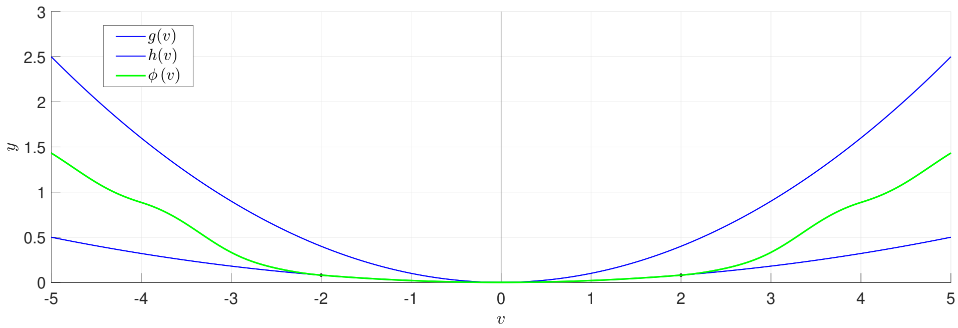

Figure 1: The graph of the function ϕ defined in Example 1

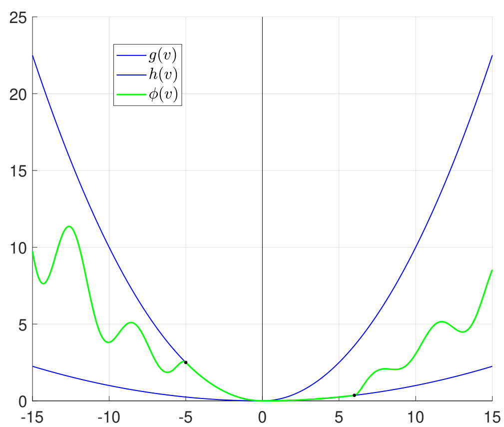

Figure 2: The graph of the function ϕ defined in Example 2

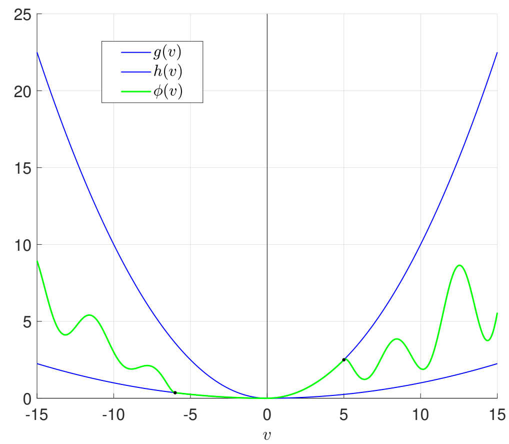

Figure 3: The graph of the function ϕ defined in Example 3

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