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ON THE NON-COERCIVE COMPETING (p,q)-LAPLACIAN DIRICHLET PROBLEM*

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Dedicated to Biagio Ricceri on the occasion of his 70th anniversary

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Abstract

We investigate the existence of generalized solutions to a non-coercive competing system driven by the (p,q)-Laplacian. In order to reach the existence result, we derive an abstract principle based on the convergence of the Galerkin scheme.

Keywords: competing operator, generalized solution, Dirichlet problem, abstract principle.

MSC: 47J05, 35J92.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Fix $p > q > 1$. We study the following problem with homogeneous Dirichlet boundary condition:

$$\begin{cases} -div\left(|\nabla u|^{p-2}\nabla u\right) + div\left(|\nabla u|^{q-2}\nabla u\right) = \lambda f(x,u,\nabla u) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\lambda > 0$ is a numerical parameter and where the convection term $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is the Carathéodory function satisfying only the following generic condition:

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(H): *There exist a nonnegative function* $\sigma \in L^{p'}(\Omega)$ *and constants* $b \geq 0$ *and* $c \geq 0$ *such that*

$$|f(x,s,\xi)| \leq \sigma(x) + b|s|^{p-1} + c|\xi|^{p-1}, \quad \textit{for a.e. } x \in \Omega, \textit{ all } s \in \mathbb{R}, \xi \in \mathbb{R}^N.$$

Due to the homogeneous Dirichlet boundary condition and since $q < p$, we consider problem (1) in the space $W_0^{1,p}(\Omega)$ which is endowed with the standard norm $\|\nabla(\cdot)\|_{L^p(\Omega)}$. The dual space of $W_0^{1,p}(\Omega)$ is denoted $W^{-1,p'}(\Omega)$. We refer to [2] for the background on Sobolev spaces. The (negative) p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is defined as follows

$$\langle -\Delta_p u, v \rangle = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x)\, \mathrm{d}x \quad \text{for all } u, v \in W_0^{1,p}(\Omega)$$

and it is uniformly monotone (hence strictly monotone), continuous, potential and bounded and therefore pseudomonotone, and also it is coercive. We refer [3], [5] for some background notions in the area of the method of monotone operators. Note that the operator $-\Delta_p + \Delta_q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ appearing on the left hand side of (1) and defined as follows for all $u, v \in W_0^{1,p}(\Omega)$

$$\langle (-\Delta_p + \Delta_q)\, u, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v\, \mathrm{d}x - \int_\Omega |\nabla u|^{q-2} \nabla u \nabla v\, \mathrm{d}x$$

is continuous, potential, bounded and due to $q < p$ also coercive on $W_0^{1,p}(\Omega)$. Nevertheless it is not pseudomonotone. Such operators are called competing operators as they lack any type of monotonicity and as their potentials lack the weak sequential lower semicontinuity. The competing operators were introduced in [6] and later on were applied for various types of problems, see [4] and references therein. The surjectivity theorem on pseudomonotone, bounded, coercive and continuous operator leads the existence of weak solutions, see [3]. If we drop the assumption that the operator is pseudomonotone, than the weak solutions may not be reached and another type of solution is needed.

The weak solution to problem (1), should it exist, is defined as a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u\,(x)\,|^{p-2} \nabla u\,(x)\, \nabla v(x)\, \mathrm{d}x - \int_\Omega |\nabla u(x)|^{q-2} \nabla u\,(x)\, \nabla v(x)\, \mathrm{d}x$$

$$= \int_\Omega f(x, u\,(x)\,, \nabla u\,(x)) v(x)\, \mathrm{d}x, \text{ for all } v \in W_0^{1,p}(\Omega).$$

Such a solution however may not be reached directly due to the mentioned lack of monotonicity of the difference of the (negative) p-Laplacian and the (negative) q-Laplacian which drives the left hand side of (1) understood in a weak sense. This prevents the usage of methods of monotone operators. Variational methods, described for example in [1], are not applicable due to the fact that the convection depends on the gradient on the one hand and due to the lack of monotonicity on the other hand.

We introduce the operator

$$A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$

which drives problem (1) in a standard manner

$$\begin{aligned} \langle A(u), v \rangle = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla v(x) \, \mathrm{d}x - \\ \int_\Omega |\nabla u(x)|^{q-2} \nabla u(x) \nabla v(x) \, \mathrm{d}x - \int_\Omega f(x, u(x), \nabla u(x)) v(x) \, \mathrm{d}x, \end{aligned} \tag{2}$$

for $u, v \in W_0^{1,p}(\Omega)$. As we will mention later on, assumption (**H**) suffices to have A continuous and bounded.

The notion of a generalized solution, introduced in [6], reads as follows: a function $u \in W_0^{1,p}(\Omega)$ is said to be a generalized solution to problem (1) if there exists a sequence $\{u_n\}_{n\geq 1}$ in $W_0^{1,p}(\Omega)$ such that
(a) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$;
(b) $\lim_{n\to\infty} \langle A(u_n), v \rangle = 0$ for each $v \in W_0^{1,p}(\Omega)$;
(c) $\lim_{n\to\infty} \langle A(u_n), u_n - u \rangle = 0$.

Generalized solutions have been obtained for problems in which the driving operator is bounded, continuous and coercive. The methods leading to their existence are described in detail in a recent book [4] and rely on the analysis of the convergence of Galerkin or, in the potential case, Ritz type schemes. The growth condition (**H**) is not sufficient to obtain the coercivity of A. Therefore, we cannot use the existing approaches concerning the existence of generalized solutions. Nevertheless we are able to derive some abstract existence result relying on the direct convergence of the Galerkin scheme and next apply it to problem under consideration in order to get the generalized solution. Precisely speaking we will determine the range of a numerical parameter for which there problem (1) has at least one generalized solution. Our contribution relies in the following observation: continuity of the operator; its boundedness and the boundedness of the Galerkin scheme lead to the existence of the generalized solution. Throughout [4] and classical applications of the Browder-Minty theorem to boundary value problems, the boundedness of the Galerkin scheme is usually obtained via the coercivity.

2 Abstract result

Assume that E is a real, separable and reflexive Banach space. Our considerations are inspired by the following result which is derived from the Browder-Minty Theorem in case the coercivity on the operator governing the equation is not necessarily imposed, see [3]:

Theorem 1. *Let $f \in E^*$ be fixed. Assume that $A : E \to E^*$ is demicontinuous (i.e. $u_n \to u_0$ in E implies that $A(u_n) \rightharpoonup A(u_0)$ in E^*), bounded and satisfies condition (M_0) and the following condition*

*(**B**): There is a number $R > 0$ such that $\langle A(u) - f, u\rangle \geq 0$ if $\|u\| = R$.*

Then the set of solutions to

$$A(u) = f$$

is non-empty and bounded.

The operator A satisfies condition (M_0), if relations $u_n \rightharpoonup u_0$ in E, $A(u_n) \rightharpoonup f$ in E^* and $\lim_{n\to\infty} \langle A(u_n), u_n\rangle = \langle f, u_0\rangle$, imply that $A(u_0) = f$. Condition (M_0) is used in order to show that the limit of the sequence of Galerkin type approximations solves the given equation.

In the case of the operator $-\Delta_p + \Delta_q$ the condition (M_0) is not satisfied. This means that we cannot use the above theorem directly, and we need to derive its counterpart concerning the existence of generalized solutions. We will need some preparation for the proof.

Remark 1. *Since E is separable, it contains a dense and countable set $\{h_k : k \in \mathbb{N}\}$. For $n \in \mathbb{N}$ define E_n as a linear hull of $\{h_1, ..., h_n\}$. The sequence of subspaces E_n has the following approximation property: for each $u \in E$ there is a sequence $\{u_n\}_{n\geq 1}$ such that $u_n \in E_n$ for $n \in \mathbb{N}$ and $u_n \to u$.*

The proof of the Browder-Minty Theorem, as seen in [3] and [4], utilizes the following finite-dimensional existence result derived from the Brouwer Fixed Point Theorem:

Lemma 1. *Assume that $f : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous mapping such that*

$$\langle f(u), u\rangle \geq 0 \text{ if } \|u\| = R,$$

for some $R > 0$. Then there is $u_0 \in \mathbb{R}^N$, $\|u_0\| \leq R$, such that $f(u_0) = 0$.

Now we turn to writing an abstract result concerning the existence of a generalized solution. Let $A : E \to E^*$ be some operator. For any fixed $f \in E^*$ we consider the following problem

$$A(u) = f. \tag{3}$$

Definition 1. *An element $u \in E$ is said to be a generalized solution to problem (3), if there exists a sequence $\{u_n\}_{n\geq 1}$ in E such that*

(a) $u_n \rightharpoonup u$ *in E as $n \to \infty$;*

(b) $\lim_{n\to\infty} \langle A(u_n) - f, v\rangle = 0$ *for each $v \in E$;*

(c) $\lim_{n\to\infty} \langle A(u_n) - f, u_n - u\rangle = 0.$

The abstract result reads:

Theorem 2. *Let $f \in E^*$ be fixed. Assume that the operator $A : E \to E^*$ is continuous, bounded and satisfies assumption (**B**). Then problem (3) has at least one generalized solution.*
If additionally the operator A satisfies the (S_+)-property, that is $u_n \rightharpoonup u$ in E and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0$ imply $u_n \to u$ in E, then problem (3) has a weak solution, i.e. there u such that

$$\langle A(u) - f, v\rangle = 0 \text{ for all } v \in E.$$

Proof. Let us fix $n \in \mathbb{N}$ and the space E_n from Remark 1. By f_n we denote the restriction of the functional f to E_n. Similarly by A_n we understand the restriction of A to E_n. Then the operator $A_n : E_n \to E_n^*$ is continuous and from assumption (**B**) we have that

$$\langle A_n(v) - f_n, v\rangle \geq 0 \text{ whenever } v \in E_n \text{ with } \|v\| = R.$$

By Lemma 1 we now see that equation

$$A_n(u) = f_n \tag{4}$$

has at least one solution u_n with $\|u_n\| \leq R$. Since the sequence $\{u_n\}_{n\geq 1}$ is bounded, there is a subsequence, which we do not renumber, converging weakly to some u. Thus we have condition (a) from Definition 1 satisfied. From (4), again possibly for a subsequence, we have

$$\lim_{n\to+\infty}\langle A_n(u_n),h\rangle=\lim_{n\to+\infty}\langle f_n,h\rangle=\langle f,h\rangle \text{ for each } h\in\bigcup_{n=1}^{\infty}E_n.$$

Since $A_n(u_n)=A(u_n)$ we obtain that condition (b) from Definition 1 is satisfied. This also means that $A(u_n)\rightharpoonup f$, since the operator A is bounded and since the sequence $\{A(u_n)\}_{n\geq 1}$ is weakly convergent, up to a subsequence. Moreover, testing (4) against u_n we see that

$$\langle A(u_n),u_n\rangle=\langle f,u_n\rangle.$$

Since obviously $\lim_{n\to+\infty}\langle A(u_n)-f,u\rangle=0$, we observe that condition (c) from Definition 1 also holds.

If additionally the operator A satisfies the (S_+)-property, then the sequence $\{u_n\}_{n\geq 1}$ converges strongly, thus the assertion follows from condition (b). □

3 Applications

Now, we apply Theorem 2 to problem (1). Recall that the first eigenvalue of $-\Delta_p$ has the variational expression

$$\lambda_1:=\inf_{u\in W_0^{1,p}(\Omega),u\neq 0}\frac{\displaystyle\int_\Omega|\nabla u(x)|^p\,\mathrm{d}x}{\displaystyle\int_\Omega|u(x)|^p\,\mathrm{d}x}.$$

We proceed with the following estimate that leads to the already mentioned continuity and boundedness of the operator of operator A given by (2). The lemma below is reformulated after Lemma 2.2 from [6]:

Lemma 2. *Under assumption **(H)** the Niemytskij operator induced by f is well defined, continuous and such that there exists a constant $C>0$ for which we have*

$$\begin{aligned}&\left|\int_\Omega f(x,u(x),\nabla u(x))v(x)\,\mathrm{d}x\right|\\ &\leq\int_\Omega|f(x,u(x),\nabla u(x))v(x)|\ \mathrm{d}x\\ &\leq C\left(\|\sigma\|_{L^{p'}(\Omega)}+\|u\|_{L^p(\Omega)}^{p-1}+\|\nabla u\|_{L^p(\Omega)}^{p-1}\right)\|\nabla v\|_{L^p(\Omega)},\end{aligned}$$

for all $u, v \in W_0^{1,p}(\Omega)$.

From Lemma 2 it follows by a direct calculation that:

$$\begin{aligned} &\int_\Omega |f(x, u(x), \nabla u(x)) u(x)| \, \mathrm{d}x \\ &\leq \left(\lambda_1^{-1} \|\sigma\|_{L^{p'}(\Omega)} \|\nabla u\|_{L^p(\Omega)} + \left(\lambda_1^{-p} b + c\right) \|\nabla u\|_{L^p(\Omega)}^p\right), \end{aligned} \tag{5}$$

for all $u \in W_0^{1,p}(\Omega)$.

Now we can proceed to the existence of generalized solutions:

Theorem 3. *Assume that condition (**H**) is satisfied. Let the number R be such that $R > (\mu(\Omega))^{1/p}$. Then, there is the parameter $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$ problem (1) has at least one generalized solution $u_0 \in W_0^{1,p}(\Omega)$ with the property $\|\nabla u_0\|_{L^p(\Omega)} \leq R$.*

Proof. From estimate (5) and from the well known relations concerning L^p spaces, we obtain for any $u \in W_0^{1,p}(\Omega)$ that:

$$\begin{aligned} &\langle A(u), u\rangle \geq \|\nabla u\|_{L^p(\Omega)}^p - (\mu(\Omega))^{1/q-1/p} \|\nabla u\|_{L^p(\Omega)}^q \\ &-\lambda\left(\lambda_1^{-1} \|\sigma\|_{L^{p'}(\Omega)} \|\nabla u\|_{L^p(\Omega)} + \lambda_1^{-p} b \|\nabla u\|_{L^p(\Omega)}^p + c \|\nabla u\|_{L^p(\Omega)}^p\right), \end{aligned}$$

where A is defined by (2). Note that since $R > (\mu(\Omega))^{1/p}$, we have

$$R^{p-1} - (\mu(\Omega))^{1/q-1/p} R^{q-1} > 0.$$

If we put

$$\lambda_0 = \frac{R^{p-1} - (\mu(\Omega))^{1/q-1/p} R^{q-1}}{\lambda_1^{-1} \|\sigma\|_{L^{p'}(\Omega)} + \left(\lambda_1^{-p} b + c\right) R^{p-1}},$$

then we see for $u \in W_0^{1,p}(\Omega)$ with $\|\nabla u\|_{L^p(\Omega)} = R$ and $\lambda \in (0, \lambda_0]$ that

$$\langle A(u), u\rangle \geq 0.$$

Hence, assumption (**B**) satisfied. Since the operator A is bounded and continuous, we can apply Theorem 2 in order to get the assertion. □

Since the standard (p, q)-Laplacian satisfies the (S_+)-property, it follows that Theorem 2 can be applied for the following classical problem

$$\begin{cases} -div\left(|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right) = \lambda f(x, u, \nabla u) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

under condition (**H**). Using the proof of Theorem 3 we obtain what follows:

Theorem 4. *Assume that condition (**H**) is satisfied. Let the number $R > 0$ be fixed. Then, there is $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0]$ problem (6) has at least one weak solution $u_0 \in W_0^{1,p}(\Omega)$ with $\|\nabla u_0\|_{L^p(\Omega)} \leq R$.*

Proof. Since $\langle -\Delta_p u - \Delta_q u, u \rangle \geq \|\nabla u\|_{L^p(\Omega)}^p$ for any $u \in W_0^{1,p}(\Omega)$, we may put

$$\lambda_0 = \frac{R^{p-1}}{\lambda_1^{-1} \|\sigma\|_{L^{p'}(\Omega)} + \left(\lambda_1^{-p} b + c\right) R^{p-1}}$$

and proceed as in the proof of Theorem 3. □

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