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ON LIPSCHITZ STABILITY OF A CLASS OF EXTENDED REAL-VALUED HEMIVARIATIONAL INEQUALITIES - APPLICATION TO A NONSMOOTH BOUNDARY VALUE PROBLEM*

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Dedicated to Biagio Ricceri on the occasion of his 70th anniversary

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Abstract

In this paper, we first present a sensitivity analysis for extended real-valued hemivariational inequalities and variational–hemivariational inequalities in reflexive Banach spaces. In particular, we provide estimates of Lipschitz type with respect to parametric perturbations in the elliptic operator and the Clarke directional derivative. Then, we apply our quantitative stability result to an elliptic scalar boundary value problem that models unilateral contact problems in solid mechanics with nonmonotone friction.

Keywords: sensitivity analysis, extended real-valued equilibria, hemivariational inequality, Clarke subdifferential, unilateral contact problem, non-monotone friction.

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1 Introduction

It is apparent today that the investigation of the influence of different parameters inherent in mathematical models from internal and/or external sources is of great interest. This actually constitutes a permanent challenge of stability analysis in both theoretical and applied models of variational analysis that deal with real world phenomena and treat concrete applications to physics and engineering problems.

There is already a rich literature on stability issues in variational analysis; without claim of completeness we can refer to [1–7, 10, 11, 15, 19, 23]. The main novelty of this paper is a study of quantitative stability in a general class of extended real-valued hemivariational inequalities with respect to parametric perturbations in the elliptic operator and the Clarke directional derivative. Moreover, we apply our stability result of Lipschitz type to an elliptic scalar boundary value problem that models unilateral contact problems in solid mechanics with non-monotone friction.

We first consider a general class of hemivariational inequalities on a reflexive Banach space V with norm $\|.\|$ and with the following ingredients following [22, Section 5.4] and [15, Section 2.4]: There are a convex closed subset $C \subset V$, a nonlinear monotone continuous operator $A_0 : V \to V^*$, a linear continuous operator $\chi : V \to X$ into a real Banach space X and a locally Lipschitz function $J_0 : X \to \mathbb{R}$ with the Clarke generalized directional derivative J_0^0. In addition we introduce an extended real-valued convex lower semicontinuous proper function $F : V \to \mathbb{R} \cup \{+\infty\}$, and consider the following extended real-valued hemivariational inequality problem: Find $u_0 \in \text{dom } F$ such that

$$\langle A_0 u_0, v - u_0 \rangle + J_0^0(\chi u_0; \chi(v - u_0)) + F(v) \geq F(u_0) \quad \forall v \in V. \qquad (HVI)$$

Further suppose that A_0 is m_0-strongly monotone ($m_0 > 0$), i.e., for all $u, v \in V$

$$\langle A_0 u - A_0 v, u - v \rangle \geq m_0 \|u - v\|^2, \tag{1}$$

and similar to [22], suppose also the one-sided Lipschitz condition for the generalized directional derivative J_0^0: There exists $\Theta_0 > 0$ such that for all $y_1, y_2 \in X$

$$J_0^0(y_1; y_2 - y_1) + J_0^0(y_2; y_1 - y_2) \leq \Theta_0 \|y_1 - y_2\|_X^2, \tag{2}$$

and in addition the smallness condition

$$\Theta_0 \|\chi\|^2 < m_0. \tag{3}$$

Under these assumptions and following [15], the problem (HVI) is uniquely solvable. Inspired by the sensitivity analysis of abstract equilibria presented in [6,21], we establish Lipschitz estimates for the parametric form of (HVI) under perturbation of the elliptic operator A_0 and the Clarke directional derivative J_0^0. The perturbation is described by a parameter μ belonging to a subset M (the space of an external perturbation) of a normed space, whose norm is denoted by $|\cdot|$. Thus, we consider a family of nonlinear operators $\{A_\mu : V \to V^*, \mu \in M\} := \{A(.,\mu) : \mu \in M\}$, and a family of locally Lipschitz continuous functions $\{J_\mu : X \to \mathbb{R}, \mu \in M\}$. For a fixed value $\bar{\mu} \in M$ of the parameter, for a given neighborhood $\mathcal{V}(\bar{\mu}) \subset M$, the parametric extended real-valued hemivariational inequality problem under study reads for each $\mu \in \mathcal{V}(\bar{\mu})$ as follows: Find $u_\mu \in \mathrm{dom}\, F$ such that

$$\langle A_\mu u_\mu, v - u_\mu\rangle + J_\mu^0(\chi u_\mu; \chi(v - u_\mu)) + F(v) \geq F(u_\mu) \quad \forall v \in V. \quad (HVI_\mu)$$

We shall consider the latter as a perturbed form of the problem (HVI). In the following, we do not need to assume that (HVI_μ) admits at least a solution u_μ; instead, solvability will be proved in the first step of our analysis. Let us denote the solution to (HVI) by $\bar{u}$, i.e., $\bar{u} := u_{\bar{\mu}} = u_0$, $A_{\bar{\mu}} := A_0$, and $J_{\bar{\mu}} := J_0$. Our analysis provides, under suitable assumptions on the parametrized families (A_μ) and (J_μ), a stability result of the following Lipschitz type: For all μ in the given neighborhood $\mathcal{V}(\bar{\mu})$

$$\|u_\mu - u_0\| \leq c_1|\mu|,$$

where $c_1 > 0$ is a constant to be specified.

The second problem in consideration is the following nonlinear variational-hemivariational inequality on a bounded Lipschitz domain $D \subset \mathbb{R}^d$ (with $d \geq 2$): Find $u_0 \in \mathrm{dom}\, F$ such that

$$\langle A_0 u_0, v - u_0\rangle + \varphi_0(u_0, v) + F(v) \geq F(u_0) \quad \forall v \in V, \qquad (VHVI)$$

where, $A_0 : V \to V^*$ is a generally nonlinear monotone operator and the bifunction φ_0 on $V \times V$ is defined by

$$\varphi_0(u, v) = \int_D j_0^0(s, \chi u(s); \chi v(s) - \chi u(s))\, ds, \quad \forall u, v \in V,$$

$j_0 : D \times \mathbb{R}^d \to \mathbb{R}$ such that $j_0(\cdot, \xi) : D \to \mathbb{R}$ is measurable on D for all $\xi \in \mathbb{R}^d$ and $j_0(s, \cdot) : \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz on $\mathbb{R}^d$ for almost all (a.a.) $s \in D$.

$\chi : V \to L^p(D)\,(1 < p < \infty)$ is a linear continuous operator. $j_0^0(s, \cdot\,;\,\cdot)$ stands for the generalized Clarke directional derivatives [9] of $j_0(s, \cdot)$.

Similarly to the problem (HVI), we are interested to the sensitivity analysis of the parametric form of ($VHVI$), under perturbation of the data A and φ, which is expressed for each $\mu \in \mathcal{V}(\bar{\mu})$ as follows: Find $u_\mu \in \text{dom } F$ such that

$$\langle A_\mu u_\mu, v - u_\mu\rangle + \varphi_\mu(u_\mu, v) + F(v) \geq F(u_\mu) \quad \forall v \in V, \qquad (VHVI_\mu)$$

where $\varphi_\mu(u, v) = \displaystyle\int_D j_\mu^0(s, \chi u(s); \chi v(s) - \chi u(s))\, ds, \quad \forall u, v \in V, \forall \mu \in \mathcal{V}(\bar{\mu}).$

The outline of the paper is as follows. The subsequent section 2 presents the sensitivity analysis of (HVI_μ) for generally nonlinear monotone operators A_0, A_μ, see Theorem 1, and also for the special case of bounded linear operators A_0, A_μ. Then the sensitivity result for ($VHVI_\mu$) is given in Theorem 2. Section 3 turns to the application to an elliptic scalar boundary value problem that models unilateral contact problems in solid mechanics with non-monotone friction and provides the sensitivity result in Theorem 3. The paper ends in section 4 with an outlook to some further directions of research.

2 Sensitivity analysis - assumptions and results

2.1 Sensitivity analysis of (HVI_μ)

Here we present our main result on sensitivity analysis for (HVI_μ), that is based on the following assumptions.

[$\mathbf{C}_{A_\mu}$] For all $\mu \in \mathcal{V}(\bar{\mu})$, $A_\mu : V \to V^*$ is such that

- **(1)** A_μ is hemicontinuous;
- **(2)** there exists $m_1 > 0$ such that for all $u, v, w \in V$ and all $\mu \in \mathcal{V}(\bar{\mu})$

$$|\langle (A_\mu u - A_0 u) - (A_\mu v - A_0 v), w\rangle| \leq m_1 |\mu| \|u - v\| \|w\|;$$

[$\mathbf{C}_{J_\mu}$] for all $\mu \in \mathcal{V}(\bar{\mu})$, $J_\mu : X \to \mathbb{R}$ is such that

- **(1)** there exists $\Theta_1 > 0$ such that for all $\mu \in \mathcal{V}(\bar{\mu})$ and all $y_1, y_2 \in X$

$$\begin{aligned}|J_\mu^0(y_1; y_2 - y_1) - J_0^0(y_1; y_2 - y_1) + J_\mu^0(y_2; y_1 - y_2)\\ - J_0^0(y_2; y_1 - y_2)| \leq \Theta_1 |\mu| \|y_1 - y_2\|_X^2;\end{aligned}$$

(2) there exists $\Theta > 0$ such that for all $\mu \in \mathcal{V}(\bar{\mu})$ and all $y_1, y_2 \in X$

$$|J_\mu^0(y_1; y_2 - y_1) - J_0^0(y_1; y_2 - y_1)| \leq \Theta|\mu| \|y_1 - y_2\|_X;$$

$[\mathbf{C}_{u_\mu}]$ boundedness of the solutions u_μ: there exists $R > 0$ such that for all $\mu \in \mathcal{V}(\bar{\mu})$

$$\|u_\mu\| \leq R;$$

$[\mathbf{C}_d]$ control of data: assume that

$$|\mu| < \frac{m_0 - \Theta_0\|\chi\|^2}{m_1 + \Theta_1\|\chi\|^2}.$$

Remark 1. *Following [7, Remark 2.1], condition* $[\boldsymbol{C}_{J_\mu}]$***(2)*** *is equivalent to: there exists* $\Theta > 0$ *such that for all* $\mu \in \mathcal{V}(\bar{\mu})$ *and all* $u \in V$,

$$\partial^c J_\mu(u) \subset \partial^c J_0(u) + \Theta|\mu| B_V,$$

where B_V *stands for the closed unit ball of* V.

Theorem 1. *Under conditions* $[\boldsymbol{C}_{A_\mu}]$, $[\boldsymbol{C}_{J_\mu}]$, $[\boldsymbol{C}_{u_\mu}]$ *and* $[\boldsymbol{C}_d]$, *for each* $\mu \in \mathcal{V}(\bar{\mu})$, *the problem* (HVI_μ) *is uniquely solvable. Moreover, the following Lipschitz estimate is satisfied: For all* $\mu \in \mathcal{V}(\bar{\mu})$,

$$\|u_\mu - u_0\| \leq \frac{\theta + \Theta\|\chi\|}{m_0 - \Theta_0\|\chi\|^2 - (m_1 + \Theta_1\|\chi\|^2)|\mu|}|\mu|, \tag{4}$$

where

$$\theta := \sup_{\mu \in \mathcal{V}(\bar{\mu})} \frac{\|A_\mu 0 - A_0 0\|_{V^*}}{|\mu|} + m_1 R.$$

Proof. We divide the proof into two steps.

Step 1: Existence and uniqueness

Assumption $[\mathbf{C}_{A_\mu}]$**(2)** implies that for all $u, v \in V$ and all $\mu \in \mathcal{V}(\bar{\mu})$

$$|\langle (A_\mu u - A_0 u) - (A_\mu v - A_0 v), u - v\rangle| \leq m_1|\mu| \|u - v\|^2. \tag{5}$$

By combining inequalities (1) and (5), we have for all $u, v \in V$ and all $\mu \in \mathcal{V}(\bar{\mu})$

$$\langle A_\mu u - A_\mu v, u - v\rangle \geq (m_0 - m_1|\mu|)\|u - v\|^2. \tag{6}$$

On the other hand, from (2), $[\mathbf{C}_{J_\mu}]$**(1)** and triangle inequality, we also have, for all $y_1, y_2 \in X$ and all $\mu \in \mathcal{V}(\bar{\mu})$

$$J_\mu^0(y_1; y_2 - y_1) + J_\mu^0(y_2; y_1 - y_2) \leq (\Theta_0 + \Theta_1|\mu|)\|y_1 - y_2\|_X^2. \tag{7}$$

Using inequality (7), we have for all $u, v \in V$ and all $\mu \in \mathcal{V}(\bar{\mu})$

$$J_\mu^0(\chi u; \chi(v-u)) + J_\mu^0(\chi v; \chi(u-v)) \leq (\Theta_0 + \Theta_1|\mu|)\|\chi\|^2\|u-v\|^2. \quad (8)$$

Since by $[\mathbf{C}_d]$, $|\mu| < \frac{m_0 - \Theta_0\|\chi\|^2}{m_1 + \Theta_1\|\chi\|^2} < \frac{m_0}{m_1}$, then $m_0 > m_1|\mu|$, we deduce from (6) that the operator A_μ is $m_\mu := (m_0 - m_1|\mu|)$-strongly monotone. On the other hand, we also have

$$(\Theta_0 + \Theta_1|\mu|)\|\chi\|^2 < m_\mu.$$

Finally, following [11, Section 4.2], the last inequality combined with (6) and (8) guarantees existence and uniqueness of a solution to the perturbed problem (HVI_μ).

Step 2: Lipschitz estimate

Now, for proving the estimation (4), we follow the quantitative stability approach of [6] for strongly monotone equilibrium problems. Let us define for all $u, v \in V$ and all $\mu \in \mathcal{V}(\bar{\mu})$

$$\Phi_\mu(u, v) := \langle A_\mu u, v - u\rangle + J_\mu^0(\chi u; \chi(v-u)).$$

Then, for all $u \in V$ and all $\mu \in \mathcal{V}(\bar{\mu})$, $\Phi_\mu(u, \cdot)$ is convex and lsc on V, and $\Phi_\mu(u, u) = 0$. In this way, the perturbed problem (HVI_μ) is equivalent to the following parametric extended real-valued equilibrium problem: for each $\mu \in \mathcal{V}(\bar{\mu})$, find $u_\mu \in V$ such that

$$\Phi_\mu(u_\mu, v) + F(v) - F(u_\mu) \geq 0, \quad \forall v \in V.$$

In a similar way, we have, for all $v \in V$

$$\Phi_0(u_0, v) + F(v) - F(u_0) \geq 0.$$

Thus, if we set $v = u_0$ in the first inequality and $v = u_\mu$ in the second one, by adding the two obtained relations, we have for all $\mu \in \mathcal{V}(\bar{\mu})$

$$0 \leq \Phi_\mu(u_\mu, u_0) + \Phi_0(u_0, u_\mu),$$

and hence

$$0 \leq \Phi_\mu(u_\mu, u_0) + \Phi_\mu(u_0, u_\mu) + \Phi_0(u_0, u_\mu) - \Phi_\mu(u_0, u_\mu).$$

Next, we first estimate using (6) and (8)

$$\begin{aligned} & \Phi_\mu(u_\mu, u_0) + \Phi_\mu(u_0, u_\mu) \\ = \ & \langle A_\mu u_\mu - A_\mu u_0, u_0 - u_\mu\rangle + J_\mu^0(\chi u_\mu; \chi(u_0 - u_\mu)) + J_\mu^0(\chi u_0; \chi(u_\mu - u_0)) \\ \leq \ & -\big((m_0 - m_1|\mu|) - (\Theta_0 + \Theta_1|\mu|)\|\chi\|^2\big)\|u_\mu - u_0\|^2 \quad \forall \mu \in \mathcal{V}(\bar{\mu}). \quad (9) \end{aligned}$$

Now, we aim to estimate $|\Phi_\mu(u_0, u_\mu) - \Phi_0(u_0, u_\mu)|$. To this end, we apply assumption $[\mathbf{C}_{A_\mu}]$**(2)** with $v = 0$, then for all $u \in V$

$$\|A_\mu u - A_0 u\|_{V^*} \leq \|A_\mu 0 - A_0 0\|_{V^*} + m_1|\mu|\|u\|.$$

Since, by $[\mathbf{C}_{u_\mu}]$, the family $\{u_\mu\}$ is uniformly bounded, we define

$$\theta := \sup_{\mu \in \mathcal{V}(\bar{\mu})} \frac{\|A_\mu 0 - A_0 0\|_{V^*}}{|\mu|} + m_1 R.$$

Then, for all $\mu \in \mathcal{V}(\bar{\mu})$

$$\|A_\mu u_\mu - A_0 u_\mu\|_{V^*} \leq \theta|\mu|. \tag{10}$$

On the other hand, by (10) and $[\mathbf{C}_{J_\mu}](2)$, we have for all $\mu \in \mathcal{V}(\bar{\mu})$

$$\begin{aligned} &|\Phi_\mu(u_0, u_\mu) - \Phi_0(u_0, u_\mu)| \\ &\leq \|A_\mu u_0 - A_0 u_0\|_{V^*}\|u_\mu - u_0\| + |J_\mu^0(\chi u_0; \chi(u_\mu - u_0)) \\ &\quad - J_0^0(\chi u_0; \chi(u_\mu - u_0))| \\ &\leq \theta|\mu|\|u_\mu - u_0\| + \Theta|\mu|\|\chi\|\|u_\mu - u_0\|. \end{aligned} \tag{11}$$

We conclude that, for all $\mu \in \mathcal{V}(\bar{\mu})$

$$|\Phi_\mu(u_0, u_\mu) - \Phi_0(u_0, u_\mu)| \leq (\theta + \Theta\|\chi\|)|\mu|\|u_\mu - u_0\|. \tag{12}$$

Using thus (9) and (12), we conclude that

$$(m_0 - m_1|\mu| - (\Theta_0 + \Theta_1|\mu|)\|\chi\|^2)\|u_\mu - u_0\|^2 \leq (\theta + \Theta\|\chi\|)|\mu|\|u_\mu - u_0\|.$$

Therefore,

$$\|u_\mu - u_0\| \leq \frac{\theta + \Theta\|\chi\|}{m_0 - m_1|\mu| - (\Theta_0 + \Theta_1|\mu|)\|\chi\|^2}|\mu|.$$

This completes the proof. □

2.2 The case of bounded linear operators

In this subsection, we consider problems of the form (*HVI*) and (HVI_μ) with bounded linear operators $A_0, A_\mu : V \to V^*$. In this case clearly assumption $[\mathbf{C}_{A_\mu}](2)$ reduces to : There exists $m_1 > 0$ such that for all $\mu \in \mathcal{V}(\bar{\mu})$

$$\|A_\mu - A_0\|_{\mathcal{L}(V,V^*)} \leq m_1|\mu|. \tag{13}$$

Moreover, similar to [15,22], in addition to the one-sided Lipschitz continuity (2), we assume that, for all $\mu \in \mathcal{V}(\bar{\mu})$, the local Lipschitz function J_μ satisfies the following growth condition

$$\|\xi\|_{X^*} \leq d_{J_\mu}(1 + \|z\|_X), \quad \forall z \in X, \xi \in \partial J_\mu(z). \tag{14}$$

Corollary 1. *Assume that assumptions $[C_{A_\mu}](1)$, (13), $[C_{J_\mu}]$ and $[C_d]$ hold. Then, for each $\mu \in \mathcal{V}(\bar{\mu})$, the problem ($HVI_\mu$) is uniquely solvable. Moreover, the following Lipschitz estimate is satisfied: for all $\mu \in \mathcal{V}(\bar{\mu})$, there exists $c_\mu > 0$ such that*

$$\|u_\mu - u_0\| \leq \frac{\theta c_\mu + \Theta\|\chi\|}{m_0 - \Theta_0\|\chi\|^2 - (m_1 + \Theta_1\|\chi\|^2)|\mu|}|\mu|. \tag{15}$$

Proof. The proof of existence and uniqueness of a solution to (HVI_μ) is similar to that of Theorem 1.

Let us prove the estimation (15). Under assumption (13), inequality (12) becomes: for all $\mu \in \mathcal{V}(\bar{\mu})$

$$|\Phi_\mu(u_0, u_\mu) - \Phi_0(u_0, u_\mu)| \leq (\theta\|u_\mu\| + \Theta\|\chi\|)|\mu|\|u - v\|. \tag{16}$$

Then, following lines of the proof of Theorem 1, we end at

$$\|u_\mu - u_0\| \leq \frac{\theta\|u_\mu\| + \Theta\|\chi\|}{m_0 - \Theta_0\|\chi\|^2 - (m_1 + \Theta_1\|\chi\|^2)|\mu|}|\mu|. \tag{17}$$

So, to conclude, we need an a priori estimate for the solution u_μ for all $\mu \in \mathcal{V}(\bar{\mu})$. For this regard, combining the $m_\mu := m_0 - m_1|\mu|$-strong monotonicity of A_μ with (8) for $u = u_\mu$ and $v = 0 \in \mathrm{dom}\, F$, we have for all $\mu \in \mathcal{V}(\bar{\mu})$

$$\begin{aligned} m_\mu\|u_\mu - 0\|^2 &\leq \langle A_\mu u_\mu - A_\mu 0, u_\mu - 0\rangle \\ &\leq J_\mu^0(\chi u_\mu; 0 - \chi u_\mu) + F(0) - F(u_\mu) \\ &\leq (\Theta_0 + \Theta_1|\mu|\|\chi\|^2\|u_\mu\|^2 - J_\mu^0(0; \chi u_\mu) + F(0) - F(u_\mu). \end{aligned}$$

On the other hand, by (14), we have

$$\begin{aligned} -J_\mu^0(0; \chi u_\mu) &\leq \max_{\xi \in \partial J_\mu(0)} \|\xi\|_{X^*}\|\chi u_\mu\|_X \\ &\leq d_{J_\mu}\|\chi\|\|u_\mu\|. \end{aligned}$$

Further, since the extended real-valued function F is conically minorized (see, [15]), that is, it enjoys the estimate

$$F(v) \geq -c_F(1 + \|v\|), \ \forall v \in V,$$

then, we have for all $\mu \in \mathcal{V}(\bar{\mu})$

$$\big(m_\mu - (\Theta_0 + \Theta_1|\mu|\big)\|\chi\|^2\|u_\mu\|^2 \leq c_F(1 + \|u_\mu\|) + d_{J_\mu}\|\chi\|\|u_\mu\| + F(0),$$

which yields

$$\|u_\mu\|^2 \leq \alpha_\mu\|u_\mu\| + \beta_\mu,$$

where $\alpha_\mu = \frac{c_F + d_{J_\mu}\|\chi\|}{m_\mu - (\Theta_0 + \Theta_1|\mu|)\|\chi\|^2} > 0$ and $\beta_\mu = \frac{c_F + F(0)}{m_\mu - (\Theta_0 + \Theta_1|\mu|)\|\chi\|^2} > 0$.
Using thus the elementary quadratic inequality estimate

$$x^2 \leq ax + b \quad \Rightarrow \quad x \leq a + \sqrt{b}, \quad \forall x, a, b \geq 0$$

to obtain, for $|\mu|$ sufficiently small, a priori estimate:

$$\|u_\mu\| \leq c_\mu := \frac{c_F + d_{J_\mu}\|\chi\|}{m_\mu - (\Theta_0 + \Theta_1|\mu|)\|\chi\|^2} + \sqrt{\frac{c_F + F(0)}{m_\mu - (\Theta_0 + \Theta_1|\mu|)\|\chi\|^2}}. \quad (18)$$

Finally, we conclude by combining (17) and (18). □

Remark 2. *1) One can consider the following sharper estimate for* $\|u_\mu\|$ *:*

$$\|u_\mu\| \leq \frac{\alpha_\mu + \sqrt{\alpha_\mu^2 + 4\beta_\mu}}{2},$$

where α_μ *and* β_μ *are the positive constants defined above. Indeed, the sharpest upper bound for* x *satisfying the quadratic inequality* $x^2 \leq ax + b$ $(a, b > 0)$ *is* $\frac{a+\sqrt{a^2+4b}}{2}$ *which is always tighter than* $x \leq a + \sqrt{b}$*.*

2) Let $C \subset V$ *be a closed convex set. To obtain a stability result applicable to the parametric form of the following problem: Find an element* $u_0 \in C$ *such that*

$$\langle A_0 u_0, v - u_0\rangle + J_0^0(\chi u_0; \chi(v - u_0)) + f(v) \geq f(u_0) \quad \forall v \in C,$$

we can simply set $F = f + I_C$ *where* I_C *is the indicator function of* C*.*

2.3 Sensitivity analysis of $(VHVI_\mu)$

In this subsection we come back to the nonlinear problem $(VHVI_\mu)$ described in the introduction section. We provide a stability result of this problem within the framework of (HVI_μ). Let V be a function space that endowed with the norm $\|\cdot\|$ is a reflexive Banach space. Further we have

a bounded Lipschitz domain $D \subset \mathbb{R}^d$ (with $d \geq 2$) and a linear continuous operator $\chi : V \to L^p(D)\,(1 < p < \infty)$. With V continuously embedded in $(L^p(D), \|.\|_p)$, there is $c_p > 0$, such that $\|u\|_p \leq c_p\|u\| \; \forall u \in V$.
Let $X := L^p(D)$ and introduce the real-valued locally Lipschitz functional

$$J_0(y) := \int_D j_0(s, y(s))\, ds \quad y \in X.$$

Then by Lebesgue's theorem of majorized convergence,

$$J_0^0(y; z) = \int_D j_0^0(s, y(s); z(s))\, ds \quad (y, z) \in X \times X.$$

Similarly, for all $\mu \in \mathcal{V}(\bar{\mu})$, set

$$J_\mu(y) := \int_D j_\mu(s, y(s))\, ds \quad y \in X.$$

Then, for all $\mu \in \mathcal{V}(\bar{\mu})$,

$$J_\mu^0(y; z) = \int_D j_\mu^0(s, y(s); z(s))\, ds \quad (y, z) \in X \times X.$$

With these notations, for all $u, v \in V$ and all $\mu \in \mathcal{V}(\bar{\mu})$, the bifunctions $\varphi_0(u, v)$ and $\varphi_\mu(u, v)$ can be respectively identified to $J_0^0(\chi u; \chi(v - u))$ and $J_\mu^0(\chi u; \chi(v - u))$. Consequently, problems $(VHVI)$ and $(VHVI_\mu)$ become equivalent to problems (HVI) and (HVI_μ), respectively.

Next suppose that the generalized directional derivative $j_0^0(s, .; .)$ satisfies the following assumption: there exists $\lambda_0 > 0$ such that there holds the Lipschitz condition: for all $\xi, \eta \in \mathbb{R}^d$

$$j_0^0(s, \xi; \eta - \xi) + j_0^0(s, \eta; \xi - \eta) \leq \lambda_0|\xi - \eta|^2. \tag{19}$$

In this case we have for all $y_1, y_2 \in X$

$$J_0^0(y_1; y_2 - y_1) + J_0^0(y_2; y_1 - y_2) \leq \lambda_0(\text{meas } (D))^{1-\frac{1}{p}}\|y_1 - y_2\|_X, \tag{20}$$

meas (D) being the measure of D. Indeed, by (19) and generalized Hölder's inequality, we have for all $y_1, y_2 \in X$

$$\begin{aligned} & J_0^0(y_1; y_2 - y_1) + J_0^0(y_2; y_1 - y_2) \\ = \; & \int_D \left(j_0^0(s, y_1(s); y_2(s) - y_1(s)) + j_0^0(s, y_2(s); y_1(s) - y_2(s))\right)\, ds \\ \leq \; & \lambda_0 \int_D |y_1(s) - y_2(s))|^2\, ds \\ \leq \; & \lambda_0 \int_D \|y_1 - y_2\|\, ds \\ \leq \; & \lambda_0(\text{meas } (D))^{1-\frac{1}{p}}\|y_1 - y_2\|_p^2. \end{aligned}$$

This means that condition (2) is satisfied with $\Theta_0 = \lambda_0(\text{meas }(D))^{1-\frac{1}{p}}$ and then, following (3), under the smallness condition

$$\lambda_0(\text{meas }(D))^{1-\frac{1}{p}}\|\chi\|^2 < m_0, \tag{21}$$

problem ($VHVI$) is uniquely solvable.

In our sensitivity analysis for the problem ($VHVI_\mu$) we require the following conditions.

[$\mathbf{C}_{j_\mu}$] for all $\mu \in \mathcal{V}(\bar{\mu})$, $j_\mu : D \times \mathbb{R}^d \to \mathbb{R}$ is such that

- **(1)** there exists $\lambda > 0$ such that there holds the Lipschitz condition: for all $\mu \in \mathcal{V}(\bar{\mu})$ and for all $\xi, \eta \in \mathbb{R}^d$

$$|j_\mu^0(s,\xi;\eta-\xi) - j_0^0(s,\xi;\eta-\xi) + j_\mu^0(s,\eta;\xi-\eta) - j_0^0(s,\eta;\xi-\eta)| \\ \leq \lambda|\mu||\xi-\eta|^2;$$

- **(2)** there exists $\delta_3 > 0$ such that for a.a. $s \in D$, all $\mu \in \mathcal{V}(\bar{\mu})$ and all $\xi, \eta \in \mathbb{R}^d$, it holds

$$|j_\mu^0(s,\xi;\eta-\xi) - j_0^0(s,\xi;\eta-\xi)| \leq \delta_3|\mu||\xi-\eta|;$$

[C'$_d$] control of data: assume that

$$|\mu| < \frac{m_0 - \lambda_0(\text{meas }(D))^{1-\frac{1}{p}}\|\chi\|^2}{m_1 + \lambda(\text{meas }(D))^{\frac{p-2}{p}}\|\chi\|^2}.$$

Remark 3. *1) Condition [$\boldsymbol{C}_{j_\mu}$](1) implies: for all $\mu \in \mathcal{V}(\bar{\mu})$ and all $y_1, y_2 \in X$*

$$|J_\mu^0(y_1;y_2-y_1) - J_0^0(y_1;y_2-y_1) + J_\mu^0(y_2;y_1-y_2) - J_0^0(y_2;y_1-y_2)| \\ \leq \lambda(\text{meas }(D))^{\frac{p-2}{p}}|\mu|\|y_1-y_2\|_X^2.$$

Indeed, by [$\boldsymbol{C}_{j_\mu}$](1) and generalized Hölder's inequality, we have for all $\mu \in \mathcal{V}(\bar{\mu})$ and all $y_1, y_2 \in X$

$$
\begin{aligned}
&|J^0_\mu(y_1;y_2-y_1)-J^0_0(y_1;y_2-y_1)+J^0_\mu(y_2;y_1-y_2)-J^0_0(y_2;y_1-y_2)|\\
&=\Big|\int_D \big(j^0_\mu(s,y_1(s);y_2(s)-y_1(s))-j^0_0(s,y_1(s);y_2(s)-y_1(s))\\
&\quad +j^0_\mu(s,y_2(s);y_1(s)-y_2(s))-j^0_0(s,y_2(s);y_1(s)-y_2(s))\big)\,ds\Big|\\
&\le \lambda|\mu|\int_D |y_1(s)-y_2(s)|^2\,ds\\
&\le \lambda|\mu|\int_D \|y_1-y_2\|^2\,ds\\
&\le \lambda|\mu|(\text{meas }(D))^{\frac{p-2}{p}}\|y_1-y_2\|_p^2.
\end{aligned}
$$

This means that assumption [$\boldsymbol{C}_{J_\mu}$](1) is satisfied with $\Theta_1=\lambda(\text{meas }(D))^{\frac{p-2}{p}}$.

2) Similar arguments justify that condition [$\boldsymbol{C}_{j_\mu}$](2) implies: for all $\mu\in\mathcal{V}(\bar\mu)$ *and all* $y_1,y_2\in X$

$$|J^0_\mu(y_1;y_2-y_1)-J^0_0(y_1;y_2-y_1)|\le \delta_3\text{meas }(D)^{1-\frac{1}{p}}|\mu|\|y_1-y_2\|_X.$$

This means that [$\boldsymbol{C}_{J_\mu}$](2) is also satisfied with $\Theta=\delta_3\text{meas }(D)^{1-\frac{1}{p}}$.

Thus, the next result is immediate from Theorem 1.

Theorem 2. *Under conditions [$\boldsymbol{C}_{A_\mu}$], [$\boldsymbol{C}_{j_\mu}$], [$\boldsymbol{C}_{u_\mu}$] and [$\boldsymbol{C'}_d$], for each* $\mu\in\mathcal{V}(\bar\mu)$*, the problem* ($VHVI_\mu$) *is uniquely solvable. Moreover, the following Lipschitz estimate is satisfied: For all* $\mu\in\mathcal{V}(\bar\mu)$,

$$\|u_\mu-u_0\| \tag{22}$$

$$\le \frac{\theta+\delta_3(\text{meas }(D))^{1-\frac{1}{p}}\|\chi\|}{m_0-\lambda_0(\text{meas }(D))^{1-\frac{1}{p}}\|\chi\|^2-(m_1+\lambda(\text{meas }(D))^{\frac{p-2}{p}}\|\chi\|^2)|\mu|}|\mu|,$$

where

$$\theta:=\sup_{\mu\in\mathcal{V}(\bar\mu)}\frac{\|A_\mu 0-A_0 0\|_{V^*}}{|\mu|}+m_1R. \tag{23}$$

3 Application to a nonsmooth boundary value problem

In this section, we consider a nonsmooth boundary value problem with unilateral and nonmonotone boundary conditions, which simplifies a bilateral obstacle problem from [15] and from [16], dispensing with random structure. This boundary value problem can be seen as a scalar model of unilateral frictional contact problems in continuum mechanics, see [12, Remark 5.1] and [18, Remark 1].

Let $D \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded Lipschitz domain with outer unit normal ν. Let

- $S_0 \in L^\infty(D)$, with $S_0(x) \geq S^* > 0$ a.e. in D,
- $R \in L^\infty(D)$,
- $T \in L^\infty(\Gamma_N \cup \Gamma_S)$, with $T(x) > 0$ a.e.

The strong form of the boundary value problem under study is to find a sufficiently smooth function $u_0 : D \to \mathbb{R}$ that satisfies the elliptic partial differential equation

$$-\operatorname{div}(S_0\, p(|\nabla u_0|)\, \nabla u_0) = R\, g \quad \text{in } D, \tag{24}$$

along with boundary conditions specified below, where $g \in L^2(D)$, and $p : [0, \infty) \to [0, \infty)$ is continuous with $t \mapsto tp(t)$ monotonically increasing.

The boundary $\partial D = \Gamma$ is partitioned into disjoint open subsets:

$$\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_{S_0} \cup \overline{\Gamma}_T \cup \overline{\Gamma}_C,$$

with $\operatorname{meas}(\Gamma_D) > 0$, where Γ_D is the Dirichlet part, Γ_N the Neumann part, Γ_S the Signorini part, Γ_T the Tresca part, and Γ_C the Clarke part. Let $h \in L^2(\Gamma_N \cup \Gamma_{S_0})$, and $k \in L^2(\Gamma_T)$, with $k > 0$ a.e.

Boundary conditions: We impose

$$\begin{aligned}
&Q_\nu := S_0\, p(|\nabla u_0|)\, \nabla u_0 \cdot \nu && \text{on } \partial D,\\
&u_0 = 0 && \text{on } \Gamma_D,\\
&Q_\nu = Th && \text{on } \Gamma_N,\\
&u_0 \leq 0, \quad Q_\nu - Th \leq 0, \quad u(Q_\nu - Th) = 0 && \text{on } \Gamma_S,\\
&|Q_\nu| \leq k, \quad u_0 Q_\nu + k|u_0| = 0 && \text{on } \Gamma_T,\\
&p(|\nabla u_0|)\, \frac{\partial u_0}{\partial \nu}\bigg|_{\Gamma_C} \in \partial j_0(\cdot, u_0|_{\Gamma_C}) && \text{on } \Gamma_C.
\end{aligned} \tag{25}$$

Here, $j_0 : \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfies:
- $j_0(\cdot, \xi)$ is measurable for all ξ,

- $j_0(s,\cdot)$ is locally Lipschitz for a.a. $s \in \Gamma_C$,
- for all $\xi \in \mathbb{R}$, and $\eta \in \partial j_0(s,\xi)$, there exist constants $c_{j_0,1}, c_{j_0,2} > 0$ such that

$$(i) \quad \eta\xi \geq -c_{j_0,2}|\xi|, \quad (ii) \quad |\eta| \leq c_{j_0,1}(1+|\xi|). \tag{26}$$

These imply:

$$j_0^0(s,\xi;-\xi) \leq c_{j_0,2}|\xi|, \quad |j_0^0(s,\xi;\varsigma)| \leq c_{j_0,1}(1+|\xi|)|\varsigma|, \quad \forall \xi,\varsigma \in \mathbb{R}.$$

Functional setting: Let $H := H_0^1(D) = \{v \in H^1(D) : v|_{\Gamma_D} = 0\}$ the separable Hilbert space with inner product and induced norm

$$\langle v,w\rangle = \int_D \nabla v \cdot \nabla w \, dx, \quad \|v\| = \|\nabla v\|_{L^2(D)}.$$

Define the closed convex subset (Signorini constraint):

$$C := \{v \in H : v|_{\Gamma_S} \leq 0\}.$$

Let $Z := L^2(\Gamma)$, and let $\gamma : H \to Z$ be the trace operator. Define:

$$K(z) := \int_{\Gamma_T} k(s)|z(s)| \, ds, \quad z \in Z,$$

$$J_0(y) := \int_{\Gamma_C} j_0(s,y(s)) \, ds, \quad y \in Z.$$

Assuming $j_0(s,\cdot)$ is locally Lipschitz, we have by Lebesgue's theorem of majorized convergence:

$$J_0^0(y;z) = \int_{\Gamma_C} j_0^0(s,y(s);z(s)) \, ds, \quad y,z \in Z.$$

Let $g(t) := \int_0^t s \cdot p(s) \, ds$, which defines the strictly convex energy functional:

$$G(u) := \int_D g(|\nabla u|) \, dx, \quad u \in H^1(D),$$

with Gateaux derivative

$$DG(u,v) = \int_D p(|\nabla u|)\nabla u \cdot \nabla v \, dx, \quad u,v \in H^1(D).$$

Assuming $p \in C^1$, $0 \leq p(t) \leq p_0$, we have

$$G(u) \leq \frac{1}{2}p_0\|u\|^2, \quad \text{and } DG \text{ is strongly monotone:}$$

$$\exists c_G > 0, \quad c_G\|u-v\|^2 \leq DG(u,u-v) - DG(v,u-v).$$

Variational formulation: In this setting it can be proved, see e.g. [18, Theorem 1] for a similar result, that the boundary value problem (24) - (25) is equivalent in the sense of distributions to the following HVI problem: Find $u_0 \in C$ such that for all $v \in C$,

$$\int_D S_0\, p(|\nabla u_0|)\nabla u_0 \cdot \nabla(v - u_0)\, dx + J_0^0(\gamma u_0; \gamma v - \gamma u_0) + K(\gamma v) - K(\gamma u_0) \geq \int_D R\, g\,(v - u_0)\, dx + \int_{\Gamma_N \cup \Gamma_S} Th(\gamma v - \gamma u_0)\, ds. \tag{27}$$

Define the nonlinear operator $A_0 : H \to H^*$ by

$$\langle A_0 u, v\rangle := \int_D S_0\, p(|\nabla u|)\nabla u \cdot \nabla v\, dx, \quad \text{for all } v \in H,$$

and the linear continuous form $l : H \to \mathbb{R}$:

$$\langle l, v\rangle = \int_D R\, g\, v\, dx \; + \int_{\Gamma_N \cup \Gamma_S} T\, h\, \gamma v\, ds.$$

Here, the pairing $\langle A_0 u, v\rangle$ is the duality between H^* and $H_0^1(D)$.

Thus, problem (27) can be written in the following form: Find $u_0 \in C$ such that for all $v \in C$,

$$\langle A_0 u_0, v - u_0\rangle + J_0^0(\gamma u_0; \gamma v - \gamma u_0) + K(\gamma v) - K(\gamma u_0) \geq \langle l, v - u_0\rangle. \tag{28}$$

Since S_0 is assumed to be bounded from below by S^*, and DG is c_G-strongly monotone, it follows that the operator A_0 is also m_0-strongly monotone with $m_0 = S^* c_G$. Combined with assumption (26), this guarantees the existence of a solution to (29) (see [13, Theorem 6.1]). For uniqueness, following [13, Remark 6.1] and condition (21), the smallness condition $\lambda_0(\text{meas }(D))^{1-\frac{1}{p}} \|\gamma\|^2_{H \to Z} < S^* c_G$, guarantees the uniqueness.

Next, we aim to establish a novel sensitivity result for the nonsmooth boundary value problem described above, under perturbations of the data S_0 and j_0.

In this context, for each $\mu \in \mathcal{V}(\bar{\mu})$, the parametric form of problem (28) that we consider is as follows: Find $u_\mu \in C$ such that for all $v \in C$,

$$\langle A_\mu u_\mu, v - u_\mu\rangle + J_\mu^0(\gamma u_\mu; \gamma v - \gamma u_\mu) + K(\gamma v) - K(\gamma u_\mu) \geq \langle l, v - u_\mu\rangle, \tag{29}$$

where, for each μ, the nonlinear operator $A_\mu : H \to H^*$ is defined by

$$\langle A_\mu u, v\rangle := \int_D S_\mu\, p(|\nabla u|)\nabla u \cdot \nabla v\, dx, \quad \text{for all } v \in H.$$

Theorem 3. *Under assumptions [C_{j_μ}], [C_{u_μ}] and by assuming further that:*

(i) for each $\mu \in \mathcal{V}(\bar{\mu})$, $S_\mu \in L^\infty(D)$, with $S_\mu(x) \geq S^* > 0$ a.e. in D;

(ii) for each $\mu \in \mathcal{V}(\bar{\mu})$, $\|S_\mu - S_0\|_{L^\infty(D)} \leq M\,|\mu|$ for some $M > 0$;

(iii) the nonlinear map $\xi \mapsto p(|\xi|)\,\xi$ is Lipschitz, i.e. there is $L > 0$ such that

$$\big|p(|\xi|)\,\xi - p(|\eta|)\,\eta\big| \leq L\,|\xi - \eta|, \quad \forall \xi, \eta \in \mathbb{R}^d;$$

(iv) control of data: assume that

$$|\mu| < \frac{S^* c_G - \lambda_0 (\text{meas }(D))^{1-\frac{1}{p}} \|\gamma\|^2}{LC + \lambda (\text{meas }(D))^{\frac{p-2}{p}} \|\gamma\|^2}.$$

Then, for each $\mu \in \mathcal{V}(\bar{\mu})$, the problem (29) is uniquely solvable. Moreover, the following Lipschitz estimate is satisfied: For all $\mu \in \mathcal{V}(\bar{\mu})$,

$$\|u_\mu - u_0\|$$

$$\leq \frac{LMR + \delta_3 (\text{meas }(D))^{1-\frac{1}{p}} \|\gamma\|}{S^* c_G - \lambda_0 (\text{meas }(D))^{1-\frac{1}{p}} \|\gamma\|^2 - (LM + \lambda (\text{meas }(D))^{\frac{p-2}{p}} \|\gamma\|^2)|\mu|} |\mu|.$$

Proof. Since the linear continuous form l is not subject to perturbation, the proof is similar to that of Theorem 2. Then it suffices to verify that the conditions of that theorem are all satisfied.

Assumption $[\mathbf{C}_{A_\mu}]\mathbf{(1)}$ on hemicontinuity of A_μ follows from (i) and the continuity of p and of the map $\xi \mapsto p(|\xi|)\,\xi$.

To check assumption $[\mathbf{C}_{A_\mu}]\mathbf{(2)}$, compute

$$\begin{aligned} &\langle (A_\mu u - A_0 u) - (A_\mu v - A_0 v), w\rangle \\ &= \int_D (S_\mu - S_0)\big[p(|\nabla u|)\nabla u - p(|\nabla v|)\nabla v\big] \cdot \nabla w dx. \end{aligned}$$

Then, using assumption (ii), we have

$$\begin{aligned} &\big|\langle (A_\mu u - A_0 u) - (A_\mu v - A_0 v), w\rangle\big| \\ &\leq \|S_\mu - S_0\|_{L^\infty} \int_D \big|p(|\nabla u|)\,\nabla u - p(|\nabla v|)\nabla v\big| |\nabla w|\,dx \\ &\leq L\|S_\mu - S_0\|_{L^\infty} \|\nabla(u-v)\|_{L^2(D)} \|\nabla w\|_{L^2(D)} \\ &\leq L\|S_\mu - S_0\|_{L^\infty} \|u-v\|\|w\| \\ &\leq LM|\mu|\|u-v\|\|w\|. \end{aligned}$$

Therefore, assumption $[\mathbf{C}_{A_\mu}]\mathbf{(2)}$ is verified with $m_1 = LM$.

On the other hand, since $\|A_\mu 0 - A_0 0\|_{H^*} = 0$, it follows that the constant θ in (23) is given by $\theta = LMR$.

- Finally, in this setting, condition (iv) becomes the control assumption in $[\mathbf{C}_d]$.

Thus, the conclusion follows directly from (22).

□

4 Some concluding remarks - an outlook

In this paper, we focused to deterministic hemivariational inequalities. By more involved arguments one could generalize some of the presented quantitative sensitivity results to more general classes of nonlinear variational inequalities and hemivariational inequalities in a random setting, see the qualitative sensitivity results in a L^2 (more generally L^P with $1 < p < \infty$) Bochner–Lebesgue space, see [14], respectively in the finer L^∞ Bochner–Lebesgue space, see [17].

Another direction of research is the extension to quasi (hemi) variational inequalities, see e.g. [8, 20].

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